Automata Theory Seminar

Büchi Complementation via Alternating Automata

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July 16, 2012

Büchi Complementation

BA $\mathcal{B}$ $\longrightarrow$ BA $\overline{\mathcal{B}}$

BA: Büchi Automaton

Büchi Complementation

$$\text{BA } \mathcal{B} \xrightarrow{2^{\Theta(n \log n)}} \text{BA } \overline{\mathcal{B}}$$

BA: Büchi Automaton

- Expensive: If $\mathcal{B}$ has n states, $\overline{\mathcal{B}}$ has $2^{\Theta(n \log n)}$ states in the worst case (Michel 1988, Safra 1988).

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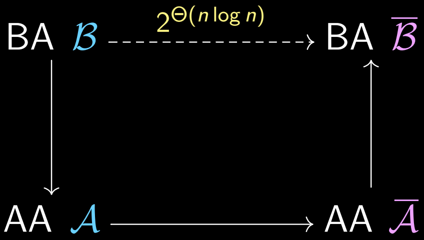
Büchi Complementation



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AA: Alternating Automaton

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- Complicated: Direct approaches are rather involved.

Consider indirect approach: detour over alternating automata.

Transition Modes (1)

Existential: some run is accepting

$$q_0 \longrightarrow q_{1_a} \longrightarrow q_{2_a} \longrightarrow q_{3_a} \longrightarrow q_{4_a} \longrightarrow q_{5_a} \longrightarrow \cdots$$

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Universal: every run is accepting

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Alternation and Complementation

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Want acceptance condition that is closed under dualization.

Outline

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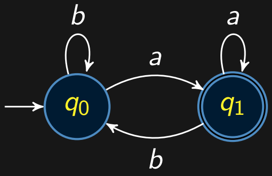
Example ($(b^* a)^\omega$)

Büchi automaton $\mathcal{B}$:



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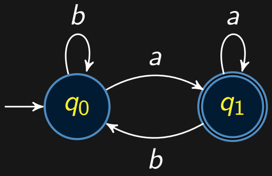
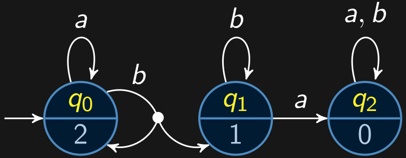


Equivalent WAPA $\mathcal{A}$:



Definition (Weak Alternating Parity Automaton)

A weak alternating parity automaton (WAPA) is a tuple

$$\mathcal{A} := \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$$

where

- Q finite set of states
- Σ finite alphabet
- q_{in} initial state

(Thomas and Löding, $\sim$2000)

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- $\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$ transition function
- q_{in} initial state
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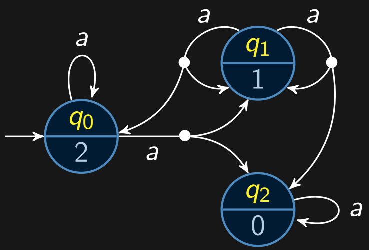
$\mathbb{B}^+(Q)$: set of all positive Boolean formulae over Q
(built only from elements in $Q \cup \{\wedge, \vee, \top, \bot\}$)

Example (a^ω)



$$\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$$

$$\langle q_0, a\rangle \mapsto q_0 \vee (q_1 \wedge q_2)$$

$$\langle q_1, a\rangle \mapsto (q_0 \wedge q_1) \vee (q_1 \wedge q_2)$$

$$\langle q_2, a\rangle \mapsto q_2$$

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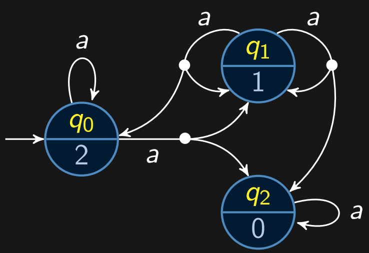

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DEFINITION (Minimal Models)

$\mathrm{Mod}_{\downarrow}(\theta) \subseteq 2^Q$: set of minimal models of $\theta \in \mathbb{B}^+(Q)$, i.e. the set of minimal subsets $M \subseteq Q$ s.t. θ is satisfied by

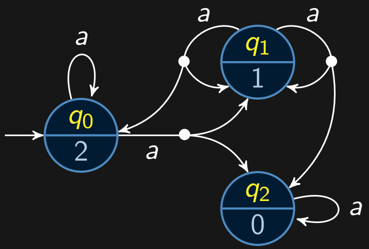
$$q \mapsto \begin{cases} \mathit{true} & \text{if } q \in M \\ \mathit{false} & \text{otherwise} \end{cases}$$

EXAMPLE

$$\mathrm{Mod}_{\downarrow}(q_0 \vee (q_1 \wedge q_2)) = \{\{q_0\}, \{q_1, q_2\}\}$$

Run Graph (1)

Example (a^ω)



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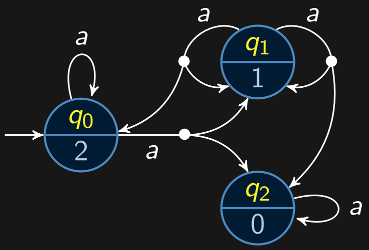

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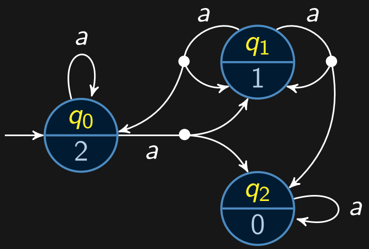

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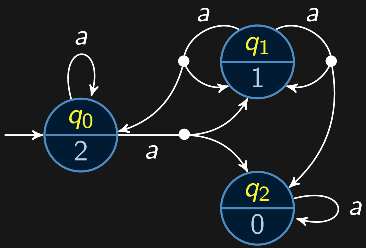

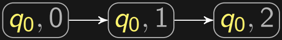

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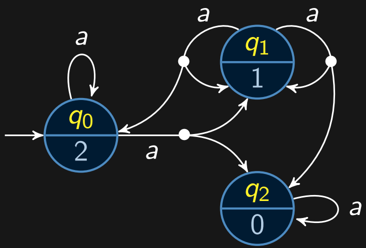

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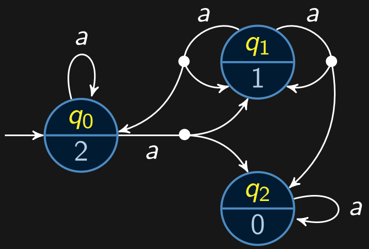

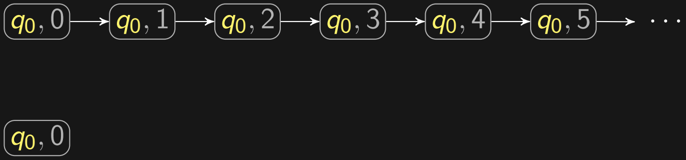

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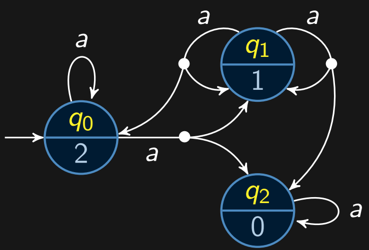

$(q_0, 0) \rightarrow (q_0, 1) \rightarrow (q_0, 2) \rightarrow (q_0, 3) \rightarrow (q_0, 4) \rightarrow (q_0, 5) \rightarrow \cdots$

$(q_0, 0) \rightarrow (q_0, 1)$

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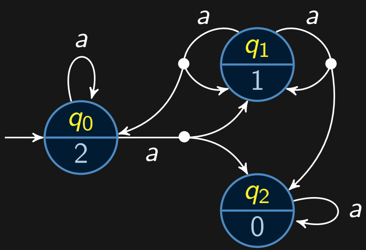

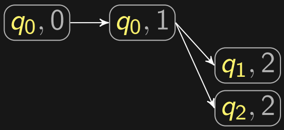

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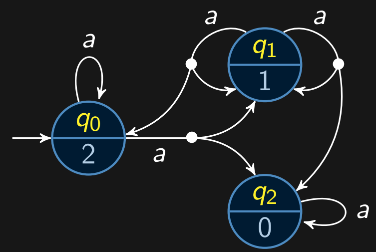

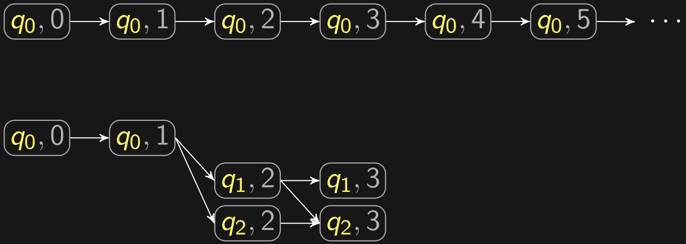

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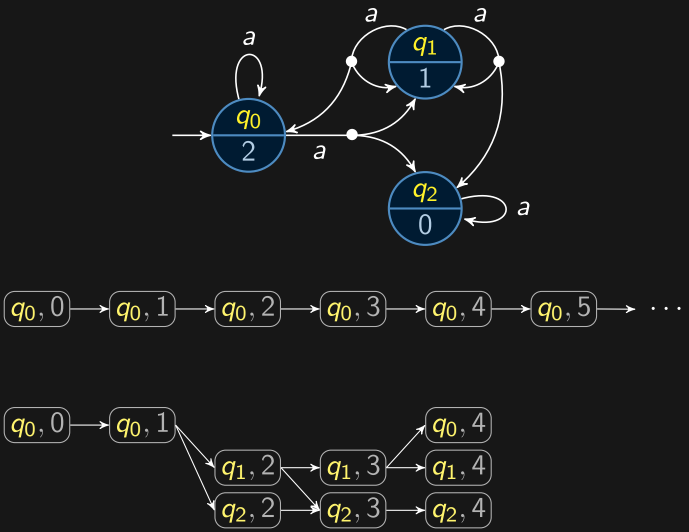

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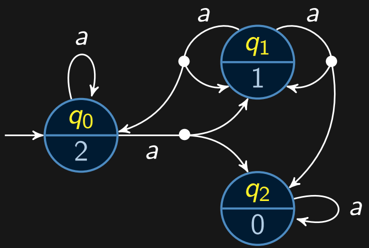

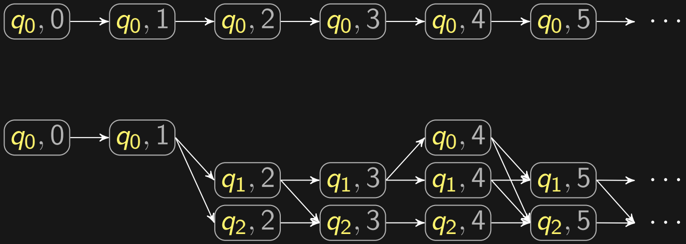

Definition (Run)

A run of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ on a word $a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed acyclic graph

$$R := \langle V, E \rangle$$

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- E contains only edges of the form $\langle \langle p, i \rangle, \langle q, i+1 \rangle \rangle$.
- For every vertex $\langle p, i \rangle \in V$ the set of successors is a minimal model of $\delta(p, a_i)$
  $$\{ q \in Q \mid \langle \langle p, i \rangle, \langle q, i+1 \rangle \rangle \in E \} \in \mathrm{Mod}_{\downarrow}(\delta(p, a_i))$$

DEFINITION (Acceptance)

Let $\mathcal{A}$ be a WAPA, $w \in \Sigma^\omega$ and $R = \langle V, E \rangle$ a run of $\mathcal{A}$ on w.

- An infinite path ρ in R satisfies the acceptance condition of $\mathcal{A}$ iff the smallest occurring parity is even, i.e.

$$\min\{\pi(q) \mid \exists\, i \in \mathbb{N} : \langle q, i \rangle \text{ occurs in } \rho\} \text{ is even.}$$

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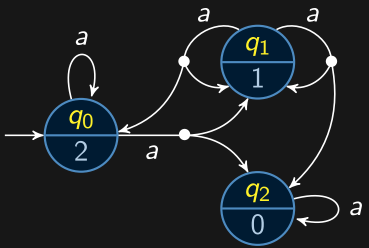
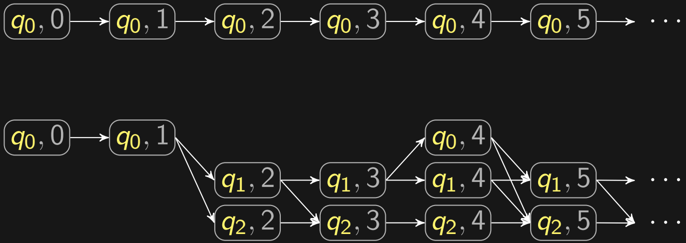
- R is an accepting run iff every infinite path ρ in R satisfies the acceptance condition.
- $\mathcal{A}$ accepts w iff there is some accepting run of $\mathcal{A}$ on w.

EXAMPLE (a^ω)





Acceptance

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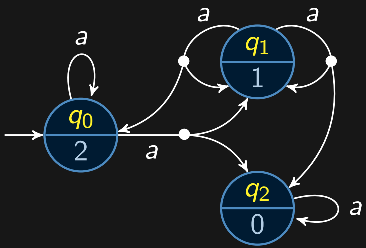
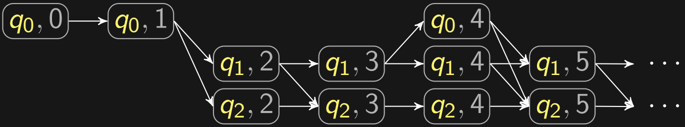


Accepting run:



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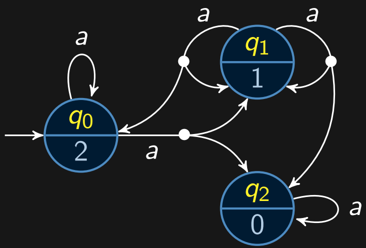
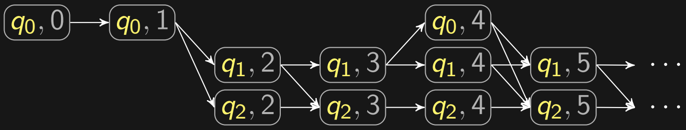


Accepting run:

Rejecting run:



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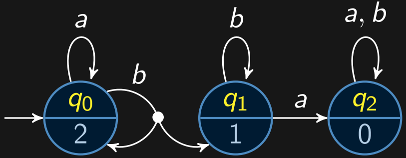

INFINITELY MANY a'S

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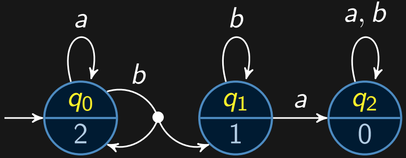
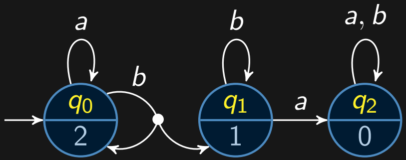


Run on b^ω:

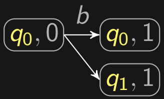
$q_0, 0$

INFINITELY MANY a'S

EXAMPLE ($(b^* a)^\omega$)



Run on b^ω:



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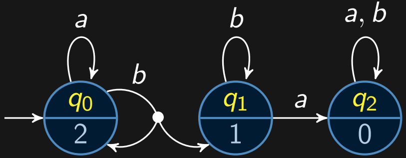

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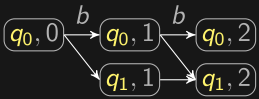

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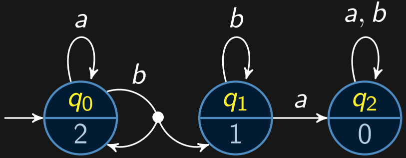

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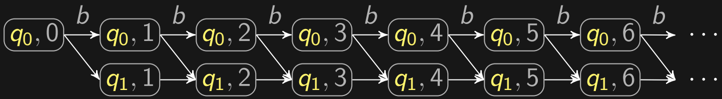

Infinitely many a's

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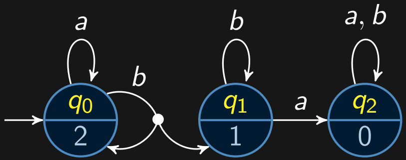

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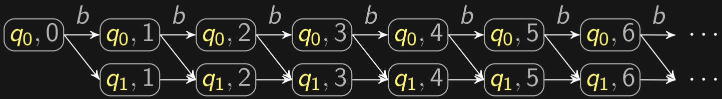

Run on $(ba)^\omega$:

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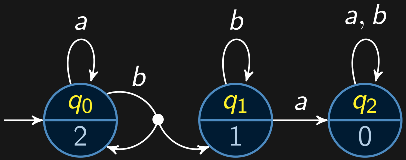

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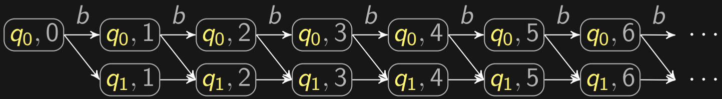

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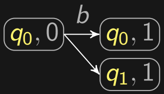

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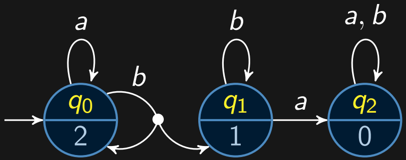

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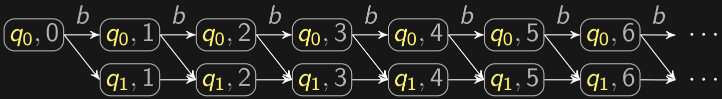

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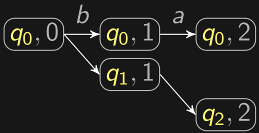

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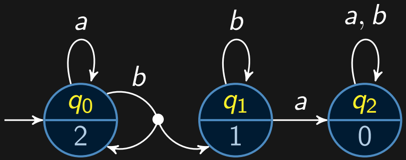

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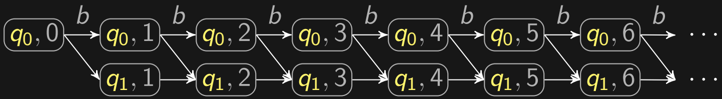

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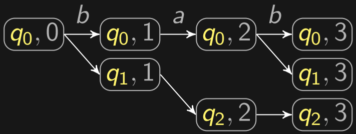

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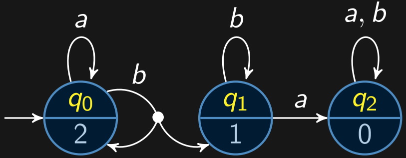

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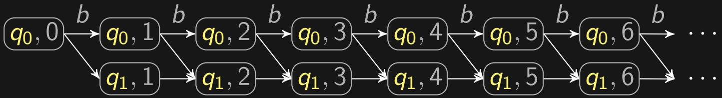

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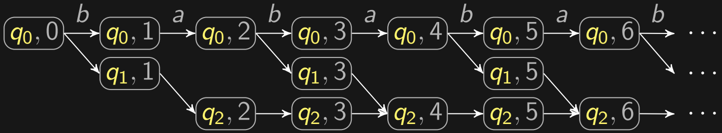

Definition (Dual Automaton)

The dual of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ is

$$\overline{\mathcal{A}} := \langle Q, \Sigma, \overline{\delta}, q_{in}, \overline{\pi} \rangle$$

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- $\overline{\delta}(q, a)$ is obtained from $\delta(q, a)$ by exchanging $\wedge, \vee$ and $\top, \bot$

for all $q \in Q$ and $a \in \Sigma$

Dual Automaton (1)

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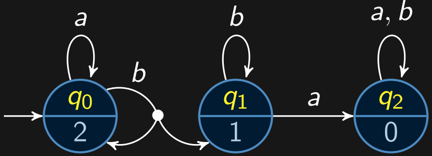
- $\overline{\delta}(q, a)$ is obtained from $\delta(q, a)$ by exchanging $\wedge, \vee$ and $\top, \bot$
- $\overline{\pi}(q) := \pi(q) + 1$

for all $q \in Q$ and $a \in \Sigma$

Dual Automaton (2)

Example ($(b^* a)^\omega$)

WAPA $\mathcal{A}$:



$\delta(q_0, a) = q_0$
$\delta(q_0, b) = q_0 \wedge q_1$

$\delta(q_1, a) = q_2$
$\delta(q_1, b) = q_1$

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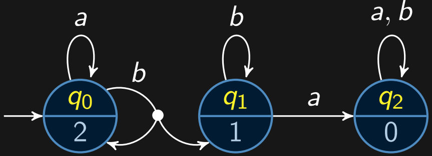

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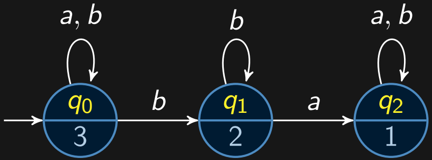
Dual $\overline{\mathcal{A}}$:



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$$\overline{\delta}(q_0, b) = q_0 \vee q_1$$
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$$\overline{\delta}(q_2, b) = q_2$$

Complementation Theorem

Main statement of this talk:

Theorem (Complementation)

The dual $\bar{\mathcal{A}}$ of a WAPA $\mathcal{A}$ accepts its complement, i.e.

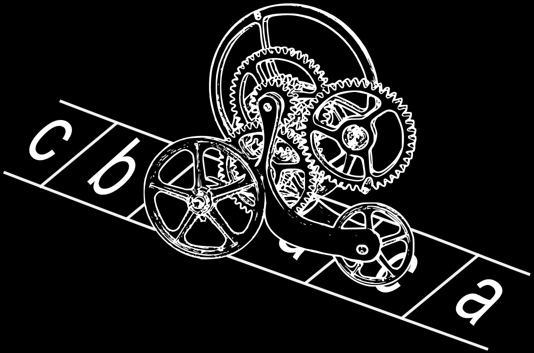
$$\mathcal{L}(\bar{\mathcal{A}}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$$

(Thomas and Löding, $\sim$2000)

Outline

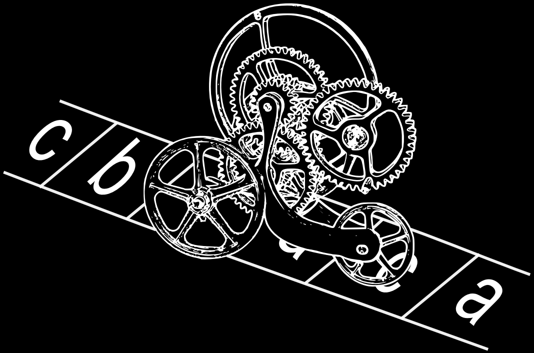
Automaton vs. Pathfinder

Automaton vs. Pathfinder



player A

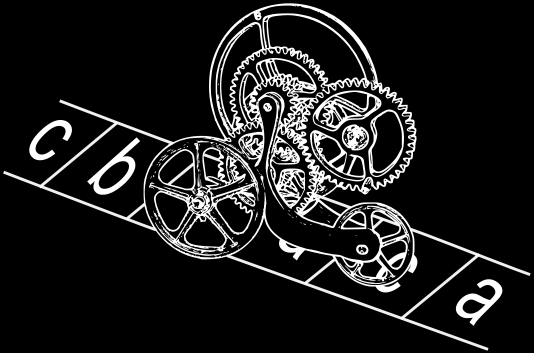
Automaton vs. Pathfinder

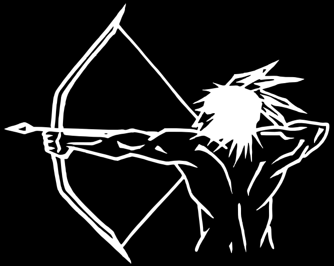


player A

find accepting run R

Automaton vs. Pathfinder



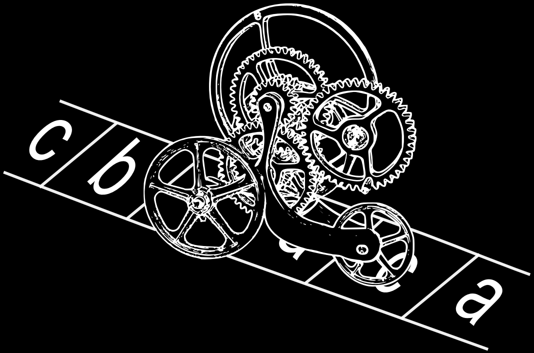


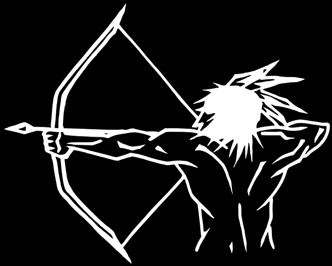
player A

player P

find accepting run R

Automaton vs. Pathfinder





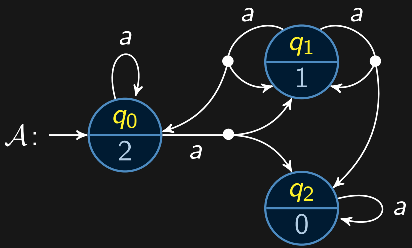
player A

find accepting run R

player P

find rejecting path in R

EXAMPLE (a^ω)



$w = a^\omega$

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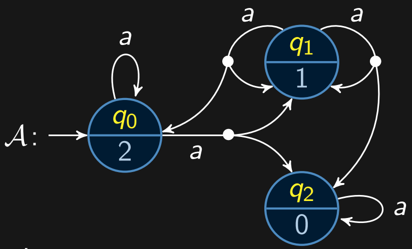

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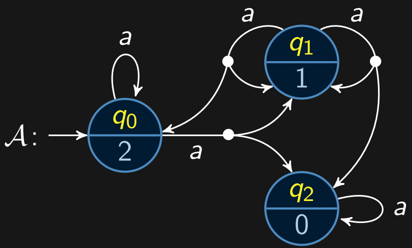
Game $G_{\mathcal{A},w}$:

$q_0, 0$

Infinite Parity Game (1)

Example (a^ω)



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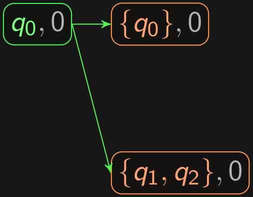

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Example (a^ω)

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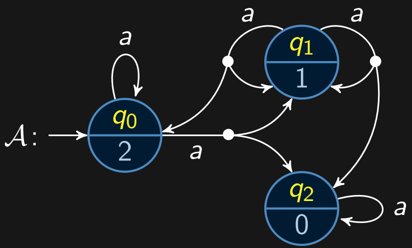

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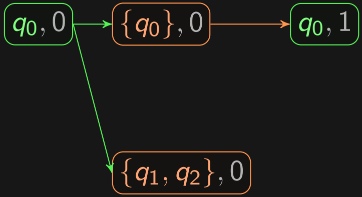

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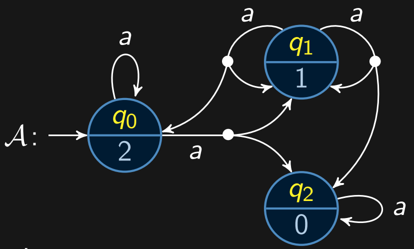

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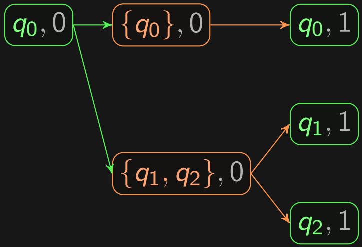

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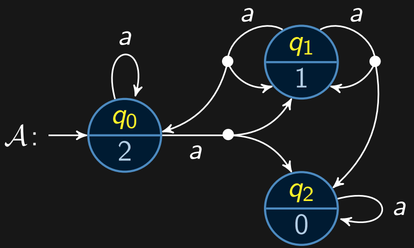

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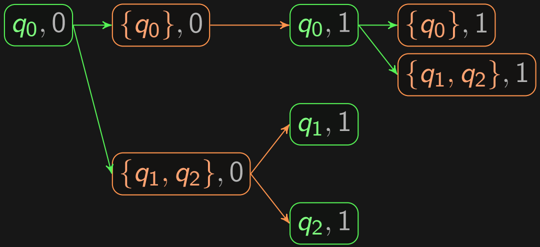

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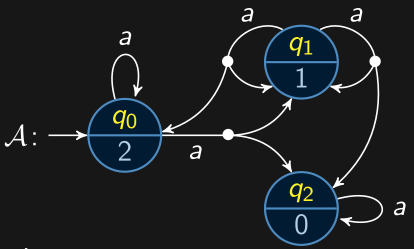

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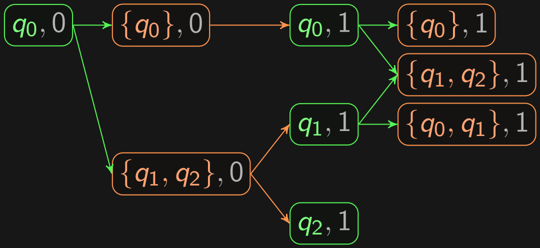

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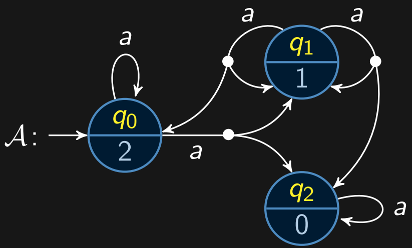

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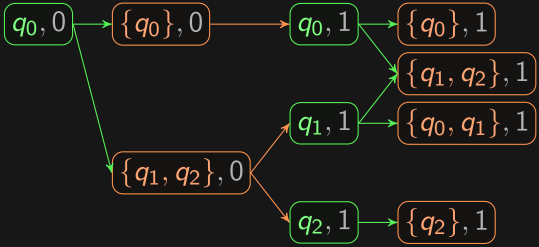

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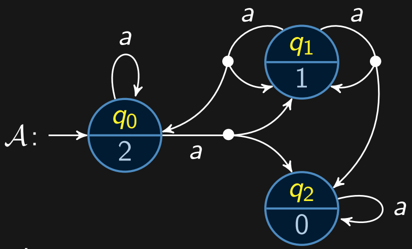

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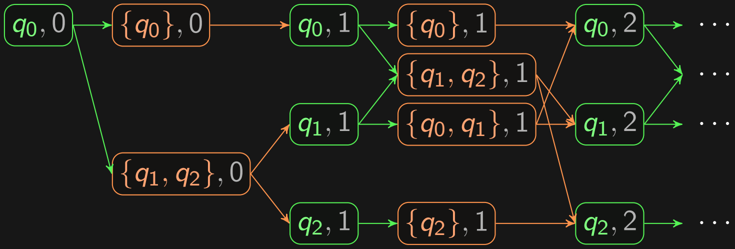

Infinite Parity Game (2)

Definition (Game)

A game for a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed graph

$$G_{\mathcal{A},w} := \langle V_A \,\dot{\cup}\, V_P, E \rangle$$

(Thomas and Löding, $\sim$2000)

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  s.t. the only contained edges are
  - $\langle \langle q, i \rangle, \langle M, i \rangle \rangle$ iff $M \in \mathrm{Mod}_{\downarrow}(\delta(q, a_i))$

  for $q \in Q$, $M \subseteq Q$, $i \in \mathbb{N}$

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  - $\langle \langle M, i \rangle, \langle q, i+1 \rangle \rangle$ iff $q \in M$

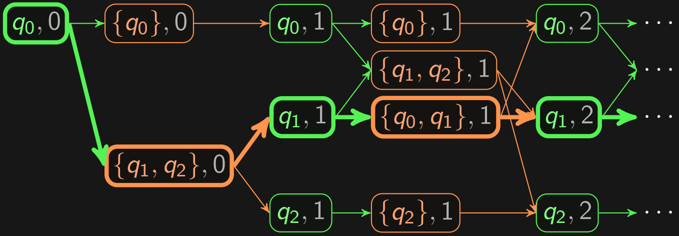
  for $q \in Q$, $M \subseteq Q$, $i \in \mathbb{N}$

(Thomas and Löding, ∼2000)

DEFINITION (Play)

A play γ in a game $G_{\mathcal{A},w}$ is an infinite path starting with $\langle q_{in}, 0\rangle$.

EXAMPLE



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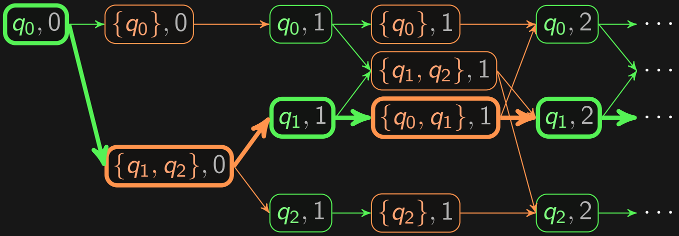
The winner of a play γ is

- player A iff the smallest parity of occurring V_A-nodes is even
- player P . . . . odd

EXAMPLE



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$X \in \{A, P\}$: a player, $\overline{X}$: its opponent

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- A strategy $f_X : V_X \to V_{\overline{X}}$ for player X selects for every decision node of player X one of its successor nodes in $G_{\mathcal{A},w}$.

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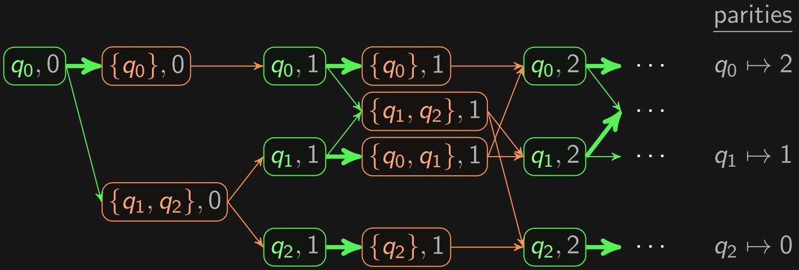
- A strategy $f_X : V_X \to V_{\overline{X}}$ for player X selects for every decision node of player X one of its successor nodes in $G_{\mathcal{A},w}$.
- f_X is a winning strategy iff player X wins every play γ that is played according to f_X.

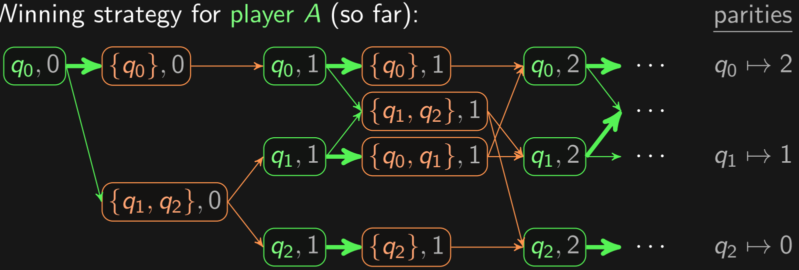
Strategies

Example



Example

Winning strategy for player A (so far):



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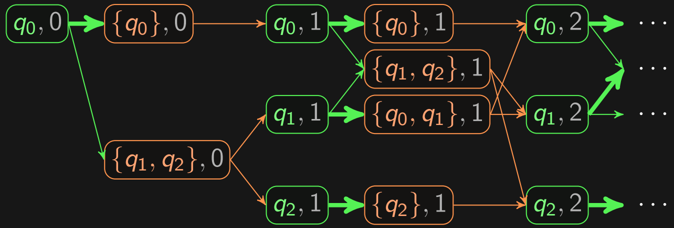


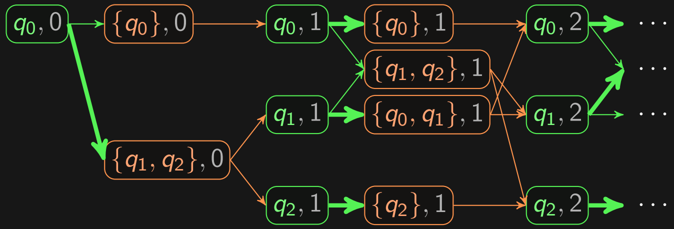
parities

$q_0 \mapsto 2$

$q_1 \mapsto 1$

$q_2 \mapsto 0$

Not a winning strategy for player A:



Outline

Lemma 1

Let $\mathcal{A}$ be a WAPA and $w \in \Sigma^\omega$.

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Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ accepts w.

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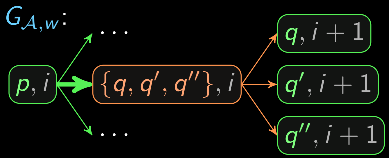
Explanation (oral):

Player A wins every play γ played according to f_A.



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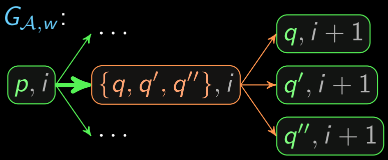
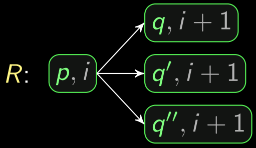


There is a run graph R in which every path ρ is accepting.



Lemma 2

Let $\mathcal{A}$ be a WAPA and $w \in \Sigma^\omega$.

Lemma 2

Player P has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ does *not* accept w.

(pointed out by Jan Leike)

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Player P has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ does *not* accept w.

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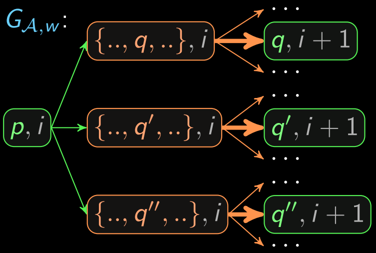

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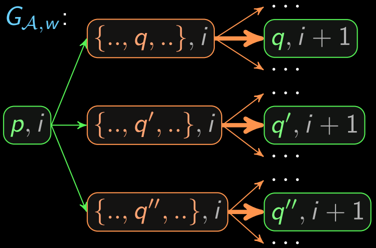
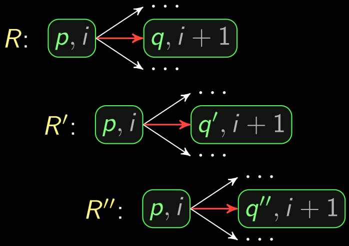


Every run graph R contains a rejecting path ρ.



Let $\theta \in \mathbb{B}^+(Q)$ be a formula over Q.

Sublemma

$S \subseteq Q$ is a model of $\overline{\theta}$ **iff** for all $M \in \mathrm{Mod}_{\downarrow}(\theta)$: $S \cap M \neq \emptyset$.

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Proof:

- W.l.o.g. θ is in DNF, i.e.

$$\theta = \bigvee_{M \in \mathrm{Mod}_{\downarrow}(\theta)} \bigwedge_{q \in M} q$$

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- Then $\overline{\theta}$ is in CNF, i.e.

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- Thus $S \subseteq Q$ is a model of $\overline{\theta}$ **iff** it contains at least one element from each disjunct of θ.

Lemma 3 (1)

Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$.

Lemma 3

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$.

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Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$.

Lemma 3

Player A has a winning strategy in $G_{\mathcal{A},w}$
iff player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$.

Proof:

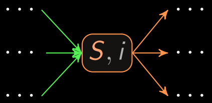
- $\Rightarrow$ Construct a winning strategy $\overline{f_P}$ for player P in $G_{\overline{\mathcal{A}},w}$.
  ...
- $\Leftarrow$ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.
  ...

Lemma 3 (2)

⇒ Construct a winning strategy $\overline{f_P}$ for player P in $G_{\overline{\mathcal{A}},w}$.

At position $\langle S, i\rangle \in V_P$

in $G_{\overline{\mathcal{A}},w}$:



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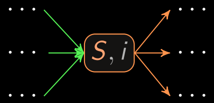

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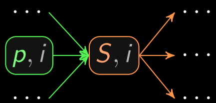

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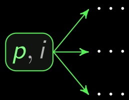

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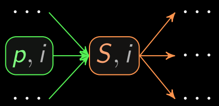

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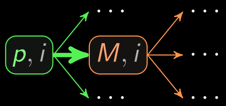

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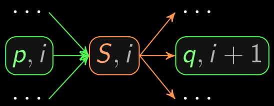

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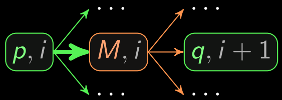

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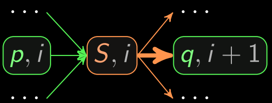

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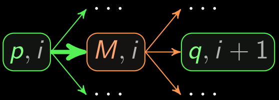

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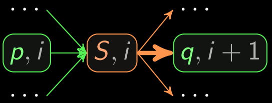

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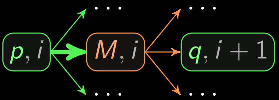

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Lemma 3 (2)

$\Rightarrow$ Construct a winning strategy $\overline{f_P}$ for player P in $G_{\overline{\mathcal{A}},w}$.

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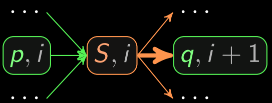

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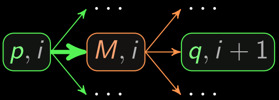

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  - Player A wins γ in $G_{\mathcal{A},w}$.

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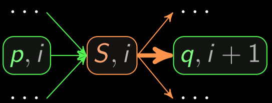

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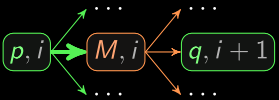

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  - Player A wins γ in $G_{\mathcal{A},w}$.
  - $\forall\, q \in Q : \overline{\pi}(q) = \pi(q) + 1$

Lemma 3 (2)

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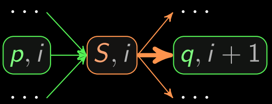
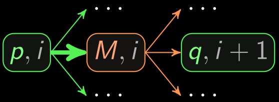


in $G_{\mathcal{A},w}$:



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  - Player A wins γ in $G_{\mathcal{A},w}$.
  - $\forall\, q \in Q : \overline{\pi}(q) = \pi(q) + 1$

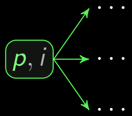
$\Rightarrow$ Player P wins $\overline{\gamma}$ in $G_{\overline{\mathcal{A}},w}$.

Lemma 3 (3)

⇐ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i \rangle \in V_A$

in $G_{\mathcal{A},w}$:



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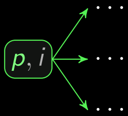
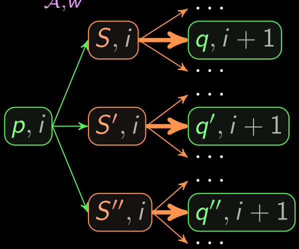


in $G_{\overline{\mathcal{A}},w}$:



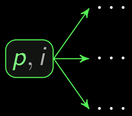
Lemma 3 (3)

$\Leftarrow$ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i\rangle \in V_A$

- $\overline{f_P}$: winning strategy for player P in $G_{\overline{\mathcal{A}},w}$
- $M^* := \{q \in Q \mid \exists\, S \in \mathrm{Mod}_{\downarrow}(\overline{\delta}(p, a_i)) : \overline{f_P}(\langle S, i\rangle) = \langle q, i+1\rangle\}$

in $G_{\mathcal{A},w}$:



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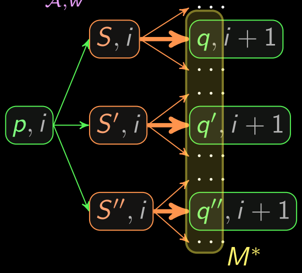

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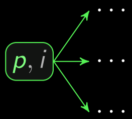

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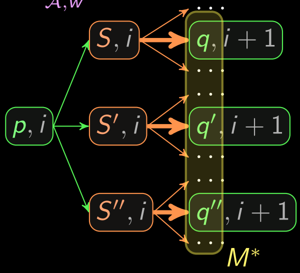

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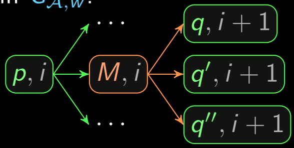
$\overset{\text{(sublemma)}}{\Longrightarrow}$ M^* is a model of $\delta(p, a_i)$.

Lemma 3 (3)

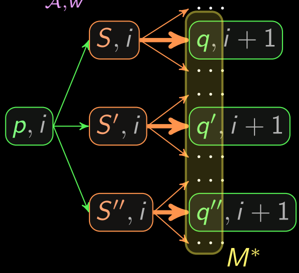
$\Leftarrow$ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i\rangle \in V_A$

in $G_{\mathcal{A},w}$:



in $G_{\overline{\mathcal{A}},w}$:



- $\overline{f_P}$: winning strategy for player P in $G_{\overline{\mathcal{A}},w}$
- $M^* := \{q \in Q \mid \exists\, S \in \mathrm{Mod}_{\downarrow}(\overline{\delta}(p, a_i)) : \overline{f_P}(\langle S, i\rangle) = \langle q, i+1\rangle\}$

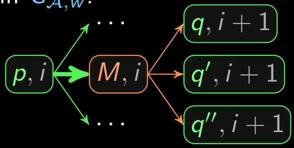
  $\overset{\text{(sublemma)}}{\Longrightarrow}$ M^* is a model of $\delta(p, a_i)$.
- M: subset of M^* that is a minimal model

  $M \subseteq M^*$, $\quad M \in \mathrm{Mod}_{\downarrow}(\delta(p, a_i))$

⇐ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i \rangle \in V_A$

in $G_{\mathcal{A},w}$:



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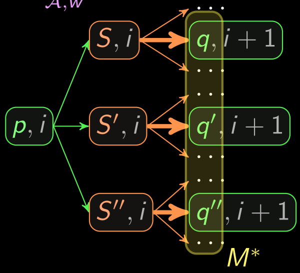

- $\overline{f_P}$: winning strategy for player P in $G_{\overline{\mathcal{A}},w}$
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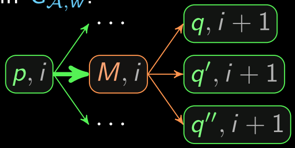
Lemma 3 (3)

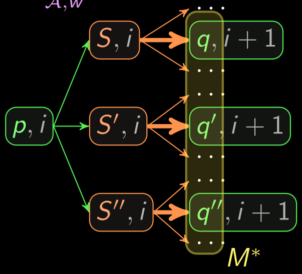
$\Leftarrow$ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i\rangle \in V_A$

in $G_{\mathcal{A},w}$:



in $G_{\overline{\mathcal{A}},w}$:



- $\overline{f_P}$: winning strategy for player P in $G_{\overline{\mathcal{A}},w}$

- $M^* := \{q \in Q \mid \exists\, S \in \mathrm{Mod}_{\downarrow}(\overline{\delta}(p, a_i)) : \overline{f_P}(\langle S, i\rangle) = \langle q, i+1\rangle\}$

  $\overset{\text{(sublemma)}}{\Longrightarrow}$ M^* is a model of $\delta(p, a_i)$.

- M: subset of M^* that is a minimal model

  $M \subseteq M^*$, $\quad M \in \mathrm{Mod}_{\downarrow}(\delta(p, a_i))$

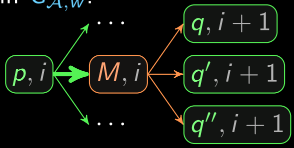
- Define $f_A(\langle p, i\rangle) := \langle M, i\rangle$

- $\forall\ \gamma$: play in $G_{\mathcal{A},w}$ played according to f_A
  $\exists\ \overline{\gamma}$: play in $G_{\overline{\mathcal{A}},w}$ played according to $\overline{f_P}$
  s.t. γ and $\overline{\gamma}$ contain the same V_A-nodes.

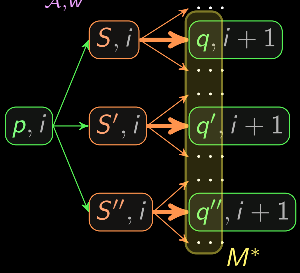
$\Leftarrow$ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i\rangle \in V_A$

in $G_{\mathcal{A},w}$:



in $G_{\overline{\mathcal{A}},w}$:



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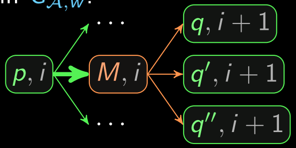
  s.t. γ and $\overline{\gamma}$ contain the same V_A-nodes.
  - Player P wins $\overline{\gamma}$ in $G_{\overline{\mathcal{A}},w}$.

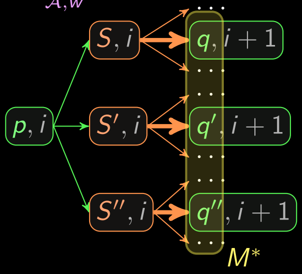
Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i\rangle \in V_A$

in $G_{\mathcal{A},w}$:



in $G_{\overline{\mathcal{A}},w}$:



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  s.t. γ and $\overline{\gamma}$ contain the same V_A-nodes.
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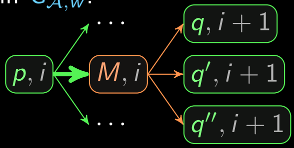
  - $\forall\, q \in Q : \pi(q) = \overline{\pi}(q) - 1$

⇐ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i\rangle \in V_A$

in $G_{\mathcal{A},w}$:



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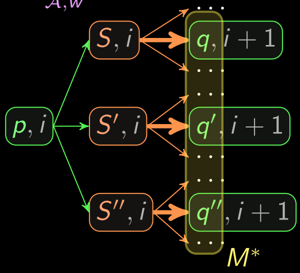

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  - $\forall\, q \in Q : \pi(q) = \overline{\pi}(q) - 1$

$\Rightarrow$ Player A wins γ in $G_{\mathcal{A},w}$.

Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w \in \Sigma^\omega$.

LEMMA 1

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ accepts w.

LEMMA 2

Player P has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ does *not* accept w.

LEMMA 3

Player A has a winning strategy in $G_{\mathcal{A},w}$
iff player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$.

Complementation Theorem

Theorem (Complementation)

The dual $\overline{\mathcal{A}}$ of a WAPA $\mathcal{A}$ accepts its complement, i.e.

$$\mathcal{L}(\overline{\mathcal{A}}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$$

(Thomas and Löding, $\sim$2000)

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The dual $\overline{\mathcal{A}}$ of a WAPA $\mathcal{A}$ accepts its complement, i.e.

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Proof:

$\mathcal{A}$ accepts $w \overset{\text{(lemma 1)}}{\iff}$ player A has a winning strategy in $G_{\mathcal{A},w}$

Theorem (Complementation)

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(Thomas and Löding, $\sim$2000)

Proof:

$\mathcal{A}$ accepts w $\overset{\text{(lemma 1)}}{\Longleftrightarrow}$ player A has a winning strategy in $G_{\mathcal{A},w}$

$\overset{\text{(lemma 3)}}{\Longleftrightarrow}$ player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$

Theorem (Complementation)

The dual $\overline{\mathcal{A}}$ of a WAPA $\mathcal{A}$ accepts its complement, i.e.

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(Thomas and Löding, $\sim$2000)

Proof:

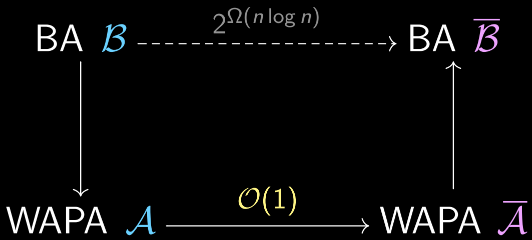
$\mathcal{A}$ accepts w $\overset{\text{(lemma 1)}}{\iff}$ player A has a winning strategy in $G_{\mathcal{A},w}$

$\overset{\text{(lemma 3)}}{\iff}$ player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$

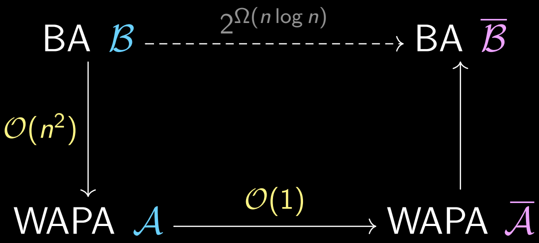
$\overset{\text{(lemma 2)}}{\iff}$ $\overline{\mathcal{A}}$ does *not* accept w

$\square$

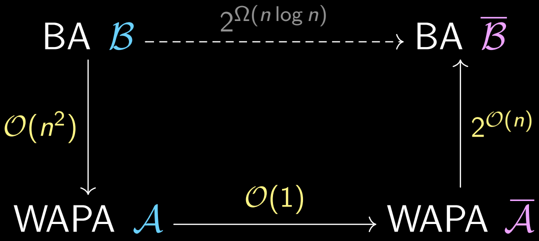
Outline


$2^{\Omega(n \log n)}$
BA $\mathcal{B}$
BA $\overline{\mathcal{B}}$
$\mathcal{O}(1)$
WAPA $\mathcal{A}$
WAPA $\overline{\mathcal{A}}$

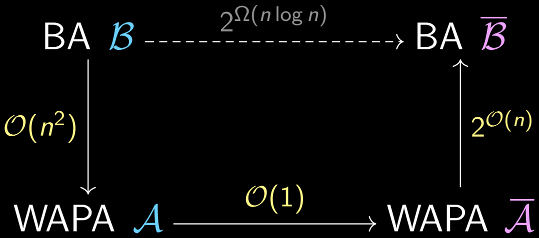
Büchi Complementation Algorithm



Büchi Complementation Algorithm

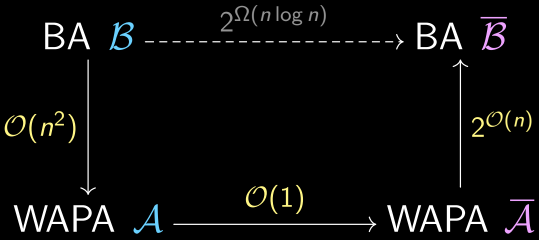


Büchi Complementation Algorithm



- Total complexity: $2^{\mathcal{O}(n^2)}$

Büchi Complementation Algorithm



- Total complexity: $2^{\mathcal{O}(n^2)}$
- Can reach $2^{\mathcal{O}(n \log n)}$ (lower bound) by improving $\overline{\mathcal{A}} \to \overline{\mathcal{B}}$.

References

Thomas, W. (1999)
Complementation of Büchi Automata Revisited.
In J. Karhumäki et al., editors, *Jewels are Forever, Contributions on Th. Comp. Science in Honor of Arto Salomaa*, pages 109–122, Springer.

Klaedtke, F. (2002)
Complementation of Büchi Automata Using Alternation.
In E. Grädel et al., editors, *Automata, Logics, and Infinite Games*, LNCS 2500, pages 61-77. Springer.

Löding, C. and Thomas, W. (2000)
Alternating Automata and Logics over Infinite Words.
In J. van Leeuwen et al., editors, *IFIP TCS 2000*, LNCS 1872, pages 521–535. Springer.

Kupferman, O. and Vardi, M. Y. (2001)
Weak Alternating Automata Are Not that Weak.
In *ACM Transactions on Computational Logic*, volume 2, No. 3, July 2001, pages 408–429.

Appendix

From BA to WAPA

Given:

- $\mathcal{B} = \langle Q, \Sigma, \delta, q_{in}, F \rangle$: BA
- $n = |Q|$

From BA to WAPA

Given:

- $\mathcal{B} = \langle Q, \Sigma, \delta, q_{in}, F \rangle$: BA
- $n = |Q|$

Construction (BA $\rightarrow$ WAPA)

$$\mathcal{A} := \langle \underbrace{Q \times \{0, \ldots, 2n\}}_{\mathcal{O}(n^2)},\ \Sigma,\ \delta',\ \langle q_{in}, 2n \rangle,\ \pi \rangle$$

(Thomas and Löding, $\sim$2000)

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where

- $\pi(\langle p, i \rangle) := i$

for $p \in Q$, $a \in \Sigma$, $i \in \{0, \ldots, 2n\}$

(Thomas and Löding, $\sim$2000)

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Given:

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where

- $\delta'(\langle p, i \rangle, a) := \begin{cases} \bigvee_{q \in \delta(p,a)} \langle q, 0 \rangle & \text{if } i = 0 \\ \bigvee_{q \in \delta(p,a)} \langle q, i \rangle \wedge \langle q, i-1 \rangle & \text{if } i \text{ even, } i > 0 \end{cases}$
- $\pi(\langle p, i \rangle) := i$

for $p \in Q, \ a \in \Sigma, \ i \in \{0, \ldots, 2n\}$

(Thomas and Löding, $\sim$2000)

From BA to WAPA

Given:

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- $\pi(\langle p, i \rangle) := i$

for $p \in Q$, $a \in \Sigma$, $i \in \{0, \ldots, 2n\}$

(Thomas and Löding, ∼2000)

Back

From WAPA to BA

Given:

- $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$: stratified WAPA, i.e.

$$\forall p \in Q \ \forall a \in \Sigma \ : \ \delta(p, a) \ \in \ \mathbb{B}^+\big(\{q \in Q \mid \pi(p) \geq \pi(q)\}\big)$$

From WAPA to BA

Given:

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- $E \subseteq Q$: all states with even parity

From WAPA to BA

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- $E \subseteq Q$: all states with even parity

Construction (WAPA $\rightarrow$ BA)

$$\mathcal{B} := \langle \underbrace{2^Q \times 2^Q}_{2^{\mathcal{O}(n)}},\ \Sigma,\ \delta',\ \langle \{q_{in}\}, \emptyset \rangle,\ 2^Q \times \{\emptyset\} \rangle$$

(Miyano and Hayashi, 1984)

From WAPA to BA

☺

Given:

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- $\delta'(\langle M, \emptyset \rangle, a) \ := \left\{ \langle M', M' \setminus E \rangle \,\middle|\, M' \in \mathrm{Mod}_{\downarrow}\left(\bigwedge_{q \in M} \delta(q,a) \right) \right\}$

for $a \in \Sigma$, $M, O \subseteq Q$, $O \neq \emptyset$

(Miyano and Hayashi, 1984)

From WAPA to BA

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- $E \subseteq Q$: all states with even parity

Construction (WAPA → BA)

$$\mathcal{B} := \langle \underbrace{2^{Q} \times 2^{Q}}_{2^{\mathcal{O}(n)}}, \ \Sigma, \ \delta', \ \langle \{q_{in}\}, \emptyset \rangle, \ 2^{Q} \times \{\emptyset\} \rangle$$

where

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- $\delta'(\langle M, O \rangle, a) := \left\{ \langle M', O' \setminus E \rangle \ \middle| \ \begin{array}{l} M' \in \mathrm{Mod}_{\downarrow}\left(\bigwedge_{q \in M} \delta(q,a) \right), \\ O' \subseteq M', \\ O' \in \mathrm{Mod}_{\downarrow}\left(\bigwedge_{q \in O} \delta(q,a) \right) \end{array} \right\}$

for $a \in \Sigma, \ \ M, O \subseteq Q, \ \ O \neq \emptyset$

(Miyano and Hayashi, 1984)