# Quasi-Dependent Variables in Hybrid Automata

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## ABSTRACT

The concept of hybrid automata provides a powerful framework to model and analyze real-world problem domains. Due to the structural complexity of hybrid systems it is important to ensure the *scalability* of analysis algorithms. We approach this problem by providing an effective extension of the recently introduced notion of *quasi-equal* clocks to hybrid systems. For this purpose, we introduce the concept of *quasi-dependent* variables. Our contribution is two-fold: we demonstrate how such variables can be automatically *detected*, and we present a *transformation* leading to an abstraction with a smaller state space which, however, still retains the same properties as an original system. We demonstrate the practical applicability of our methods on a range of industrial benchmarks.

## 1. INTRODUCTION

Real-time systems often employ distributed architectures where every component uses an independent clock for internal purposes. The classes of TDMA- [2] and EPL-based [3] protocols are prominent examples of this particular class of systems. In such settings, the information exchange among the components proceeds periodically; components are assigned to certain time ranges for communication. In the end of every period, the components reset their clocks to ensure correct system behavior. This leads to an exponential blow-up of the set of states to be considered due to the interleaving semantics of distributed resets in timed automata.

We have attacked this problem in our previous work [4–6] by proposing the notion of *quasi-equal* clocks for timed automata to efficiently reduce the system complexity. Intuitively, two clocks are *quasi-equal* if they always agree on their values except possibly when those clocks are reset.

Hybrid automata provide an expressive framework to model and analyze real-world systems exhibiting complex continuous behavior described by differential equations. Therefore, the handling of continuous dynamics takes an essential part of the whole analysis run-time. Due to this reason, the reduction of the number of variables in the system and interleaving resets might lead to crucial performance improvements. To this end, we propose to move from an *equality* relation between quasi-equal clocks to a general *dependency* relation and thus *quasi-dependent* variables. Therefore, the notion of quasi-equality is a special case of quasi-dependency of variables.

We present an approach consisting of two parts. First, we detect quasi-dependent variables. Here, in some sense, we transfer the idea of variable relations already explored in program verification [7, 8] into the context of hybrid systems. Second, based on this knowledge, we reduce the original system. This line of research is particularly motivated by industrial time-based protocols where the timers of the individual components advance at *different* rates.

The crux behind our detection algorithm is the application of abstraction techniques while analyzing the relations between variables. Both continuous and discrete evolution can break the quasi-dependency between variables. However, interestingly, we only need to be precise while considering discrete jumps whereas in case of continuous evolution we can just check which quasi-dependencies still hold in the considered location. The intuition is that we can abstract away constraints other than the ones reflecting the quasi-dependencies because if a quasi-dependency holds before computing continuous successors (which is checked without precision loss in the preceding discrete step) then the continuous evolution obeys the quasi-dependency. The quasi-dependent variables can also be detected by a hybrid model checker, however, it would explore the state space of the system in a great detail, whereas our detection algorithm conducts a coarse analysis with the focus on the state space parts relevant for establishing quasi-dependencies. These ideas lead to an abstraction that is rather imprecise, yet results in a dramatic performance improvement of quasi-dependent variables detection.

Our transformation replaces quasi-dependent variables by one representative variable and updates the system structure appropriately. Properties of the original system are preserved, that is, a forbidden configuration is reachable in the transformed system if and only if it is reachable in the original system. System complexity is reduced in two ways. First, we reduce the number of variables in the system. This step alone leads to a more efficient system handling as the underlying data structures become more compact. Furthermore, in many cases we can achieve an even larger performance boost by completely avoiding interleavings of resets of quasi-dependent variables.

The paper is organized as follows. Section 2 introduces necessary preliminaries. Section 3 describes and discusses the concept of quasi-dependency. The detection algorithm is discussed and evaluated in Section 4, the transformation in Section 5. We conclude the paper with Section 6.

*Related Work.* The state space reduction for different classes of systems by deriving dependencies between their variables has been an active research topic since the inception of the area of verification. In particular, deriving (affine) relationships between variables of a program is classical since [7,8]. This idea has also been successfully applied to the domain of timed automata. Daws et al. [9] suggest an approach to reduce the number of clocks in a single automaton based on the notion of active and equal clocks. This work was extended by Daws et al. [10] to handle networks of automata. Active clocks were used by Étienne [11] to analyze parametric timed automata. The work [12] proposes an abstraction method for a restricted class of systems with multiple clocks being activated in a sequential manner for some bounded time. Clearly, with our approach we can handle timed automata as a subclass of hybrid automata, however, we impose weaker structural requirements compared to the above mentioned works.

In the scope of hybrid automata dependency detection and appropriate reduction becomes especially complicated. The main efforts in this area were concentrated towards reducing hybrid automata by finding appropriate (bi-)simulation relations. In particular, Pappas [13] and van der Schaft [14] presented a theoretical framework of bisimulation for linear systems. The idea of simulation was extended to its approximate version by Girard et al. [15] where the distance between the observed behavior of the reduced and original should be bounded. Finally, the notion of approximate simulations was lifted to hybrid automata by Girard et al. [16].

In this paper, we pose weaker requirements on the structure of the reduced system. Departing from strong bisimulation towards *weak* bisimulation between the reduced and original systems allows us to completely eliminate some configurations which are induced by interleavings. In order to ensure the preservation of the property to be checked, we rewrite the property itself. In this way, we keep the information which was lost during the system reduction.

## 2. PRELIMINARIES

In the following, we recall the definitions of hybrid automata [17], (non-Zeno) run, and observable behaviour for self-containedness. In addition, we introduce the notions of update point and instantaneous valuations. The latter denotes the non-empty, possibly non-singleton set of valuations observable at a given point in time.

Given a set of real-valued variables $Var$, we use $V$ to denote the set $(Var \rightarrow \mathbb{R})$ of *valuations* of $Var$. A *hybrid automaton* is a tuple $\mathcal{H} = (Loc, Var, Lab, Edge, Act, Inv, Init)$ where $Loc$ is a finite set of *locations*, $Var$ is a finite set of real-valued variables, $Lab$ is a set of *synchronisation labels* including the *stutter label* $\tau \in Lab$, $Edge \subseteq Loc \times Lab \times 2^{V^2} \times Loc$ is a finite set of directed *edges* including a *stutter edge* $(\ell, \tau, id, \ell)$ for each location $\ell \in Loc$. An edge $(\ell, a, \mu, \ell')$ from location $\ell$ to $\ell'$ is labelled with a label $a \in Lab$ and a *conditional update* $\mu$. $Act : Loc \rightarrow 2^{\mathbb{R}_0^+ \rightarrow V}$ is a function which assigns a set of *activities* $f : \mathbb{R}_0^+ \rightarrow V$ to each location. The activity sets are time-invariant, i.e., for each $t' \in \mathbb{R}_0^+$, $f \in Act(\ell)$ implies $(f + t) \in Act(\ell)$, where $(f + t)(t') = f(t + t')$. $Inv : Loc \rightarrow 2^V$ is a function which assigns an *invariant* $Inv(\ell) \subseteq V$ to
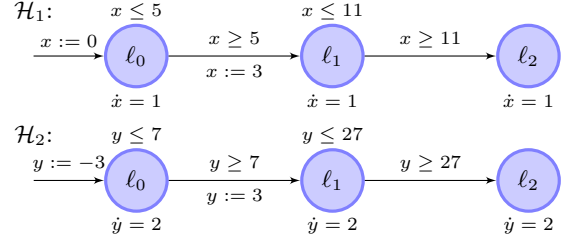


Figure 1: Example of a network $\mathcal{N}$ of hybrid automata.

each location $\ell \in Loc$, and $Init : Loc \rightarrow 2^V$ is a function which assigns an *initial value invariant* $Init(\ell) \subseteq V$ to each location $\ell \in Loc$. Locations $\ell \in Loc$ with $Init(\ell) \neq \emptyset$ are called *initial locations*.

A network $\mathcal{N} = \{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$ is a finite set of hybrid automata each with variables $Var$ and with pairwise disjoint sets of locations. We write $\mathcal{H} \in \mathcal{N}$ if and only if $\mathcal{H} \in \{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$. We use $Lab(\mathcal{H}), Edge(\mathcal{H})$, etc. to denote the set of labels, edges, etc. of automaton $\mathcal{H}$.

The *operational semantics* of a network $\mathcal{N}$ is defined by the (labelled) transition system

$$\mathcal{T}(\mathcal{N}) = (Conf(\mathcal{N}), \Lambda, \{\xrightarrow{\lambda} | \lambda \in \Lambda\}, C_{ini})$$

where $Conf(\mathcal{N}) = Loc \times V$, and $\Lambda := (\mathbb{R}_0^+ \times Act(\mathcal{N})) \cup \bigcup_{1 \leq i \leq n} Lab_i$ is the set of transition labels. The set of configurations $Conf(\mathcal{N})$ consists of pairs of *location vectors* $\vec{\ell} = \langle \ell_1, \ldots, \ell_N \rangle \in \times_{i=1}^n Loc(\mathcal{H}_i)$ and valuations. We write $\ell_{s,i}$, $1 \leq i \leq N$, to denote the location that automaton $\mathcal{H}_i$ assumes in a configuration $s = \langle \vec{\ell}_s, \nu_s \rangle$ and $\nu_{s,i}$ to denote $\nu_s|_{Var(\mathcal{H}_i)}$. The set of initial configurations is $C_{ini} = \{\langle \vec{\ell}, \nu \rangle \in Conf(\mathcal{N}) \mid \nu \in \cap_{1 \leq i \leq n} Init_i(\ell_i)\}$. There is a *discrete transition* $s \xrightarrow{a} s'$ from configuration $s = \langle (\ell_1, \ldots, \ell_n), \nu \rangle$ to configuration $s' = \langle (\ell'_1, \ldots, \ell'_n), \nu' \rangle$ if and only if there are edges $(\ell_i, a_i, \mu_i, \ell'_i) \in Edge_i$, $1 \leq i \leq n$, such that for each $i$, either $(\ell_i, a, \mu_i, \ell'_i) \in Edge_i$ or $(\ell_i, \tau, \mu_i, \ell'_i) \in Edge_i$ and $a \notin Lab_i$, and the edges are enabled by valuation $\nu$ and $\nu'$ is an effect of the updates applied to $\nu$, i.e. $(\nu, \nu') \in \bigcap_{1 \leq i \leq n} \mu_i$, $\nu' \in \bigcap_{1 \leq i \leq n} Inv(\ell_i)$. There is a *time transition* $s \xrightarrow{d,f} s'$ with delay $d \in \mathbb{R}_0^+$ from configuration $s = \langle \vec{\ell}, \nu \rangle$ to configuration $s' = \langle \vec{\ell}, \nu' \rangle$ if and only if action $f$ is in $\bigcap_{1 \leq i \leq n} Act(\ell_i)$ and $f(0) = \nu$, $f(d) = \nu'$, $\forall 0 \leq t \leq d \bullet f(t) \in \bigcap_{1 \leq i \leq n} Inv(\ell_i)$.

Note that multiple automata in a network can take labelled edges simultaneously; asynchronous labelled edges are only possible for local labels, i.e., labels that only occur in the set of labels of a single hybrid automaton $\mathcal{H} \in \mathcal{N}$.

A sequence $\pi = \langle \ell_0, \nu_0 \rangle, t_0 \xrightarrow{\lambda_1} \langle \ell_1, \nu_1 \rangle, t_1 \xrightarrow{\lambda_2} \cdots$ of time-stamped configurations is called *run* of $\mathcal{H}$ if and only if $\langle \ell_0, \nu_0 \rangle \in C_{ini}$, $t_0 = 0$, and for all $i \in \mathbb{N}_0$, we have $\lambda_{i+1} \in Lab$ and $t_i = t_{i+1}$ or $\lambda_{i+1} \in \{t_{i+1} - t_i\} \times Act$, and the transition relation $\langle \ell_i, \nu_i \rangle \xrightarrow{\lambda_{i+1}} \langle \ell_{i+1}, \nu_{i+1} \rangle$. We use $\Pi(\mathcal{H})$ to denote the set of all runs of $\mathcal{H}$; a hybrid automaton need not have a run.

One run of network $\mathcal{N}$ in Figure 1 is $\pi = \langle (\ell_0, \ell_0), (0, -3) \rangle$, $0 \xrightarrow{5,f} \langle (\ell_0, \ell_0), (5, 7) \rangle, 5 \xrightarrow{\tau} \langle (\ell_1, \ell_0), (3, 7) \rangle, 5 \xrightarrow{\tau} \langle (\ell_1, \ell_1), (3, 3) \rangle, 5 \xrightarrow{2,f} \langle (\ell_1, \ell_1), (7, 5) \rangle, 7 \cdots$
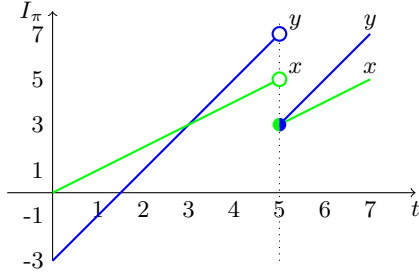
Figure 2: Observable behavior of variables $x$ and $y$ from $\mathcal{N}$.

In the following, we assume infinite runs without Zeno behaviour, i.e. where the sum of the time transition labels diverges. A configuration $s$ is called *reachable* in $\mathcal{H}$ if and only if there is a run $\pi \in \Pi(\mathcal{H})$ such that $s$ occurs in $\pi$.

Figure 1 shows an example of a network $\mathcal{N}$ consisting of the hybrid automata $\mathcal{H}_1$ and $\mathcal{H}_2$ with variables $x$ and $y$, respectively. Variables $x$ and $y$ with respective initial values 0 and $-3$ are set to 3 at the point in time 11 and 27, respectively. Yet, in the strict interleaving semantics of networks of hybrid automata, these updates occur one after the other.

The *observable behaviour* of run $\pi$ is a function $\mathcal{I}^\pi : \mathbb{R}_0^+ \to V$ which assigns to each point in time the valuation defined by an action during a non-zero delay in $\pi$, i.e. $\mathcal{I}^\pi(t) := f_{i+1}(t - t_i)$ where $i \in \mathbb{N}_0$ is the (unique) index such that $t_i \leq t < t_{i+1}$ and $\lambda_{i+1} = (d_{i+1}, f_{i+1})$. We use $\mathcal{I}_x^\pi : \mathbb{R}_0^+ \to \mathbb{R}$ to denote the function which assigns to each point in time the value of variable $x \in Var$ at that point in time, i.e. $\mathcal{I}_x^\pi$ is point-wise defined as $\mathcal{I}_x^\pi(t) := \mathcal{I}^\pi(t)(x)$ for $t \in \mathbb{R}_0^+$. We may omit the superscript if it is clear from the context. Figure 2 shows the observable behaviour of the network in Figure 1 corresponding to the example run $\pi$ above.

A point in time $t \in \mathbb{R}_0^+$ is called *update point* of $\pi$ if and only if a discrete transition occurs in $\pi$ at $t$, i.e. if

$$\exists i \in \mathbb{N}_0 \bullet \langle \ell_i, \nu_i \rangle, t_i \xrightarrow{a_{i+1}} \langle \ell_{i+1}, \nu_{i+1} \rangle, t_{i+1} \wedge t = t_{i+1}.$$

We use $\mathcal{U}^\pi$ to denote the set of update points of $\pi$, the index may be omitted if clear from context. The *instantaneous valuations* of $\pi$ is a function $\mathcal{V}^\pi : \mathbb{R}_0^+ \to 2^V \setminus \{\emptyset\}$ which assigns to each point in time the (non-empty) set of valuations that occurs at that point in time, i.e.

$$\mathcal{V}^\pi(t) := \{\mathcal{I}^\pi(t)\} \cup \{\nu_i \mid t_i = t\}.$$

We use $\mathcal{V}_x^\pi : \mathbb{R}_0^+ \to 2^\mathbb{R} \setminus \{\emptyset\}$ to denote the function which assigns to each point in time the set of values that variable $x \in Var$ assumes at that point in time, i.e. $\mathcal{V}_x^\pi$ is point-wise defined as $\mathcal{V}_x^\pi := \{\nu(x) \mid \nu \in \mathcal{V}^\pi(t)\}$ for $t \in \mathbb{R}_0^+$. We may omit the superscript if it is clear from the context.

In the following we assume that a set of *forbidden configurations* for network $\mathcal{N}$ is given in form of a *configuration formula* $CF$. $CF$ is any logical connection of *basic formulae* over $\mathcal{N}$ which are given by the grammar $\beta ::= \ell \mid \neg \ell \mid \varphi$ where $\ell \in L$, $1 \leq i \leq n$, and where $\varphi$ is any arithmetical constraint over the variables in $x$. A basic formula $\beta$ is *satisfied* by a configuration $s \in Conf(\mathcal{N})$ if and only if $\ell_{s,i} = \ell$, $\ell_{s,i} \neq \ell$, and $\nu_s \models \varphi$, respectively. We say $CF$ is *reachable*

in $\mathcal{N}$ if only if there is a configuration $s$ reachable in $\mathcal{T}(\mathcal{N})$ which satisfies $CF$.

## 3. QUASI-DEPENDENT VARIABLES

In the following, we define quasi-dependency between variables in terms of the observable behaviour of a hybrid automaton. We discuss two alternative characterisations of quasi-dependency and show how quasi-dependency induces an equivalence relation.

*Definition 1.* (Quasi-$f$-Dependent Variables) Let $x, y \in Var$ be two variables of the hybrid automaton $\mathcal{H}$. We say $x$ *quasi-depends on* $y$ via function $f$, denoted by $x \simeq_f y$, if and only if for each run of $\mathcal{H}$ and for each point in time, the value of $x$ in the observable behaviour is the value of $f$ applied to the value of $y$ in the observable behaviour, i.e. if

$$\forall \pi \in \Pi(\mathcal{H}) \; \forall t \in \mathbb{R}_0^+ \bullet \mathcal{I}_x^\pi(t) = f(\mathcal{I}_y^\pi(t)).$$

$f$ is called *dependency function* for $x$ wrt. $y$. $\diamond$

In the example of the network $\mathcal{N}$ from Figure 1, the variables $x$ and $y$ are quasi-dependent with dependency function $f(z) = 2z - 3$. At time 5, there are configurations where their values don't satisfy $f(y) = 2x - 3$, but these configurations do not contribute to the observable behaviour as there is no delay possible afterwards (cf. Figure 2).

Note that dependency functions are in general not unique. For example, if only value $a$ occurs in the observable behaviour of variable $x$ and only value $b$ for variable $y$, then $x$ and $y$ are quasi-dependent via any function $f$ with $f(a) = b$.

We can alternatively state quasi-dependency using instantaneous valuations and update points. The instantaneous values of quasi-dependent variables are related by the dependency function $f$ everywhere except for update points. At update points, the values given by the observable behaviour have to adhere to $f$.

LEMMA 3.1. *Let $x, y \in Var(\mathcal{H})$ be two variables of hybrid automaton $\mathcal{H}$ such that $x \simeq_f y$ with dependency function $f$. Then $x \simeq_f y$ if and only if $\forall \pi \in \Pi(\mathcal{H}) \bullet$*

$$\mathcal{V}_x^\pi|_{\mathbb{R}_0^+ \setminus \mathcal{U}} = (f \circ \mathcal{V}_y^\pi)|_{\mathbb{R}_0^+ \setminus \mathcal{U}} \wedge \forall t \in \mathcal{U} \; \bullet \; \mathcal{I}_x^\pi(t) = f(\mathcal{I}_y^\pi(t))$$

*where $f \circ \mathcal{V}_y$ is the set-valued function which yields the set of values obtained by applying $f$ to all values in $\mathcal{V}_y$.* $\diamond$

The following lemma states a third approach to quasi-dependency: two variables are quasi-dependent if the accumulated duration where their values are not related by the dependency function is 0.

LEMMA 3.2. *Let $x, y \in Var(\mathcal{H})$ be two variables of hybrid automaton $\mathcal{H}$. Let $\pi \in \Pi(\mathcal{H})$ be a run of $\mathcal{H}$. Let $\Delta : \mathbb{R}_0^+ \to \{0, 1\}$ be the characteristic function of the points in time where the value of $x$ is not obtained by applying the dependency function $f$ to the value of $y$, i.e.*

$$\Delta(t) = \begin{cases} 1, & \text{if there exist } v \in \mathcal{V}_x^\pi(t), w \in \mathcal{V}_y^\pi(t) \text{ s.t. } v \neq f(w) \\ 0, & \text{otherwise.} \end{cases}$$

*Then $x \simeq_f y$ implies $\int_0^\infty \Delta(t) dt = 0$.* $\qquad\qquad\qquad\diamond$

Note that Lemma 3.2 is only an implication. The opposite direction does not hold, e.g., with actions that non-continuously change their value at isolated points during a time transition. The integral is blind for such changes, in the observable behaviour (and thus for Definition 1) they are present.

*Definition 2.* (Quasi-Dependent Variables) Let $x, y \in Var$ be variables of hybrid automaton $\mathcal{H}$. We say $x$ *quasi-depends on* $y$, denoted by $x \equiv y$, if and only if there exist dependency functions $f$ and $g$ such that $x \simeq_f y$ and $y \simeq_g x$. $\qquad\diamond$

Quasi-dependency induces an equivalence relation as follows.

LEMMA 3.3 (EQUIVALENCE RELATION). *Let $\mathcal{H}$ be a hybrid automaton. Quasi-dependency $\equiv$ is an equivalence relation on the variables $Var$ of $\mathcal{H}$.* $\qquad\diamond$

In the following, we use $\mathcal{EC}_\mathcal{N}$ to denote the set $\{Y \in Var/\equiv \mid 1 < |Y|\}$ of equivalence classes of quasi-dependent variables of $\mathcal{N}$ with at least two elements. For each $Y \in \mathcal{EC}_\mathcal{N}$, we assume a designated representative variable denoted by $rep(Y)$. For each $x \in Y$, we use $rep(x)$ to denote $rep(Y)$.

LEMMA 3.4. *Let $x, y \in Var$ be variables of hybrid automaton $\mathcal{H}$ such that $x \simeq_f y$ and $y \simeq_g x$. Then $f$ is the inverse of $g$ in the sense that*

$$\forall \pi \in \Pi(\mathcal{H}) \; \forall t \in \mathbb{R}_0^+ \bullet g(f(\mathcal{I}_y^\pi(t))) = \mathcal{I}_y^\pi(t).$$

Note that the notion of quasi-dependency of variables substantially generalizes the notion of quasi-equal clocks from [4]. Firstly, timed-automata are a subclass of hybrid automata where all continuous variables increase with rate 1 and can only be assigned to 0. Two clocks in a timed-automaton are quasi-equal if and only if they are quasi-dependent via the identity function.

## 4. QUASI-DEPENDENCY DETECTION

For our detection of quasi-dependent variables in hybrid automata, we consider the well-known symbolic semantics of hybrid automata where sets of valuations are represented by so called *regions*. For the rest of this section, let $\mathcal{H}$ denote the hybrid automaton $(Loc, Var, Lab, Edge, Act, Inv, Init)$.

*Definition 3.* (Symbolic Semantics) The *symbolic semantics* of a hybrid automaton $\mathcal{H}$ is defined by the region transition system $\mathcal{T}(\mathcal{H}) = (Conf'(\mathcal{H}), \Longrightarrow, C_{ini})$ where

- $Conf'(\mathcal{H}) = Loc \times (2^V \setminus \{\emptyset\})$ is the set of configurations, consisting of pairs of a location and a set of valuations,

- $C_{ini} = \{\langle \ell, Z \rangle \in Conf'(\mathcal{H}) \mid Z \subseteq Init(\ell)\}$ is the set of initial configurations, and

- $\Longrightarrow$ is the transition relation where
  - there is a *discrete transition* $\langle \ell, Z \rangle \Longrightarrow \langle \ell', Z' \rangle$ induced by edge $(\ell, a, \mu, \ell') \in Edge$ if $Z' = \{\nu' \mid \exists \nu \in Z \bullet (\nu, \nu') \in \mu$ and $\nu' \in Inv(\ell')\}$, and
  - there is a *time transition* $\langle \ell, Z \rangle \Longrightarrow \langle \ell, Z' \rangle$ if $Z' = \{\nu' \mid \exists f \in Act(\ell), \nu \in Z, t \in \mathbb{R}_0^+ \bullet f(0) = \nu \wedge f(t) = \nu' \wedge \forall 0 \le t' \le t \bullet f(t') \in Inv(\ell)\}$. $\qquad\diamond$

We call configurations where no flow of positive duration is possible, i.e. the only possible flow has duration zero, zero time configurations.

*Definition 4.* (Zero time configuration) A configuration $\langle \ell, Z \rangle$ is called *zero time*, if and only if the invariant of $\ell$ does not allow time to elapse from any valuation in $Z$, i.e. if

$$\forall f \in Act(\ell), \nu \in Z, t \in \mathbb{R}_0^+ \bullet$$
$$f(0) = \nu \wedge f(t) \in Inv(\ell)$$
$$\wedge \, \forall 0 \le t' \le t \bullet f(t') \in Inv(\ell) \implies t = 0.$$

We write $\mathsf{zt}(\ell, Z)$ if and only if $\langle \ell, Z \rangle$ is zero time. $\qquad\diamond$

Note that the values of quasi-dependent variables are *not* related by the dependency function at most in zero time configurations, and that they may assume any value during zero-time configurations without violating quasi-dependency. The other way round, quasi-dependecy is violated in each non-zero time configuration where the values of the variables are not related by the given dependency function. Thus our analysis tries to establish that for each transition sequence involving only zero-time configurations, whenever two variables are related by the quasi-dependency function when entering a zero-time configuration, they will be related by the quasi-dependency function again when entering a non zero-time configuration.

To this end, our analysis uses precise values for variables only during zero-time configurations. For non zero-time configurations, the *relax operator* only preserves the information which variables are related by which dependency function.

*Definition 5.* (Relax operator) The *relax operator* $\mathsf{rlx}$ applied to the configuration $\langle \ell, Z \rangle$ of $\mathcal{H}$ over-approximates the region $Z$ by a conjunction of the quasi-dependent functions it entails. Formally, $\mathsf{rlx}(\ell, Z) := \langle \ell, Z' \rangle$ where

$$Z' = \bigwedge \{y = f(x) \mid x, y \in Var \text{ and } \forall g \in Act(\ell), \nu \in Z \bullet$$
$$g(0)(x) = \nu(x) \wedge g(0)(y) = \nu(y)$$
$$\implies \forall t \in \mathbb{R}_+^0 \bullet g(t)(y) = f(g(t)(x))\}.$$

The abstraction function depends on whether the given configuration is zero time. If it is zero time, then the abstraction goes as precise as possible by preserving all the information in the given region. If the configuration is not zero time, then the abstraction will preserve only quasi-dependency relations. In practice, the number of zero time configurations is often small in comparison with the number of non-zero
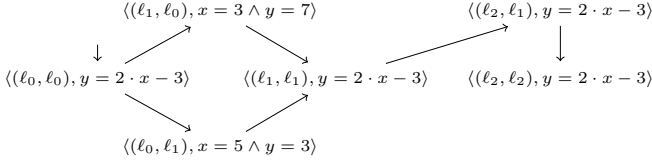
$\langle(\ell_1, \ell_0), x = 3 \wedge y = 7\rangle$ $\langle(\ell_2, \ell_1), y = 2 \cdot x - 3\rangle$

$\langle(\ell_0, \ell_0), y = 2 \cdot x - 3\rangle$ $\langle(\ell_1, \ell_1), y = 2 \cdot x - 3\rangle$ $\langle(\ell_2, \ell_2), y = 2 \cdot x - 3\rangle$

$\langle(\ell_0, \ell_1), x = 5 \wedge y = 3\rangle$

Figure 3: Abstract region graph of network $\mathcal{N}$ from Figure 1.

time configurations. Since for every non-zero time configuration the abstraction function maps given regions to much bigger ones, the size of the resulting abstract system is then much smaller than the corresponding concrete one.

*Definition 6.* (Zero time abstraction function) The abstraction function $\alpha_{\mathsf{zt}}$ applied to a configuration $\langle \ell, Z \rangle$ leaves the configuration unaffected if the configuration is zero time and it abstracts it using the relax operator otherwise, i.e.

$$\alpha_{\mathsf{zt}}(\langle \ell, Z \rangle) := \begin{cases} \langle \ell, Z \rangle & \text{if } \mathsf{zt}(\ell, Z) \\ \mathsf{rlx}(\ell, Z) & \text{otherwise} \end{cases}$$

The following lemma ensures that the abstraction function applied to a configuration produces a configuration with a bigger or equal region.

LEMMA 4.1. *The abstraction function is increasing and idempotent, i.e., let $\langle \ell, Z \rangle \in Conf'(\mathcal{H})$ be a configuration.*

1. $Z \subseteq Z'$ *if* $\langle \ell, Z' \rangle = \alpha_{\mathsf{zt}}(\langle \ell, Z \rangle)$, *and*

2. $\alpha_{\mathsf{zt}}(\alpha_{\mathsf{zt}}(\langle \ell, Z \rangle)) = \alpha_{\mathsf{zt}}(\langle \ell, Z \rangle)$. $\diamond$

Our technique generates an abstract region graph, where quasi-dependent variables can be soundly detected by traversing the graph. The following definition is constructive in the sense that it tells how to construct the abstract region graph: compute the initial configurations firstly and then for each configuration, compute its successors using the abstraction function $\alpha_{\mathsf{zt}}$. It is important to note that there are only discrete transitions in the abstract region graph. This is because, by definition of the abstraction function there are two possibilities: either a configuration is zero time, meaning it does not have time successors, or the configuration is non-zero time. In the latter case the relax operator is applied, and yields a region which includes all time successors.

*Definition 7.* (Abstract region graph) $\mathcal{H}$ induces the abstract region graph $\mathcal{T}^{\#}(\mathcal{H}) = (Conf'(\mathcal{H}), \Longrightarrow^{\#}, C_0^{\#})$ where

- $\Longrightarrow^{\#} \subseteq Conf'(\mathcal{H}) \times Conf'(\mathcal{H})$ is the transition relation. There is a transition $\langle \ell, Z \rangle \Longrightarrow^{\#} \alpha_{\mathsf{zt}}(\langle \ell', F \rangle)$ if there is an edge $(\ell, a, \mu, \ell') \in E$ such that $F = \{\nu' \mid \exists \nu \in Z \bullet (\nu, \nu') \in \mu$ and $\nu' \in Inv(\ell')\}$ is not empty, and

- $C_0^{\#} = \{\alpha_{\mathsf{zt}}(\langle \ell, Z \rangle) \mid \langle \ell, Z \rangle \in Conf'(\mathcal{H}) \wedge Z = Init(\ell)\}$ is the set of initial configurations. $\diamond$

Figure 3 shows the abstract region graph for the running example. The abstract region graph simulates the region transition system thus quasi-dependency detection on the abstract region graph is sound.

*Definition 8.* (Simulation relation) A simulation relation $\leq$ for two region transition systems $\mathcal{T}_i = (Conf'(\mathcal{H}), \Longrightarrow_i, C_{ini_i})$, $i \in \{1, 2\}$, is a binary relation on $Conf'(\mathcal{H})$ satisfying the following properties:

1. for all $c_{ini_1} \in C_{ini_1}$ there exists $c_{ini_2}$ with $c_{ini_1} \leq c_{ini_2}$,

2. $\langle \ell_1, Z_1 \rangle \leq \langle \ell_2, Z_2 \rangle$ implies $\ell_1 = \ell_2$, $Z_1 \subseteq Z_2$, and

3. $\langle \ell_1, Z_1 \rangle \leq \langle \ell_2, Z_2 \rangle$ and $\langle \ell_1, Z_1 \rangle \Longrightarrow_1 \langle \ell_1', Z_1' \rangle$ implies that there exists a configuration $\langle \ell_2', Z_2' \rangle$ such that $\langle \ell_2, Z_2 \rangle \Longrightarrow_2 \langle \ell_2', Z_2' \rangle$ and $\langle \ell_1', Z_1' \rangle \leq \langle \ell_2', Z_2' \rangle$.

We say that $\mathcal{T}_2$ *simulates* $\mathcal{T}_1$ if and only if there exists a simulation relation for $\mathcal{T}_1, \mathcal{T}_2$. $\diamond$

THEOREM 4.2. $\mathcal{T}^{\#}(\mathcal{H})$ *simulates* $\mathcal{T}(\mathcal{H})$. $\diamond$

Quasi-dependency of variables $x$ and $y$ via dependency function $f$ is checked on the abstract region graph by checking, for each non-zero time configuration $\langle \ell, Z \rangle$ whether $Z$ entails $x = f(y)$. If there is a (possibly spurious) counter-example for the quasi-dependency, the corresponding application of the relax operator removes this dependency from the region.

Note that Definition 5 needs a given dependency function candidate $f$. For *linear* hybrid automata, also known as *multi-rate* automata, dependency function candidates can easily be constructed from locations' actions (rates) and initial values of variables. Further note that two variables $x, y$ having the same rate in each location does not imply quasi-dependency between $x$ and $y$; similarly, having different rates for $x$ and $y$ in some (even reachable) location does not imply that there is no quasi-dependency between them.

## 4.1 Experimental Results

We have implemented our approach in our tool sAsEt. We have implemented regions as predicates over reals and use the SMT solver Z3 [18] to compute the result of the relax operator, whether a configuration is zero time, and whether guards are satisfied. As Z3 is restricted to linear arithmetics, the scope of our implementation is restricted to *linear* hybrid automata. sAsEt is based on the Jahob system [19, 20]. It takes as input a network of hybrid automata in the SpaceEx [21] format and computes successors on the fly.

Table 1 compares the run-time of sAsEt with SpaceEx on two artificial benchmarks which represent two typical cases for quasi-dependent variables in hybrid models. As SpaceEx does not support the detection of quasi-dependency directly, we introduced an auxiliary continuous variable $t$ with rate $\dot{t} = 1$ in all locations and which is set to 0 on all edges. Then checking for quasi-equality, e.g., for the running example would amount to checking $(y < 2x + 3 \vee y > 2x + 3) \wedge t > 0$.

---

[1]AMD Opteron 6174, 2.2GHz, 16GB, Linux 2.6.32-5-amd64; SpaceEx 0.9.7c (32bit); $t(s)$ gives user+system time.
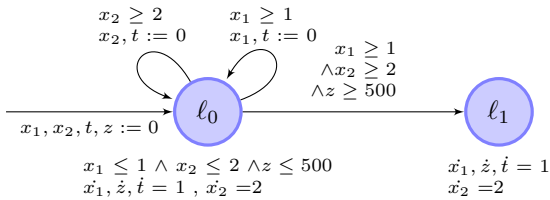
Figure 4: Class B benchmark: $x_1 \simeq_f x_2$ via $f(x) = 0.5 \cdot x$.

| | | sAsEt | | | SpaceEx | |
|---|---|---|---|---|---|---|
| Cl. | $n$ | SMT-calls | states | $t$ (s) | iterat. | $t$ (s) |
| A | 1 | 4 | 3 | 0.34 | 1003 | 2.35 |
| | 2 | 5 | 4 | 0.64 | 1503 | 4.21 |
| | 3 | 6 | 5 | 1.00 | 2003 | 6.79 |
| | 4 | 7 | 6 | 1.48 | 2503 | 10.23 |
| B | 1 | 17 | 4 | 0.91 | 1502 | 7.52 |
| | 2 | 71 | 8 | 5.12 | 3502 | 49.14 |
| | 3 | 278 | 16 | 25.72 | 7502 | 283.70 |
| | 4 | 1255 | 35 | 142.20 | - | - |

Table 1: $x_0 \simeq_{f_i} x_i$ via $f_i(x) = x/i + 1$. Timeout 1200 sec.[1]

In the Class A benchmark, there exists an ordering on the assignments of quasi-dependent variables. This yields a linear increase of the number of zero time configurations. Both sAsEt and SpaceEx scale, but the non quasi-dependent variables are early abstracted away by our approach yielding a very small abstract region graph in comparison with the one obtained by SpaceEx. In the Class B benchmark (cf. Figure 4), there is non-determinism between the assignments of quasi-dependent variables which yields an exponential growth of the number of zero time configurations. Hence the number of SMT calls grow exponentially. Yet in addition of being faster, the abstraction is more space efficient than SpaceEx. Note that the use of SMT calls amounts to 80% of the computation time.

## 5. TRANSFORMATION

In this section we present our tranformation for networks of hybrid automata which reduces a given set of quasi-dependent variables and reflects reachability of forbidden configurations. For simplicity, we impose a set of syntactical criteria called well-formedness rules on networks.

Firstly, we introduce some notions which we need to state our rules. Let $W \subseteq V$ be a set of valuations. We use $vars(W) \subseteq Var$ to denote the set of variables that are constrained by $W$, i.e. $vars(W) = \{x \in Var \mid \{\nu(x) \mid \nu \in W\} \neq \mathbb{R}\}$. Analogously, for update $\mu \subseteq V \times V$ and action $f$, we use $vars(\mu)$ and $vars(f)$, to denote the set of variables that are constrained by $\mu$ and $f$, respectively. We use $\mathcal{V}(\mathcal{H})$ to denote the set of variables that are constrained by some initial condition, location invariant, update, or action in $\mathcal{H}$.

We use $\mathcal{SE}_Y(\mathcal{H})$ to denote the set of *simple resetting edges* of hybrid automaton $\mathcal{H}$ where only quasi-dependent variables from $Y$ are constrained by the conditional update and which have action $\tau$, i.e., $\mathcal{SE}_Y(\mathcal{H}) = \{(\ell, \tau, \mu, \ell') \in Edge(\mathcal{H}) \mid vars(\mu) \subseteq Var(\mathcal{H}) \cap Y\}$.

We use $\mathcal{CE}_Y(\mathcal{H})$ to denote the set of *complex resetting edges* of $\mathcal{H}$ where quasi-dependent and non-quasi-dependent variables occur in the conditional update, or which have a label different from $\tau$, i.e., $\mathcal{CE}_Y(\mathcal{H}) = \{(\ell, a, \mu, \ell') \in Edge(\mathcal{H}) \mid vars(\mu) \cap Y \neq \emptyset \wedge (vars(\mu) \setminus Y \neq \emptyset \vee a \neq \tau)\}$. We use $\mathcal{LS}_Y(\mathcal{H})$ and $\mathcal{LC}_Y(\mathcal{H})$ to respectively denote the set of source and destination locations of simple and complex resetting edges of $\mathcal{H}$. We use $\mathcal{E}_Y(\mathcal{H}) = \mathcal{SE}_Y(\mathcal{H}) \cup \mathcal{CE}_Y(\mathcal{H})$ to denote the set of resetting edges of $\mathcal{H}$ wrt. $Y$, and $\mathcal{RES}_Y(\mathcal{N})$ to denote the set of automata in $\mathcal{N}$ which have a resetting edge, i.e., $\mathcal{RES}_Y(\mathcal{N}) = \{\mathcal{H} \in \mathcal{N} \mid \mathcal{E}_Y(\mathcal{H}) \neq \emptyset\}$. A location $\ell$ ($\ell'$) is called is called *reset (successor) location* wrt. $Y$ in $\mathcal{N}$ if and only if there is a resetting edge in $\mathcal{E}_Y(\mathcal{H})$ from (to) $\ell$ ($\ell'$). We use $\mathcal{RL}_Y$ ($\mathcal{RL}_Y^+$) to denote the set of reset (successor) locations wrt. $Y$ in $\mathcal{N}$ and we set $\mathcal{RL}_\mathcal{N} := \bigcup_{Y \in \mathcal{EC}_\mathcal{N}} \mathcal{RL}_Y$ and similarly $\mathcal{RL}_\mathcal{N}^+$.

*Definition 9.* (Well-formed Network) A network $\mathcal{N}$ is called *well-formed* if and only if it satisfies the following restrictions for each set of quasi-dependent variables $Y \in \mathcal{EC}_\mathcal{N}$:

(R1) All resetting edges update at most one variable $x \in Y$, and, given $x \simeq_f y$, the guard of the edge and the invariant of its source location and the assignment, resp., determine unique values $C_Y, O_Y \in \mathbb{R}$, i.e.

$$\exists C_Y, O_Y \in \mathbb{R} \, \forall (\ell, a, \mu, \ell') \in \mathcal{E}_Y(\mathcal{N}) \, \exists x \in Y \bullet$$
$$\forall (\nu, \nu') \in \mu \bullet \nu|_{\mathcal{V} \setminus \{x\}} = \nu'|_{\mathcal{V} \setminus \{x\}}$$
$$\wedge \, \forall \nu \in (\mu \downarrow_1) \cap Inv(\ell) \bullet \nu(x) = C_Y$$
$$\wedge \, \forall \nu \in (\mu \downarrow_2) \bullet \nu(x) = O_Y,$$

where $\mu \downarrow_i$ denotes projection onto the $i$-th component.

(R2) There are no two resetting edges from one location, i.e.

$$\forall e_1 = (\ell_1, a_1, \mu_1, \ell_1'), e_2 = (\ell_2, a_2, \mu_2, \ell_2') \in \mathcal{E}_Y(\mathcal{N}) \bullet$$
$$e_1 \neq e_2 \implies \ell_1 \neq \ell_2.$$

(R3) All edges that synchronise on some channel either all reset a clock from $Y$ or none does, i.e.,

$$\forall a \in Lab \bullet (\exists e = (\ell, a, \mu, \ell') \in Edge(\mathcal{N}) \bullet e \in \mathcal{E}_Y(\mathcal{N}))$$
$$\implies (\forall e = (\ell, a, \mu, \ell') \in Edge(\mathcal{N}) \bullet e \in \mathcal{E}_Y(\mathcal{N})).$$

(R4) No guard relates two or more variables from $Y$, i.e.

$$\forall e = (\ell, a, \mu, \ell') \in Edge(\mathcal{N}) \, \forall x \in Y \bullet (\mu \downarrow_1)|_{\{x\}} \subsetneq \mathbb{R}$$
$$\implies \forall y \in Y \setminus \{x\} \bullet (\mu \downarrow_1)|_{\{y\}} = \mathbb{R}.$$

Note that network $\mathcal{N}$ from Figure 1 is well-formed.

In the following we describe our transformation procedure $\mathcal{K}$. It works with two given inputs, a well-formed network $\mathcal{N} = \{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$ and a set of equivalence classes $\mathcal{EC}_\mathcal{N}$ of quasi-dependent variables. The output is the transformed network $\mathcal{N}' = \{\mathcal{H}_1', \ldots, \mathcal{H}_n'\} \cup \{\mathcal{R}_Y \mid Y \in \mathcal{EC}_\mathcal{N}\}$ which consists of hybrid automata $\mathcal{H}_i' = (Loc'(\mathcal{H}_i), Var', Lab'(\mathcal{H}_i), Edge'(\mathcal{H}_i), Act'(\mathcal{H}_i), Inv'(\mathcal{H}_i), Init'(\mathcal{H}_i))$ which are obtained as modifications of the $\mathcal{H}_i$, and one *resetter* automaton for each equivalence class $Y \in \mathcal{EC}_\mathcal{N}$ (cf. Appendix A).

The common set of variables $Var'$ consists of the non-quasi-dependent variables in $\mathcal{N}$, one representative $rep(Y)$ for each equivalence class, and bookkeeping variables $rst_Y^{I_\mathcal{H}}$ and $rst_Y^{O_\mathcal{H}}$, $\mathcal{H} \in \mathcal{N}$. For each equivalence class $Y \in \mathcal{RES}_{\mathcal{EC}_\mathcal{N}}(\mathcal{N})$, the set of labels of $\mathcal{H}$ is extended by the fresh label $reset_Y$ if $\mathcal{H} \in \mathcal{RES}_Y(\mathcal{N})$.

For each automaton $\mathcal{H} \in \mathcal{N}$, and for each complex edge $e \in \mathcal{CE}_Y(\mathcal{H})$, the set of locations is extended by one fresh location $\ell_{\xi_{Y,e}}$ and the set of edges is extended by one edge with label $reset_Y$ and no guard and no assignment from the source of $e$ to $\ell_{\xi_{Y,e}}$. The source location of $e$ is changed to be $\ell_{\xi_{Y,e}}$, any guard on and assignment of quasi-dependent variables is removed from its $\mu$. For each simple edge, its label $\tau$ is replaced by $reset_Y$ and any guard on and assignment of quasi-dependent variables is removed from its $\mu$.

In order to keep track on whether a hybrid automaton is ready to reset a quasi-dependent variable or has just reset its variable, the assignments of all edges with a reset location as destination are changed such that $rst_Y^{I_\mathcal{H}}$ is assigned 1, the assignments of all non-resetting edges with a reset location as source are changed such that $rst_Y^{I_\mathcal{H}}$ is assigned 0, and the assignments of all resetting edges (including those redirected to have source $\ell_{\xi_{Y,e}}$) is changed such that $rst_Y^{O_\mathcal{H}}$ is assigned 0. Note that $rst_Y^{I_\mathcal{H}}$ may also be assigned 1 on resetting edges in case of loops.

For each location, any action $f \in Act(\mathcal{H})$ is restricted to $Var'$ and modified such that $rst_Y^{I_\mathcal{H}}$ and $rst_Y^{O_\mathcal{H}}$ keep their value over time, and such that the evolution of the representative variables is unconstrained. Their evolution is determined by the resetter automata. The actions of the locations $\ell_{\xi_{Y,e}}$ keep all values constant. For each location, the valuations in the invariant and the initial valuations set are modified such that they express the original guard in terms of the representative variable and the dependency function.

For example, applying $\mathcal{K}$ to network $\mathcal{N}$ from Figure 1 with the set of quasi-dependent variables $Y = \{x, y\}$ yields the automata $\mathcal{H}_1'$ and $\mathcal{H}_2'$ as shown in Figure 5. Note that only the representative variable of $Y$ remains, namely variable $x$.

The *resetter* for equivalence class $Y$ is the hybrid automaton $\mathcal{R}_Y$ with two locations $\ell_{ini,\mathcal{R}_Y}$ and $\ell_{nst,Y}$. and two edges $(\ell_{ini,\mathcal{R}_Y}, reset_Y, \mu_1, \ell_{nst,Y})$ and $(\ell_{nst,Y}, \tau, \mu_2, \ell_{ini,\mathcal{R}_Y})$. The fist edge obtains a guard $\mu_1$ which checks whether the representative variable of equivalence class $Y$ is $C_Y$ and $rst_Y^{I_\mathcal{H}}$ evaluates to $n_Y := |\mathcal{RES}_Y(\mathcal{N})|$, the number of automata that reset the clocks of $Y$. $\mu_1$ assigns all $rst_Y^{I_\mathcal{H}}$ to 0 and the representative variable to $O_Y$. The second edge obtains a guard $\mu_2$ which checks whether $rst_Y^{O_\mathcal{H}}$ evaluates to 0, and assigns $n_Y$ to $rst_Y^{O_\mathcal{H}}$.

In location $\ell_{ini,\mathcal{R}_Y}$, the invariant is not constraining any variable, in $\ell_{nst,Y}$ we require that the value of the representative is $O_Y$. The variables $rst_Y^{I_\mathcal{H}}$ and $rst_Y^{O_\mathcal{H}}$ are initialized to $iL_Y := |\{\mathcal{H} \in \mathcal{N} \mid \ell_{ini,\mathcal{H}} \in \mathcal{RL}_Y\}|$, the number of automata whose initial location is a reset location of $Y$, $n_Y$, respectively, in $\ell_{ini,\mathcal{R}_Y}$; $\ell_{nst,Y}$ is not an initial location. Both locations inherit the actions for the representative variable $x$ from the automata constraining $x$.
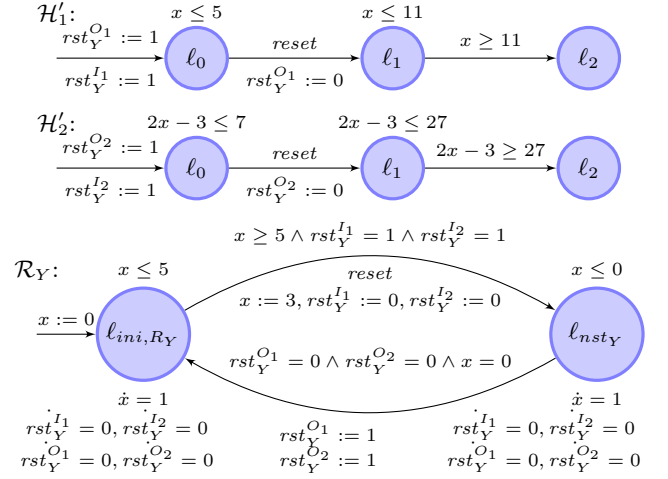


Figure 5: Network $\mathcal{N}'$ after applying the transformation.

See Figure 5 for an example of a resetter. Note that and the update operation is delegated to the resetter. Note that well-formedness together with the counter variables $rst_Y^{I_\mathcal{H}}$ enforces blocking multicast synchronisation, i.e., always all automata from $\mathcal{RES}_Y(\mathcal{N})$ participate in the reset.

To support all possible queries, we use the location $\ell_{nst,Y}$ to summarise information from configurations that are induced by interleaving updates of quasi-dependent variables. The variables $rst_Y^{O_\mathcal{H}}$ are introduced to indicate automaton $\mathcal{R}_Y$ how many automata still need to take their reset edges. The following function $\Omega$ syntactically transforms properties over a well-formed network $\mathcal{N}$ into properties over $\mathcal{N}' = \mathcal{K}(\mathcal{N}, \mathcal{EC}_\mathcal{N})$. Given a configuration formula $CF$, it outputs an equivalent property which can be verified in $\mathcal{N}'$.

Function $\Omega$ treats queries for source or destination locations of resetting edges special and instead refers to the summary location $\ell_{nst,Y}$. For instance, for a simple resetting edge $e \in \mathcal{SE}_Y(\mathcal{H})$ of some $\mathcal{H} \in \mathcal{N}$, the source location $\ell$ of $e$ can be assumed in $\mathcal{N}$ in different configurations: either the reset time is not yet reached, or the reset time is reached but $\mathcal{H}$ did not reset yet, while other automata in $\mathcal{RES}_Y(\mathcal{N})$ may have reset their quasi-dependent variables already. In $\mathcal{N}'$, all edges resulting from simple edges fire at once on the synchronisation on $reset$, so all source locations are left together. With this synchronisation, the resetter moves to $\ell_{nst,Y}$, which represents all similar configurations of $\mathcal{N}$ where all simple edges are in their source or destination location and the variables updated on these edges have value $C_Y$ or $O_Y$. Thus the location $\ell$ is reachable in $\mathcal{N}$ if and only if, for $\mathcal{N}'$, (i) $\ell_{nst,Y}$ is reachable, or (ii) $\ell$ is reached while being stable, i.e., not being in $\ell_{nst,Y}$.

Note that, although involving the two choices of location and variable value, there are actually only two cases (not four) summarised by $\ell_{nst,Y}$: assuming the source location of a simple resetting edge implies that the representative has value $C_Y$, assuming the destination location implies the value $O_Y$. Furthermore, we can only assume either the source or the destination location of the simple resetting edge. To this end, we introduce fresh existentially quantified variables $\tilde{\ell}$ and $\ell x$ in $\Omega_0$ and conjoin them with a feasibility conjunc-

$$\Omega_0(\beta) = \begin{cases} (\ell \wedge \neg \ell_{nst,Y}) \vee (\ell_{nst,Y} \wedge \tilde{\ell}) & \text{, if } \beta = \ell, \ell \in \mathcal{LS}_Y(\mathcal{N}) \cup (\mathcal{LC}_Y(\mathcal{N}) \cap \mathcal{RL}_Y), Y \in \mathcal{EC}_\mathcal{N}. \\ (\neg \ell \wedge \neg \ell_{nst,Y}) \vee (\ell_{nst,Y} \wedge \neg \tilde{\ell}) & \text{, if } \beta = \neg \ell, \ell \in \mathcal{LS}_Y(\mathcal{N}) \cup (\mathcal{LC}_Y(\mathcal{N}) \cap \mathcal{RL}_Y), Y \in \mathcal{EC}_\mathcal{N}. \\ (\Gamma(\varphi) \wedge \neg \ell_{nst,Y}) \vee (\ell_{nst,Y} \wedge \tilde{\varphi}) & \text{, if } \beta = \varphi, \tilde{\varphi} = \varphi[x/\tilde{x} \mid x \in Var], Y \in \mathcal{EC}_\mathcal{N}. \\ \beta & \text{, otherwise} \end{cases}$$

$$\Omega(CF) = \exists \tilde{x}_1, \ldots, \tilde{x}_k \, \exists \tilde{\ell}_1, \ldots, \tilde{\ell}_m \bullet$$

$$\Omega_0(CF) \wedge \bigwedge_{\substack{1 \le i \le k, 1 \le j \le m, \\ x_j \in \mathcal{X}_p \cap Y, 1 \le p \le n, \\ \ell_i \in L_p \cap (\mathcal{RL}_Y \setminus \mathcal{RL}_Y^+)}} (\tilde{\ell}_i \implies \tilde{x}_j = C_Y) \wedge \bigwedge_{\substack{1 \le i \le k, 1 \le j \le m, \\ x_j \in \mathcal{X}_p \cap Y, 1 \le p \le n, \\ \ell_i \in L_p \cap (\mathcal{RL}_Y^+ \setminus \mathcal{RL}_Y)}} (\tilde{\ell}_i \implies \tilde{x}_j = O_Y) \wedge \bigwedge_{\substack{(\ell, a, \mu, \ell') \\ \in \mathcal{SE}_Y(\mathcal{H}), \\ \ell_i \in \{\ell, \ell'\}}} (\tilde{\ell}_i \implies \ell') \wedge \bigwedge_{\substack{(\ell, a, \mu, \ell') \\ \in \mathcal{CE}_Y(\mathcal{H}), \\ \ell_i = \ell}} (\tilde{\ell}_i \implies \ell_{\xi_{Y,e}}) \wedge \bigwedge_{\substack{1 \le i \ne j \le m, \\ 1 \le p \le n \\ \ell_i, \tilde{\ell}_j \in L_p,}} \neg(\tilde{\ell}_i \wedge \tilde{\ell}_j)$$

Table 2: Formula transformation function $\Omega$. $\Gamma(\varphi)$ denotes the substitution of variables by their representative.

tion. By *R1*, we only need to consider, e.g., $O_Y$ and $C_Y$ as values of $\tilde{x}$, thus the existential quantification reduces to acan be rewritten into a big disjunction, and hence is a proper configuration formula.

*Definition 10.* (Function $\Omega$) Let $CF$ be configuration formula over a well-formed network $\mathcal{N}$. The formula tranformation function $\Omega$ is defined in Table 2, where $\Omega_0(CF)$ denotes the application of $\Omega_0$ to all basic formulae in $CF$. ◇

Following our discussion, we can distinguish two kinds of configurations in the transition system of a well-formed network $\mathcal{N}$. A configuration is stable wrt. $Y \in \mathcal{EC}_\mathcal{N}$ if the value of all variables from $Y$ can directly be obtained from the value of the representative variable of $Y$ by applying the dependency function, and unstable otherwise. For the transformed network $\mathcal{N}' = \mathcal{K}(\mathcal{N}, Y)$, stable configurations wrt. $Y$ are those where the initial location of the resetter $\mathcal{R}_Y$ occurs, unstable otherwise.

*Definition 11.* (Stable/Unstable Configurations) Let $\mathcal{N}$ be a network and let $Y \in \mathcal{EC}_\mathcal{N}$ be a set of quasi-dependent variables. Let $\mathcal{N}' = \mathcal{K}(\mathcal{N}, \mathcal{EC}_\mathcal{N})$. A configuration $s \in Conf(\mathcal{N})$ is called *stable wrt.* $Y \in \mathcal{EC}_\mathcal{N}$ if and only if $\forall x \in Y, x \simeq_f rep(x) \bullet \nu_s(x) = f(\nu_s(rep(x)))$. We use $\mathcal{SC}_\mathcal{N}^Y$ to denote the set of all configurations that are stable wrt. $Y$ and $\mathcal{SC}_\mathcal{N}$ to denote the set $\bigcap_{Y \in \mathcal{EC}_\mathcal{N}} \mathcal{SC}_\mathcal{N}^Y$ of *globally stable* configurations of $\mathcal{N}$. Configurations not in $\mathcal{SC}_\mathcal{N}$ are called *unstable*.

A configuration $r \in Conf(\mathcal{N}')$ is called *stable wrt.* $Y$ if and only if the initial location $\ell_{ini,\mathcal{R}_Y}$ of resetter $\mathcal{R}_Y \in \mathcal{N}'$ occurs in $r$, i.e., if $\ell_r = \ell_{ini,\mathcal{R}_Y}$. We use $\mathcal{SC}_{\mathcal{N}'}^Y$ to denote the set of all configurations that are stable wrt. $Y$ and $\mathcal{SC}_{\mathcal{N}'}$ to denote the set $\bigcap_{Y \in \mathcal{EC}_\mathcal{N}} \mathcal{SC}_{\mathcal{N}'}^Y$ of *globally stable* configurations of $\mathcal{N}'$. We call the configurations in $Conf(\mathcal{N}') \setminus \mathcal{SC}_{\mathcal{N}'}$ *unstable*. ◇

In the following, we observe that our transformation yields a network whose stable configurations directly relate to the stable configurations of $\mathcal{N}$ (one-to-one). For unstable configurations from both networks the relation between them is more involved (one-to-many), i.e., this relation depends on information from simple and complex resetting edges, as well as on values from *rst*-variables. To this end, we introduce definitions that allow us to define a weak bisimulation relation between $\mathcal{N}$ and $\mathcal{N}'$.

*Definition 12.* (Delayed Edge) An edge $e$ of a timed automaton $\mathcal{H}$ in network $\mathcal{N}$ is called *delayed* if and only if time must pass before $e$ can be taken, i.e., if $\forall s_0 \xrightarrow{\lambda_1}_{E_1} \ldots s_{n-1} \xrightarrow{\lambda_n}_{E_n} s_n \in \Pi(\mathcal{H}) \bullet e \in E_n \implies \exists 0 \le j < n \bullet \lambda_j \in \mathbb{R}_0^+ \setminus \{0\} \wedge \forall j \le i < n \bullet Edge(\mathcal{H}) \cap E_i = \emptyset$.

Here, we write $s_i \xrightarrow{\lambda_i}_{E_i} s_{i+1}$, $i \in \mathbb{N}^+$, to denote that the transition $s_i \xrightarrow{\lambda_i} s_{i+1}$ is justified by the set of edges $E_i$; $E_i$ is empty for time transitions, i.e., if $\lambda_i \in \mathbb{R}_0^+ \times Lab$. ◇

*Definition 13.* (Reset Pre/Post Delay) Let $\mathcal{N}$ be a *well-formed* network. We say $\mathcal{EC}_\mathcal{N}$-*reset edges are pre/post delayed in network* $\mathcal{N}$ if and only if all edges originating in reset (successor) locations are delayed, i.e., if $\forall e = (\ell, a, \mu, \ell') \in Edge(\mathcal{N}) \bullet \ell \in \mathcal{RL}_\mathcal{N} \cup \mathcal{RL}_\mathcal{N}^+ \implies e$ is delayed. ◇

There are *sufficient* syntactic criteria for an edge $e = (\ell_1, a_1, \mu_1, \ell_2)$ being delayed. For instance, if $(\ell_0, a_0, \mu_0, \ell_1)$ is the only incoming edge to $\ell_1$ and if $\varphi_0 = (x \ge C \wedge x \le C)$ and $\varphi_1 = (x \ge D \wedge x \le D)$ and $C < D$, then $e$ is delayed. It is also delayed if $(\ell_0, a_0, \mu_0, \ell_1)$ is the only incoming edge to $\ell_1$, $\vec{r}_0$ is resetting $x$, and $\varphi_1 = (x > 0)$.

Both patterns occur, e.g., in the FSN case-study (cf. Section 5.1). There, the reset location is entered via an edge following the former pattern, and the edges originating at the reset successor location follow the latter pattern. Thus $\mathcal{EC}_\mathcal{N}$-reset edges are pre/post delayed in FSN.

LEMMA 5.1. *(Weak Bisimulation) Any well-formed network* $\mathcal{N}$ *where* $\mathcal{EC}_\mathcal{N}$-*reset edges are pre/post delayed, is* weakly bisimilar *to* $\mathcal{N}' = \mathcal{K}(\mathcal{N}, \mathcal{EC}_\mathcal{N})$, *i.e., there is a* weak bisimulation relation $\mathcal{S} \subseteq Conf(\mathcal{N}) \times Conf(\mathcal{N}')$ *such that*

1. $\forall s \in Conf(\mathcal{N}) \exists r \bullet (s, r) \in \mathcal{S}$ and $\forall r \in Conf(\mathcal{N}) \exists s \bullet (s, r) \in \mathcal{S}$.

2. *For all configuration formulae $CF$ over* $\mathcal{N}$, $\forall (s, r) \in \mathcal{S} \bullet s \models CF \implies r \models \Omega(CF)$ *and* $\forall r \in CONS_{\mathcal{EC}_\mathcal{N}} \bullet r \models \Omega(CF) \implies \exists s \in Conf(\mathcal{N}) \bullet (s, r) \in \mathcal{S} \wedge s \models CF$.

3. *For all* $(s, r) \in \mathcal{S}$, *if* $s \xrightarrow{\lambda} s'$ *with*

(a) $s, s' \notin \mathcal{SC}_\mathcal{N}^Y$, where $Y \in \mathcal{EC}_\mathcal{N}$, and justified by a simple resetting edge, or $s \notin \mathcal{SC}_\mathcal{N}^Y$, $s' \in \mathcal{SC}_\mathcal{N}^Y$, where $Y \in \mathcal{EC}_\mathcal{N}$, and justified by a simple resetting edge, then $r \xrightarrow{0} r$ and $(s', r) \in \mathcal{S}$.

(b) $s \in \mathcal{SC}_\mathcal{N}^Y$, $s' \notin \mathcal{SC}_\mathcal{N}^Y$, where $Y \in \mathcal{EC}_\mathcal{N}$, and justified by the set $CE_Y^1 \subseteq \mathcal{CE}_Y(\mathcal{N})$ of complex resetting edges wrt. $Y$, or $s, s' \in \mathcal{SC}_\mathcal{N}^Y$, where $Y \in \mathcal{EC}_\mathcal{N}$, and justified by $CE_Y \subseteq \mathcal{CE}_Y(\mathcal{N})$, or $s, s' \in \mathcal{SC}_\mathcal{N}^Y$, $\ell_r = \ell_{nst, \mathcal{R}_Y}$ for some $Y \in \mathcal{EC}_\mathcal{N}$, and $\lambda = d > 0$, then there exist $r', r''$ such that $r \xrightarrow{\tau} r' \xrightarrow{\lambda} r''$ and $(s, r'), (s', r'') \in \mathcal{S}$.

(c) Otherwise there is $r'$ s.t. $r \xrightarrow{\lambda} r'$ and $(s', r') \in \mathcal{S}$.

and if $r \xrightarrow{\lambda} r'$ with

(a) $r \in \mathcal{SC}_{\mathcal{N}'}^Y, r' \notin \mathcal{SC}_{\mathcal{N}'}^Y$, where $Y \in \mathcal{EC}_\mathcal{N}$, $\nu_{r'}(rst_Y^{\mathcal{OH}}) < N$, where $N = \nu_r(rst_Y^{\mathcal{OH}})$, there exist $s_1, \ldots, s_n$ where $n = N - \nu_{r'}(rst_Y^{\mathcal{OH}})$, such that $s \xrightarrow{\tau} s_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} s_n$ and $(s_i, r') \in \mathcal{S}$, $1 \le i \le n$.

(b) $r \in \mathcal{SC}_{\mathcal{N}'}^Y, r' \notin \mathcal{SC}_{\mathcal{N}'}^Y$, $\nu_{r'}(rst_Y^{\mathcal{OH}}) = \nu_r(rst_Y^{\mathcal{OH}})$, where $Y \in \mathcal{EC}_\mathcal{N}$, or $\ell_r = \ell_{nst, \mathcal{R}_Y}$, $\ell_{r'} \neq \ell_{nst, \mathcal{R}_Y}$, $Y \in \mathcal{EC}_\mathcal{N}$, then $s \xrightarrow{0} s$ and $(s, r') \in \mathcal{S}$.

(c) Otherwise there is $s'$ s.t. $s \xrightarrow{\lambda} s'$ and $(s', r') \in \mathcal{S}$.

THEOREM 5.2. *Let $CF$ be a configuration formula over well-formed network $\mathcal{N}$ with pre/ post delayed $\mathcal{EC}_\mathcal{N}$-resets. $CF$ is reachable in $\mathcal{N}$ iff $\Omega(CF)$ is reachable in $\mathcal{K}(\mathcal{N}, \mathcal{EC}_\mathcal{N})$.*

## 5.1 Experimental Results

For our experimental evaluation we have chosen two industrial time-based protocols for wireless sensor networks where the timers of the individual sensors advance at different rates (each rate being constant). In our case studies we have verified queries which were proposed by the respective authors. Our first benchmark, the FSN fire alarm protocol from [22] belongs to the class of TDMA protocols. It consists of one controller and, in our setting, from 10 to 130 sensors. Each sensor has only one (simple) resetting edge. We verified that the communication between controller and sensors is successful, i.e., each message received by the controller is replied to and reaches the transmitting sensor. The results are reported in Table 3; we have used the same machine as in Section 4.1. We compare the number of iterations and analysis runtime needed by SpaceEx for the original and the transformed systems (the latter denoted in the table by the suffix K). Furthermore, we report time spent in the detection phase. It is important to note that the detection can be done in a compositional manner by considering only *pairs* of sensors and their variables. These detections can run in parallel; the final set of quasi-dependent variables can be derived using transitivity. Therefore, in column "Detect" of Table 3, we show the results of pair-wise detection. Overall, our approach shows better performance starting from the smallest instances. Furthermore, our combined approach uses much less memory thus we can treat larger system instances. Our approach scales up to 130 sensors whereas the analysis of the original system runs out of memory already for the instance with 12 sensors.

Our second case study [3] belongs to the class of EPL protocols. Once again there is one controller with multiple sensors, each sensor has a single (complex) resetting edge where quasi-dependent and non-quasi-dependent variables are updated. We verified that all sensors receive and acknowledge each message sent by the controller. The results for our settings range from 7 to 11 sensors and are presented in Table 4. Although the analysis of the transformed system by SpaceEx always takes less time, we observe that the speed-up is smaller than for the FSN benchmark. This can be explained by the fact that we are able to completely remove the interleaving "diamond" in case of FSN because of only simple resetting edges, whereas for EPL we only reduce the number of variables while still having to analyze the interleavings induced by the update of non-quasi-dependent variables in complex resetting edges. Those interleavings also have an impact on the detection phase performance as we need to consider more paths in the abstract region graph. Still, we can observe that the analysis of the original system runs out of memory for 10 sensors, our approach can analyze larger instances.

As SpaceEx supports only continuous variables, we have, for the sake of efficiency, represented values of the discrete counter variables $rst_Y^{\mathcal{IH}}$ and $rst_Y^{\mathcal{OH}}$ by corresponding locations in the resetter automata.

## 6. CONCLUSION

In this paper, we have presented techniques to efficiently exploit the dependencies between continuous variables in hybrid automata.

The technical contribution comprises two methods: an automatic detection of quasi-dependent variables and a method to automatically transform and reduce the original system. The current prototypical implementation of our detection algorithm is backed by a linear SMT solver. Although using a rather straight-forward, constraint-based representation of regions in the abstract region graph, we have been able to achieve a significant performance speed-up. Thus, by using elaborated data structures to represent regions we can expect to achieve even better performance. Our transformation technique shrinks the set of states emerging during discrete updates of quasi-dependent variables due to the interleaving semantics. This leads to an abstracted hybrid automaton with a much smaller state space, which, however, preserves all properties of the original system.

Overall, a combination of our methods, i.e. the detection part and consequent transformation, shows very promising results compared to the application of the hybrid model checker SpaceEx to original models. Although we have only evaluated our approach on linear hybrid automata motivated by industrial, time-based protocols where the timers of the individual sensors advance at measurably different rates, our theory is generic as it uses the SMT solver (for the detecting part) and SpaceEx (for the analysis of a reduced hybrid automaton) as black-boxes. Therefore, by putting in routines suitable for larger classes of systems, we will be able to directly apply our approach to those systems.

| | Detect | | Check | | |
|---|---|---|---|---|---|
| | pairs | $\Sigma\,t$(s) | iter. | $t$(s) | $\Sigma$ |
| 010 | | | 2086 | 47.18 | 47.18 |
| 010_K | 9 | 27.68 | 41 | 0.14 | 27.82 |
| 011 | | | 4138 | 134.53 | 134.53 |
| 011_K | 10 | 30.74 | 45 | 0.17 | 30.91 |
| 012 | | | - | - | - |
| 012_K | 11 | 33.78 | 49 | 0.19 | 33.97 |
| 110_K | 109 | 334.28 | 441 | 46.35 | 380.63 |
| 120_K | 119 | 364.93 | 481 | 59.72 | 424.65 |
| 130_K | 129 | 395.48 | 521 | 75.55 | 471.03 |

| | Detect | | Check | | |
|---|---|---|---|---|---|
| | pairs | $\Sigma\,t$(s) | iter. | $t$(s) | $\Sigma$ |
| 07 | | | 396 | 5.45 | 5.45 |
| 07_K | 5 | 48.53 | 408 | 2.05 | 50.58 |
| 08 | | | 781 | 15.51 | 15.51 |
| 08_K | 6 | 58.21 | 794 | 5.28 | 63.49 |
| 09 | | | 1550 | 43.75 | 43.75 |
| 09_K | 7 | 68.25 | 1564 | 14.27 | 82.52 |
| 10 | | | - | - | - |
| 10_K | 8 | 77.85 | 3102 | 41.89 | 119.74 |
| 11_K | 9 | 87.49 | 6176 | 139.91 | 227.40 |
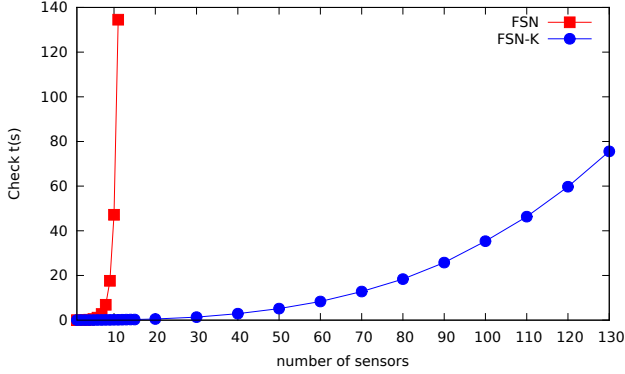


Table 3: Benchmark FSN: For each pair of quasi-dependent variables, detect needs 41 SMT calls, 14 states.[1]
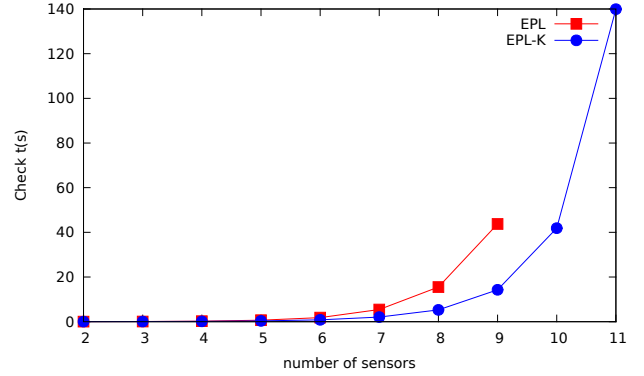


Table 4: Benchmark EPL: For each pair of quasi-dependent variables, detect needs 120 SMT calls, 34 states.[1]

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# APPENDIX

## A. TRANSFORMATION PROCEDURE

Let $\mathcal{N}$ be a well-formed network with the set $\mathcal{EC}_\mathcal{N}$ of equivalence classes of quasi-dependent variables. Applying our transformation procedure on $\mathcal{N}$ yields the transformed network $\mathcal{N}' = \{\mathcal{H}'_1, \ldots, \mathcal{H}'_n\} \cup \{\mathcal{R}_Y \mid Y \in \mathcal{EC}_\mathcal{N}\}$.

The $\mathcal{H}' = (Loc'(\mathcal{H}),\ Var',\ Lab'(\mathcal{H}),\ Edge'(\mathcal{H}),\ Act'(\mathcal{H}),$ $Inv'(\mathcal{H}),\ Init'(\mathcal{H}))$ in the transformed network are obtained as follows.

- $Loc'(\mathcal{H}) = Loc(\mathcal{H}) \cup \bigcup_{Y \in \mathcal{EC}_\mathcal{N}} \Xi_Y(\mathcal{H})$,
  with $\Xi_Y(\mathcal{H}) = \{\ell_{\xi_{Y,e}} \mid e \in \mathcal{CE}_Y(\mathcal{H}), Y \in \mathcal{EC}_\mathcal{N}\}$,

- $Var' = (Var \setminus \mathcal{EC}_\mathcal{N}) \cup \{rep(Y) \mid Y \in \mathcal{EC}_\mathcal{N}\}$
  $\cup \{rst_Y^{I_\mathcal{H}}, rst_Y^{O_\mathcal{H}} \mid \mathcal{H} \in \mathcal{RES}_Y(\mathcal{N})\}$,

- $Lab'(\mathcal{H}) = Lab(\mathcal{H}) \cup \{reset_Y \mid \mathcal{H} \in \mathcal{RES}_Y(\mathcal{N})\}$,

- $Act'(\mathcal{H})(\ell) = \{(f\!\uparrow)|_{Var'} \mid f \in Act(\ell)\}$ for $\ell \in Loc(\mathcal{H})$,
  where $f\!\uparrow$ denotes the the (unconstrained) extension of action $f$ for the variables $rst_Y^{I_\mathcal{H}}, rst_Y^{O_\mathcal{H}}$, $\mathcal{H} \in \mathcal{N}$,

- $Inv'(\mathcal{H})(\ell) = \{\nu' \mid \exists\, \nu \in Inv(\ell)\ \forall x \simeq_f y \bullet$
  $\qquad\qquad \nu = f(\nu'(rep(x)))\}, \ell \in Loc(\mathcal{H})$,
  $Inv'(\mathcal{H})(\ell) = \{\nu' \mid \nu'(rep(Y)) = O_Y\}, \ell \in \Xi_Y(\mathcal{H})$,

- $Init'(\mathcal{H})(\ell) = \{\nu' \mid \exists\, \nu \in Init(\ell)\ \forall x \in Y, x \simeq_f y \bullet$
  $\qquad\qquad \nu = f(\nu'(rep(x)))\}$ for $\ell \in Loc(\mathcal{H})$,
  $Init'(\mathcal{H})(\ell) = \emptyset$ for $\ell \in \Xi_Y(\mathcal{H})$,

- $Edge'(\mathcal{H}) =$

  $\{(\ell, a, \mu_{\mathcal{EC}_\mathcal{N}} \cap \mu_e^{\ell,\ell'}, \ell') \mid$
  $\qquad e = (\ell, a, \mu, \ell') \in Edge(\mathcal{H}) \setminus \mathcal{E}_Y(\mathcal{H})\}$

  $\cup\, \{(\ell, reset_Y, (\mu\!\uparrow)|_{Var'} \cap \mu_e^{\ell,\ell'}, \ell') \mid$
  $\qquad e = (\ell, \tau, \mu, \ell') \in \mathcal{SE}_Y(\mathcal{H})\}$

  $\cup\, \{(\ell, reset_Y, \mu_{id}, \ell_{\xi_{Y,e}}), (\ell_{\xi_{Y,e}}, a, (\mu\!\uparrow)|_{Var'} \cap \mu_e^{\ell,\ell'}, \ell') \mid$
  $\qquad e = (\ell, a, \mu, \ell') \in \mathcal{CE}_Y(\mathcal{H})\}$,

  where

  $$\mu_{id} = \{(\nu, \nu') \mid \nu'(x) = \nu(x), x \in Var'\},$$

  $$\mu_{\mathcal{EC}_\mathcal{N}} = \{(\nu, \nu') \mid \exists\, (\nu_0, \nu'_0) \in \mu \bullet$$
  $$\nu|_{Var' \setminus \mathcal{EC}_\mathcal{N}} = \nu_0|_{Var' \setminus \mathcal{EC}_\mathcal{N}}$$
  $$\wedge\, \nu'|_{Var' \setminus \mathcal{EC}_\mathcal{N}} = \nu'_0|_{Var' \setminus \mathcal{EC}_\mathcal{N}}$$
  $$\wedge\, \forall x \simeq_f rep(x), x \in \mathcal{EC}_\mathcal{N} \bullet$$
  $$\nu(rep(x)) = f(\nu_0(x))\},$$

  and $\mu_e^{\ell,\ell'}$ denotes the pairs $(\nu, \nu')$ of valuations where

  - $\nu'(rst_Y^{I_\mathcal{H}}) = \nu(rst_Y^{I_\mathcal{H}}) + 1$, if $\ell' \in \mathcal{RL}_Y$,
  - $\nu'(rst_Y^{I_\mathcal{H}}) = \nu(rst_Y^{I_\mathcal{H}}) - 1$,
    $\qquad\qquad$ if $e \notin \mathcal{E}_Y(\mathcal{H})$ and $\ell \in \mathcal{RL}_Y$,
  - $\nu'(rst_Y^{I_\mathcal{H}}) = \nu(rst_Y^{I_\mathcal{H}})$, otherwise,
  - $\nu'(rst_Y^{O_\mathcal{H}}) = \nu(rst_Y^{O_\mathcal{H}}) - 1$, if $e \in \mathcal{E}_Y(\mathcal{H})$,
  - $\nu'(rst_Y^{O_\mathcal{H}}) = \nu(rst_Y^{O_\mathcal{H}})$, otherwise.

For each equivalence class $Y \in \mathcal{EC}_\mathcal{N}$ the automaton $\mathcal{R}_Y$ consists of the following elements. Two locations $\ell_{ini,\mathcal{R}_Y}$ and $\ell_{nst,Y}$. and two edges $(\ell_{ini,\mathcal{R}_Y}, reset_Y, \mu_1, \ell_{nst,Y})$ and $(\ell_{nst,Y}, \tau, \mu_2, \ell_{ini,\mathcal{R}_Y})$. The fist edge obtains a guard

$$\mu_1 = \{(\nu, \nu') \mid \nu(rst_Y^{I_\mathcal{H}}) = n_Y \wedge \nu(rep(Y)) = C_Y)$$
$$\wedge\, \nu'(rst_Y^{I_\mathcal{H}}) = 0 \wedge \nu'(rep(Y)) = O_Y\}$$

which checks whether the representative variable of equivalence class $Y$ has the value $C_Y$ from *R1* and $rst_Y^{I_\mathcal{H}}$ evaluates to $n_Y := |\mathcal{RES}_Y(\mathcal{N})|$, the number of automata that reset the clocks of $Y$. $\mu_1$ assigns all $rst_Y^{I_\mathcal{H}}$ to 0 and the representative variable to $O_Y$ from *R1*. The second edge obtains a guard

$$\mu_2 = \{(\nu, \nu') \mid \nu(rst_Y^{O_\mathcal{H}}) = 0 \wedge \nu(rep(Y)) = O_Y \wedge$$
$$\nu'(rst_Y^{O_\mathcal{H}}) = n_Y\}$$

which checks whether $rst_Y^{O_\mathcal{H}}$ evaluates to 0, and assigns $n_Y$ to $rst_Y^{O_\mathcal{H}}$.

In location $\ell_{ini,\mathcal{R}_Y}$, the invariant is not constraining any variable, in $\ell_{nst,Y}$ we require that the value of the representative is $O_Y$. The variables $rst_Y^{I_\mathcal{H}}$ and $rst_Y^{O_\mathcal{H}}$ are initialized to

$$iL_Y := |\{\mathcal{H} \in \mathcal{N} \mid \ell_{ini,\mathcal{H}} \in \mathcal{RL}_Y\}|,$$

the number of automata whose initial location is a reset location of $Y$, $n_Y$, respectively, in $\ell_{ini,\mathcal{R}_Y}$; $\ell_{nst,Y}$ is not an initial location. Both locations inherit the actions for the representative variable $x$ from the automata constraining $x$.

The actions of $\ell \in Loc(\mathcal{R}_Y)$ only require the $rst_Y^{I_\mathcal{H}}$ and $rst_Y^{O_\mathcal{H}}$ variables to stay constant, i.e.

$Act(\ell) = \{f : \mathbb{R}_0^+ \to V \mid \forall t \in \mathbb{R}_0^+ \bullet$
$\quad f(t)(rst_Y^{I_\mathcal{H}}) = f(0)(rst_Y^{I_\mathcal{H}}) \wedge f(t)(rst_Y^{O_\mathcal{H}}) = f(0)(rst_Y^{O_\mathcal{H}})$.

In location $\ell_{ini,\mathcal{R}_Y}$, the invariants are $Inv(\ell_{ini,\mathcal{R}_Y}) = V$ and

$$Inv(\ell_{nst,Y}) = \{\nu \mid \nu(rep(Y)) = O_Y\}.$$

The variables $rst_Y^{I_\mathcal{H}}$, $rst_Y^{O_\mathcal{H}}$, $rep(Y)$ are initialized to $iL_Y$ and $n_Y$, respectively. Location $\ell_{nst,Y}$ is not an initial location, i.e. $Inv(\ell_{nst,Y}) = \emptyset$.