Büchi Complementation

BA $\mathcal{B}$ $\longrightarrow$ BA $\overline{\mathcal{B}}$

BA: Büchi Automaton

Expensive: If $\mathcal{B}$ has $n$ states, $\mathcal{B}$ has $2^{\Theta(n \log n)}$ states in the worst case (Michel 1988, Safra 1988).

Complicated: Direct approaches are rather involved. Consider indirect approach: detour over alternating automata.
Büchi Complementation

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Büchi Complementation

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Complicated: Direct approaches are rather involved.

Consider indirect approach: detour over alternating automata.
**Transition Modes (1)**

**Existential:** some run is accepting

- $q_0 \rightarrow q_{1a} \rightarrow q_{2a} \rightarrow q_{3a} \rightarrow q_{4a} \rightarrow q_{5a} \rightarrow \cdots$
- $q_0 \rightarrow q_{1b} \rightarrow q_{2b} \rightarrow q_{3b} \rightarrow q_{4b} \rightarrow q_{5b} \rightarrow \cdots$
- $q_0 \rightarrow q_{1c} \rightarrow q_{2c} \rightarrow q_{3c} \rightarrow q_{4c} \rightarrow q_{5c} \rightarrow \cdots$
- $q_0 \rightarrow q_{1d} \rightarrow q_{2d} \rightarrow q_{3d} \rightarrow q_{4d} \rightarrow q_{5d} \rightarrow \cdots$
- $q_0 \rightarrow q_{1e} \rightarrow q_{2e} \rightarrow q_{3e} \rightarrow q_{4e} \rightarrow q_{5e} \rightarrow \cdots$
**Transition Modes (1)**

**Existential:** some run is accepting

\[
q_0 \rightarrow q_{1a} \rightarrow q_{2a} \rightarrow q_{3a} \rightarrow q_{4a} \rightarrow q_{5a} \rightarrow \cdots
\]

\[
q_0 \rightarrow q_{1b} \rightarrow q_{2b} \rightarrow q_{3b} \rightarrow q_{4b} \rightarrow q_{5b} \rightarrow \cdots
\]

\[
q_0 \rightarrow q_{1c} \rightarrow q_{2c} \rightarrow q_{3c} \rightarrow q_{4c} \rightarrow q_{5c} \rightarrow \cdots
\]

\[
q_0 \rightarrow q_{1d} \rightarrow q_{2d} \rightarrow q_{3d} \rightarrow q_{4d} \rightarrow q_{5d} \rightarrow \cdots
\]

\[
q_0 \rightarrow q_{1e} \rightarrow q_{2e} \rightarrow q_{3e} \rightarrow q_{4e} \rightarrow q_{5e} \rightarrow \cdots
\]

**Universal:** every run is accepting

\[
q_0 \rightarrow q_{1a} \rightarrow q_{2a} \rightarrow q_{3a} \rightarrow q_{4a} \rightarrow q_{5a} \rightarrow \cdots
\]

\[
q_0 \rightarrow q_{1b} \rightarrow q_{2b} \rightarrow q_{3b} \rightarrow q_{4b} \rightarrow q_{5b} \rightarrow \cdots
\]

\[
q_0 \rightarrow q_{1c} \rightarrow q_{2c} \rightarrow q_{3c} \rightarrow q_{4c} \rightarrow q_{5c} \rightarrow \cdots
\]

\[
q_0 \rightarrow q_{1d} \rightarrow q_{2d} \rightarrow q_{3d} \rightarrow q_{4d} \rightarrow q_{5d} \rightarrow \cdots
\]

\[
q_0 \rightarrow q_{1e} \rightarrow q_{2e} \rightarrow q_{3e} \rightarrow q_{4e} \rightarrow q_{5e} \rightarrow \cdots
\]
Transition Modes (2)

Alternating: in **some** set of runs **every** run is accepting

- $q_0 \rightarrow q_{1a} \rightarrow q_{2a} \rightarrow q_{3a} \rightarrow q_{4a} \rightarrow q_{5a} \rightarrow \cdots$
- $q_0 \rightarrow q_{1b} \rightarrow q_{2b} \rightarrow q_{3b} \rightarrow q_{4b} \rightarrow q_{5b} \rightarrow \cdots$
- $q_0 \rightarrow q_{1c} \rightarrow q_{2c} \rightarrow q_{3c} \rightarrow q_{4c} \rightarrow q_{5c} \rightarrow \cdots$
- $q_0 \rightarrow q_{1d} \rightarrow q_{2d} \rightarrow q_{3d} \rightarrow q_{4d} \rightarrow q_{5d} \rightarrow \cdots$
- $q_0 \rightarrow q_{1e} \rightarrow q_{2e} \rightarrow q_{3e} \rightarrow q_{4e} \rightarrow q_{5e} \rightarrow \cdots$
- $q_0 \rightarrow q_{1f} \rightarrow q_{2f} \rightarrow q_{3f} \rightarrow q_{4f} \rightarrow q_{5f} \rightarrow \cdots$
- $q_0 \rightarrow q_{1g} \rightarrow q_{2g} \rightarrow q_{3g} \rightarrow q_{4g} \rightarrow q_{5g} \rightarrow \cdots$
- $q_0 \rightarrow q_{1h} \rightarrow q_{2h} \rightarrow q_{3h} \rightarrow q_{4h} \rightarrow q_{5h} \rightarrow \cdots$
- $q_0 \rightarrow q_{1i} \rightarrow q_{2i} \rightarrow q_{3i} \rightarrow q_{4i} \rightarrow q_{5i} \rightarrow \cdots$
Special case: $\mathcal{A}$ in existential mode

$\mathcal{A}$ accepts iff $\exists$ run $\rho : \rho$ fulfills acceptance condition of $\mathcal{A}$
Special case: $A$ in existential mode

- $A$ accepts iff $\exists$ run $\rho : \rho$ fulfills acceptance condition of $A$

- $\bar{A}$ accepts iff $\forall$ run $\rho : \neg(\rho$ fulfills acceptance condition of $A)$
Special case: $A$ in existential mode

- $A$ accepts iff $\exists$ run $\rho : \rho$ fulfills acceptance condition of $A$

- $\bar{A}$ accepts iff $\forall$ run $\rho : \neg(\rho$ fulfills acceptance condition of $A)$
  iff $\forall$ run $\rho: \rho$ fulfills dual acceptance condition of $A$
**Alternation and Complementation**

**Special case: \( A \) in existential mode**

- \( A \) accepts iff \( \exists \) run \( \rho \) : \( \rho \) fulfills acceptance condition of \( A \)

- \( \overline{A} \) accepts iff \( \forall \) run \( \rho \) : \( \neg(\rho \) fulfills acceptance condition of \( A \)\)

\( \Rightarrow \) complementation \( \overset{\approx}{=} \) dualization of:

- transition mode
- acceptance condition
Special case: $\mathcal{A}$ in existential mode

- $\mathcal{A}$ accepts iff $\exists \text{ run } \rho : \rho$ fulfills acceptance condition of $\mathcal{A}$
- $\overline{\mathcal{A}}$ accepts iff $\forall \text{ run } \rho : \neg(\rho$ fulfills acceptance condition of $\mathcal{A}$) iff $\forall \text{ run } \rho : \rho$ fulfills dual acceptance condition of $\mathcal{A}$

$\Rightarrow$ complementation $\cong$ dualization of:
- transition mode
- acceptance condition

Want acceptance condition that is closed under dualization.
Outline

1. Weak Alternating Parity Automata
2. Infinite Parity Games
3. Proof of the Complementation Theorem
4. Büchi Complementation Algorithm
1 Weak Alternating Parity Automata
   ■ Definitions and Examples
   ■ Dual Automaton

2 Infinite Parity Games

3 Proof of the Complementation Theorem

4 Büchi Complementation Algorithm
**Example** \( ((b^* a)^\omega) \)

Büchi automaton \( \mathcal{B} \):

![Büchi automaton diagram](image-url)
**Example** \(((b^*a)^\omega)\)

Büchi automaton \(\mathcal{B}\):

![Büchi automaton \(\mathcal{B}\)](image)

Equivalent WAPA \(\mathcal{A}\):

![Equivalent WAPA \(\mathcal{A}\)](image)
**Definition (Weak Alternating Parity Automaton)**

A weak alternating parity automaton (WAPA) is a tuple

\[ \mathcal{A} := \langle Q, \Sigma, \delta, q_{in}, \pi \rangle \]

where

- \( Q \) finite set of states
- \( \Sigma \) finite alphabet
- \( q_{in} \) initial state

(Thomas and Löding, ~2000)
Weak Alternating Parity Automaton

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- \( \pi : Q \rightarrow \mathbb{N} \) parity function

(Thomas and Löding, ~2000)
**Definition (Weak Alternating Parity Automaton)**

A weak alternating parity automaton (WAPA) is a tuple

$$\mathcal{A} := \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$$

where

- $Q$: finite set of states
- $\Sigma$: finite alphabet
- $\delta : Q \times \Sigma \rightarrow \mathbb{B}^+(Q)$: transition function
- $q_{in}$: initial state
- $\pi : Q \rightarrow \mathbb{N}$: parity function

($\mathbb{B}^+(Q)$: set of all positive Boolean formulae over $Q$
(built only from elements in $Q \cup \{\land, \lor, T, \bot\}$))

(Thomas and Löding, ~2000)
**Example** $(a^\omega)$

\[
\begin{align*}
\delta : Q \times \Sigma &\rightarrow \mathbb{B}^+(Q) \\
\langle q_0, a \rangle &\mapsto q_0 \lor (q_1 \land q_2) \\
\langle q_1, a \rangle &\mapsto (q_0 \land q_1) \lor (q_1 \land q_2) \\
\langle q_2, a \rangle &\mapsto q_2
\end{align*}
\]
**Example** 

\( q^0 \)

\[ a \]

\[ q^1 \]

\[ a \]

\[ q^2 \]

\[ a \]

\( \delta : Q \times \Sigma \rightarrow \mathbb{B}^+(Q) \)

\( \langle q^0, a \rangle \mapsto q^0 \lor (q^1 \land q^2) \)

\( \langle q^1, a \rangle \mapsto (q^0 \land q^1) \lor (q^1 \land q^2) \)

\( \langle q^2, a \rangle \mapsto q^2 \)

**Definition** (Minimal Models)

\( \text{Mod} \downarrow(\theta) \subseteq 2^Q \): set of minimal models of \( \theta \in \mathbb{B}^+(Q) \), i.e. the set of minimal subsets \( M \subseteq Q \) s.t. \( \theta \) is satisfied by

\[ q \mapsto \begin{cases} 
\text{true} & \text{if } q \in M \\
\text{false} & \text{otherwise}
\end{cases} \]

**Example**

\( \text{Mod} \downarrow(q^0 \lor (q^1 \land q^2)) = \{ \{q^0\}, \{q^1, q^2\} \} \)
Example ($a^\omega$)

Run Graph (1)
Run Graph (1)

Example \((a^\omega)\)

- Accepting run: 
  - \(q_0, 0\)
  - \(q_0, 1\)
  - \(q_0, 2\)
  - \(q_0, 3\)
  - \(q_0, 4\)
  - \(q_0, 5\)
  - \(q_0, 6\)
  - \(q_0, 7\)
  - \(q_0, 8\)
  - \(q_0, 9\)
  - \(q_0, 10\)

- Rejecting run:
  - \(q_0, 0\)
  - \(q_1, 2\)
  - \(q_1, 3\)
  - \(q_1, 4\)
  - \(q_2, 2\)
  - \(q_2, 3\)
  - \(q_2, 4\)
  - \(q_2, 5\)
  - \(q_2, 6\)
  - \(q_2, 7\)
  - \(q_2, 8\)
  - \(q_2, 9\)
  - \(q_2, 10\)
Example ($a^\omega$)

Accepting run:
- $q_0, 0 \rightarrow q_0, 1 \rightarrow \ldots$

Rejecting run:
- $q_0, 0 \rightarrow q_1, 1 \rightarrow q_2, 2 \rightarrow \ldots$
Example \( (a^\omega) \)

Run Graph (1)

Accepting run:
- \( q_0, 0 \) to \( q_0, 2 \)

Rejecting run:
- \( q_0, 0 \) to \( q_1, 2 \)
- \( q_1, 2 \) to \( q_2, 2 \)

State transitions:
- \( q_0 \) to \( q_0 \) on \( a \)
- \( q_0 \) to \( q_1 \) on \( a \)
- \( q_1 \) to \( q_1 \) on \( a \)
- \( q_1 \) to \( q_2 \) on \( a \)
- \( q_2 \) to \( q_2 \) on \( a \)
Example ($a^\omega$)

Accepting run: $q_0, 0 \rightarrow q_0, 1 \rightarrow q_0, 2 \rightarrow q_0, 3 \rightarrow q_0, 4 \rightarrow q_0, 5 \rightarrow \cdots$

Rejecting run: $q_0, 0 \rightarrow q_1, 2 \rightarrow q_1, 3 \rightarrow q_1, 4 \rightarrow q_1, 5 \rightarrow \cdots$
Run Graph (1)

Example \((a^\omega)\)

Accepting run:

\(q_0, 0\), \(q_0, 1\), \(q_0, 2\), \(q_0, 3\), \(q_0, 4\), \(q_0, 5\) \(\ldots\)

Rejecting run:

\(q_0, 0\), \(q_1, 2\), \(q_1, 3\), \(q_1, 4\), \(q_2, 2\), \(q_2, 3\), \(q_2, 4\) \(\ldots\)
Example ($a^\omega$)

Run Graph (1)

Accepting run:

$q_0, 0 \rightarrow q_0, 1 \rightarrow q_0, 2 \rightarrow q_0, 3 \rightarrow q_0, 4 \rightarrow q_0, 5 \rightarrow \ldots$

Rejecting run:

$q_0, 0 \rightarrow q_1, 1 \rightarrow q_1, 2 \rightarrow q_1, 3 \rightarrow q_1, 4 \rightarrow q_2, 5 \rightarrow \ldots$

$q_0, 0 \rightarrow q_0, 1$
**Run Graph (1)**

**Example (a^\omega)**

- **Accepting run:**
  - $q_0, 0 \rightarrow q_0, 1 \rightarrow q_0, 2 \rightarrow q_0, 3 \rightarrow q_0, 4 \rightarrow q_0, 5 \rightarrow \ldots$

- **Rejecting run:**
  - $q_0, 0 \rightarrow q_0, 1 \rightarrow q_1, 2 \rightarrow q_2, 2 \rightarrow \ldots$
Example ($a^\omega$)

Accepting run:
$q_0, 0 \rightarrow q_0, 1 \rightarrow q_0, 2 \rightarrow q_0, 3 \rightarrow q_0, 4 \rightarrow q_0, 5 \rightarrow \ldots$

Rejecting run:
$q_0, 0 \rightarrow q_0, 1 \rightarrow q_1, 2 \rightarrow q_1, 3 \rightarrow q_2, 2 \rightarrow q_2, 3 \rightarrow q_1, 4 \rightarrow q_1, 5 \rightarrow q_2, 4 \rightarrow q_2, 5 \rightarrow \ldots$
Run Graph (1)

Example ($a^\omega$)

\[ a q_0 2 q_1 a q_2 a \]

Accepting run:
\[ q_0, 0, q_0, 1, q_0, 2, q_0, 3, q_0, 4, q_0, 5, \ldots \]

Rejecting run:
\[ q_0, 0, q_0, 1, q_1, 2, q_1, 3, q_1, 4, q_2, 2, q_2, 3, q_2, 4, \ldots \]
Example ($a^\omega$)

Run Graph (1)
**Definition (Run)**

A run of a WAPA \( \mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle \) on a word \( a_0 a_1 a_2 \ldots \in \Sigma^\omega \) is a directed acyclic graph

\[
R := \langle V, E \rangle
\]
**Definition (Run)**

A run of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ on a word $a_0a_1a_2\ldots \in \Sigma^\omega$ is a directed acyclic graph

$$R := \langle V, E \rangle$$

where

- $V \subseteq Q \times \mathbb{N}$ with $\langle q_{in}, 0 \rangle \in V$
**Definition (Run)**

A run of a WAPA $A = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ on a word $a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed acyclic graph

$$R := \langle V, E \rangle$$

where

- $V \subseteq Q \times \mathbb{N}$ with $\langle q_{in}, 0 \rangle \in V$
- $V$ contains only vertices reachable from $\langle q_{in}, 0 \rangle$. 

$\Rightarrow R$ is a directed acyclic graph.
**Definition (Run)**

A run of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ on a word $a_0a_1a_2\ldots \in \Sigma^\omega$ is a directed acyclic graph

$$R := \langle V, E \rangle$$

where

- $V \subseteq Q \times \mathbb{N}$ with $\langle q_{in}, 0 \rangle \in V$
- $V$ contains only vertices reachable from $\langle q_{in}, 0 \rangle$.
- $E$ contains only edges of the form $\langle \langle p, i \rangle, \langle q, i + 1 \rangle \rangle$. 
Definition (Run)

A run of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{\text{in}}, \pi \rangle$ on a word $a_0a_1a_2 \ldots \in \Sigma^\omega$
is a directed acyclic graph

$$ R := \langle V, E \rangle $$

where

- $V \subseteq Q \times \mathbb{N}$ with $\langle q_{\text{in}}, 0 \rangle \in V$
- $V$ contains only vertices reachable from $\langle q_{\text{in}}, 0 \rangle$.
- $E$ contains only edges of the form $\langle \langle p, i \rangle, \langle q, i + 1 \rangle \rangle$.
- For every vertex $\langle p, i \rangle \in V$ the set of successors is a minimal model of $\delta(p, a_i)$

$$ \{ q \in Q \mid \langle \langle p, i \rangle, \langle q, i + 1 \rangle \rangle \in E \} \in \text{Mod}_{\downarrow}(\delta(p, a_i)) $$
Definition (Acceptance)

Let $\mathcal{A}$ be a WAPA, $w \in \Sigma^\omega$ and $R = \langle V, E \rangle$ a run of $\mathcal{A}$ on $w$.

- An infinite path $\rho$ in $R$ satisfies the acceptance condition of $\mathcal{A}$ iff the smallest occurring parity is even, i.e.
  \[
  \min\{\pi(q) \mid \exists i \in \mathbb{N} : \langle q, i \rangle \text{ occurs in } \rho\} \text{ is even.}
  \]
**Definition (Acceptance)**

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- $R$ is an accepting run iff every infinite path $\rho$ in $R$ satisfies the acceptance condition.
**Definition (Acceptance)**

Let \( \mathcal{A} \) be a WAPA, \( w \in \Sigma^\omega \) and \( R = \langle V, E \rangle \) a run of \( \mathcal{A} \) on \( w \).

- An infinite path \( \rho \) in \( R \) satisfies the acceptance condition of \( \mathcal{A} \) iff the smallest occurring parity is even, i.e.
  \[
  \min\{\pi(q) \mid \exists i \in \mathbb{N} : \langle q, i \rangle \text{ occurs in } \rho\}
  \]
  is even.

- \( R \) is an accepting run iff every infinite path \( \rho \) in \( R \) satisfies the acceptance condition.

- \( \mathcal{A} \) accepts \( w \) iff there is some accepting run of \( \mathcal{A} \) on \( w \).
**Acceptance**

**Example** $(a^\omega)$

![Diagram](image-url)

Accepting run:

$q_0, 0 \rightarrow q_0, 1 \rightarrow q_0, 2 \rightarrow q_0, 3 \rightarrow q_0, 4 \rightarrow q_0, 5 \rightarrow \cdots$

Rejecting run:

$q_0, 0 \rightarrow q_0, 1 \rightarrow q_1, 2 \rightarrow q_1, 3 \rightarrow q_1, 4 \rightarrow q_1, 5 \rightarrow \cdots$

$q_2, 2 \rightarrow q_2, 3 \rightarrow q_2, 4 \rightarrow q_2, 5 \rightarrow \cdots$
Acceptance

Example \(a^\omega\)

Accepting run:

\[
\begin{align*}
q_0, 0 &\rightarrow q_0, 1 \\
q_0, 1 &\rightarrow q_0, 2 \\
q_0, 2 &\rightarrow q_0, 3 \\
q_0, 3 &\rightarrow q_0, 4 \\
q_0, 4 &\rightarrow q_0, 5 \\
\end{align*}
\]

\[
\begin{align*}
q_0, 0 &\rightarrow q_0, 1 \\
q_0, 1 &\rightarrow q_1, 2 \\
q_1, 2 &\rightarrow q_1, 3 \\
q_1, 3 &\rightarrow q_1, 4 \\
q_1, 4 &\rightarrow q_1, 5 \\
\end{align*}
\]

Rejecting run:
**Acceptance**

**Example** ($a^\omega$)

Accepting run:

$$q_0, 0 \rightarrow q_0, 1 \rightarrow q_0, 2 \rightarrow q_0, 3 \rightarrow q_0, 4 \rightarrow q_0, 5 \rightarrow \ldots$$

Rejecting run:

$$q_0, 0 \rightarrow q_0, 1 \rightarrow q_1, 2 \rightarrow q_1, 3 \rightarrow q_1, 4 \rightarrow q_1, 5 \rightarrow \ldots$$

$$q_0, 0 \rightarrow q_0, 1 \rightarrow q_2, 2 \rightarrow q_2, 3 \rightarrow q_2, 4 \rightarrow q_2, 5 \rightarrow \ldots$$
Acceptance

Alternate: in some set of runs every run is accepting

$q_0 \rightarrow q_{1a} \rightarrow q_{2a} \rightarrow q_{3a} \rightarrow q_{4a} \rightarrow q_{5a} \rightarrow \cdots$
$q_0 \rightarrow q_{1b} \rightarrow q_{2b} \rightarrow q_{3b} \rightarrow q_{4b} \rightarrow q_{5b} \rightarrow \cdots$
$q_0 \rightarrow q_{1c} \rightarrow q_{2c} \rightarrow q_{3c} \rightarrow q_{4c} \rightarrow q_{5c} \rightarrow \cdots$
$q_0 \rightarrow q_{1d} \rightarrow q_{2d} \rightarrow q_{3d} \rightarrow q_{4d} \rightarrow q_{5d} \rightarrow \cdots$
$q_0 \rightarrow q_{1e} \rightarrow q_{2e} \rightarrow q_{3e} \rightarrow q_{4e} \rightarrow q_{5e} \rightarrow \cdots$
$q_0 \rightarrow q_{1f} \rightarrow q_{2f} \rightarrow q_{3f} \rightarrow q_{4f} \rightarrow q_{5f} \rightarrow \cdots$
$q_0 \rightarrow q_{1g} \rightarrow q_{2g} \rightarrow q_{3g} \rightarrow q_{4g} \rightarrow q_{5g} \rightarrow \cdots$
$q_0 \rightarrow q_{1h} \rightarrow q_{2h} \rightarrow q_{3h} \rightarrow q_{4h} \rightarrow q_{5h} \rightarrow \cdots$
$q_0 \rightarrow q_{1i} \rightarrow q_{2i} \rightarrow q_{3i} \rightarrow q_{4i} \rightarrow q_{5i} \rightarrow \cdots$
Infininitely many a’s

Example \(((b^*a)\omega)\)
INFINITELY MANY a’S

Example: $((b^a)\omega)$

Run on $b^\omega$: $q_0, 0$
Infinitely many a's

Example \(( (b^*a)^\omega ) \)

Run on \( b^\omega \):

- \( q_0, 0 \) to \( q_0, 1 \)
- \( q_1, 1 \) to \( q_0, 1 \)

Diagram:

- \( q_0 \):
  - \( a \) to state 2
  - \( b \) to state 1
- \( q_1 \):
  - \( b \) to state 1
  - \( a \) to state 0
- \( q_2 \):
  - \( a \) to state 0
  - \( b \) to state 1
**INFINITELY MANY a’S**

**EXAMPLE** \(((b^*a)^\omega)\)

Run on \(b^\omega\):

\[
\begin{align*}
q_0,0 & \xrightarrow{b} q_0,1 & q_0,1 & \xrightarrow{b} q_0,2 \\
q_1,1 & \xrightarrow{b} q_1,2 & q_1,1 & \xrightarrow{b} q_1,2
\end{align*}
\]
Infinitely many $a$'s

Example ($(b^*a)^\omega$)

Run on $b^\omega$:
Infinitely many $a$’s

**Example** $((b^*a)\omega)$

Run on $b^\omega$:

Run on $(ba)^\omega$:
INFINITELY MANY a’S

Example $((b^*a)\omega)$

Run on $b^\omega$:

Run on $(ba)^\omega$: 
Inﬁnitely many a’s

Example ((b^*a)^\omega)

Run on b^\omega:

Run on (ba)^\omega:
**Example** \(( (b^* a)^\omega ) \)

**Run on** \(b^\omega\):

- \(q_0, 0\) → \(q_0, 1\) → \(q_0, 2\) → \(q_0, 3\) → \(q_0, 4\) → \(q_0, 5\) → \(q_0, 6\) → \(...\)
- \(q_1, 1\) → \(q_1, 2\) → \(q_1, 3\) → \(q_1, 4\) → \(q_1, 5\) → \(q_1, 6\) → \(...\)

**Run on** \((ba)^\omega\):

- \(q_0, 0\) → \(q_0, 1\) → \(q_0, 2\) → \(q_0, 3\) → \(...\)
- \(q_1, 1\) → \(q_1, 2\) → \(q_1, 3\) → \(...\)
- \(q_2, 2\) → \(q_2, 3\) → \(...\)
INFINITELY MANY a’S

EXAMPLE \(( (b^* a)^\omega ) \)

Run on \( b^\omega \):

Run on \( (ba)^\omega \):
Definition (Dual Automaton)

The dual of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{\text{in}}, \pi \rangle$ is

$\overline{\mathcal{A}} := \langle Q, \Sigma, \overline{\delta}, q_{\text{in}}, \overline{\pi} \rangle$
**Definition (Dual Automaton)**

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where

- $\overline{\delta}(q, a)$ is obtained from $\delta(q, a)$ by exchanging $\wedge, \vee$ and $\top, \bot$

for all $q \in Q$ and $a \in \Sigma$
**Definition (Dual Automaton)**

The dual of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{\text{in}}, \pi \rangle$ is

$$\overline{\mathcal{A}} := \langle Q, \Sigma, \overline{\delta}, q_{\text{in}}, \overline{\pi} \rangle$$

where

- $\overline{\delta}(q, a)$ is obtained from $\delta(q, a)$ by exchanging $\wedge, \vee$ and $\top, \bot$
- $\overline{\pi}(q) := \pi(q) + 1$

for all $q \in Q$ and $a \in \Sigma$
**Example** \( ((b^*a)^\omega) \)

**WAPA** \( A \):

\[
\begin{align*}
\delta(q_0, a) &= q_0 \\
\delta(q_0, b) &= q_0 \land q_1 \\
\delta(q_1, a) &= q_2 \\
\delta(q_1, b) &= q_1 \\
\delta(q_2, a) &= q_2 \\
\delta(q_2, b) &= q_2
\end{align*}
\]
**Example** $((b^* a)^\omega)$

**WAPA** $\mathcal{A}$:

Dual $\overline{\mathcal{A}}$:

$\delta(q_0, a) = q_0$
$\delta(q_0, b) = q_0 \land q_1$
$\delta(q_1, a) = q_2$
$\delta(q_1, b) = q_1$
$\delta(q_2, a) = q_2$
$\delta(q_2, b) = q_2$

$\overline{\delta}(q_0, a) = q_0$
$\overline{\delta}(q_0, b) = q_0 \lor q_1$
$\overline{\delta}(q_1, a) = q_2$
$\overline{\delta}(q_1, b) = q_1$
$\overline{\delta}(q_2, a) = q_2$
$\overline{\delta}(q_2, b) = q_2$
Main statement of this talk:

**Theorem (Complementation)**

The dual $\overline{A}$ of a WAPA $A$ accepts its complement, i.e.

$$\mathcal{L}(\overline{A}) = \Sigma^\omega \setminus \mathcal{L}(A)$$

(Thomas and Löding, ~2000)
Outline

1. Weak Alternating Parity Automata
2. Infinite Parity Games
3. Proof of the Complementation Theorem
4. Büchi Complementation Algorithm
Automaton vs. Pathfinder

player A finds accepting run R

player P finds rejecting path in R
Automaton vs. Pathfinder

player A
Automaton vs. Pathfinder

player A

find accepting run $R$
Automaton vs. Pathfinder

player A
find accepting run $R$

player $P$
Automaton vs. Pathfinder

player A
find accepting run $R$

player P
find rejecting path in $R$
Infinite Parity Game (1)

Example \((a^\omega)\)

\[
A: \quad w = a^\omega
\]

\[
q_0 \quad 2
\]

\[
q_1 \quad 1
\]

\[
q_2 \quad 0
\]
Infinite Parity Game (1)

Example \((a^\omega)\)

Game \(G_{A,w}:\)

\(q_0, 0\)

\[w = a^\omega\]
**Example** ($a^\omega$)

Game $G_{A,w}$:

- $q_0, 0$ to $\{q_0\}, 0$
- $\{q_1, q_2\}, 0$
Infinite Parity Game (1)

Example ($a^\omega$)

Game $G_{A,w}$:

- Initial state: $q_0, 0$
- Accepting states: $\{q_0\}, 0$ and $q_0, 1$
- Transitions:
  - From $q_0, 0$ to $\{q_0\}, 0$ on $a$
  - From $\{q_1, q_2\}, 0$ to $q_0, 0$ on $a$
  - From $q_0, 1$ to $q_2, 0$ on $a$

$w = a^\omega$
**Infinite Parity Game (1)**

**Example ($a^\omega$)**

Game $G_{A,w}$:

- $A$: $q_0, 2 \rightarrow a \rightarrow q_1, 1 \rightarrow a \rightarrow q_2, 0 \rightarrow a \rightarrow \cdots$
- $w = a^\omega$

- $\{q_0\}, 0 \rightarrow q_0, 1$
- $\{q_1, q_2\}, 0 \rightarrow q_1, 1$
- $q_2, 1$

**Game $G_{A,w}$**
Example \((a^\omega)\)

Game \(G_{\mathcal{A},w}\):

\[
\begin{align*}
& q_0, 0 \rightarrow \{q_0\}, 0 \rightarrow q_0, 1 \rightarrow \{q_0\}, 1 \rightarrow \{q_1, q_2\}, 1 \\
& \quad \downarrow \quad \downarrow \quad \downarrow \\
& \quad q_1, 1 \rightarrow \{q_1, q_2\}, 0 \\
& \quad \downarrow \\
& \quad q_2, 1
\end{align*}
\]
**Example \(a^\omega\)**

Game \(G_{A,w}\):

\[
\begin{align*}
\{q_0, 0\} & \rightarrow \{q_0\}, 0 \rightarrow q_0, 1 \rightarrow \{q_0\}, 1 \\
\{q_1, q_2\}, 0 & \rightarrow q_1, 1 \rightarrow \{q_1, q_2\}, 1 \\
q_2, 1 & \rightarrow \{q_0, q_1\}, 1
\end{align*}
\]
Infinite Parity Game (1)

Example ($a^\omega$)

Game $G_{A,w}$:
Infinite Parity Game (1)

Example \(a^\omega\)

Game \(G_{A,w}\):

- \(q_0, 0 \rightarrow \{q_0\}, 0 \rightarrow q_0, 1 \rightarrow \{q_0\}, 1 \rightarrow q_0, 2 \rightarrow \ldots\)
- \(q_1, 1 \rightarrow \{q_1, q_2\}, 1 \rightarrow q_1, 2 \rightarrow \ldots\)
- \(q_2, 1 \rightarrow \{q_2\}, 1 \rightarrow q_2, 2 \rightarrow \ldots\)

- \(\{q_1, q_2\}, 0 \rightarrow q_1, 1 \rightarrow \{q_0, q_1\}, 1 \rightarrow q_1, 2 \rightarrow \ldots\)

- \(A: w = a^\omega\)
Definition (Game)

A game for a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed graph

$$G_{\mathcal{A}, w} := \langle V_A \cup V_P, E \rangle$$

(Thomas and Löding, ∼ 2000)
**Definition (Game)**

A game for a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0a_1a_2\ldots \in \Sigma^\omega$ is a directed graph

$$G_{\mathcal{A},w} := \langle V_{\mathcal{A}} \cup V_P, E \rangle$$

where

- $V_{\mathcal{A}} := Q \times \mathbb{N}$ (decision nodes of player $A$)

(Thomas and Löding, ∼ 2000)
**Definition (Game)**

A game for a WAPA \( \mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle \) and \( w = a_0 a_1 a_2 \ldots \in \Sigma^\omega \) is a directed graph

\[
G_{\mathcal{A}, w} := \langle V_{\mathcal{A}} \cup V_P, E \rangle
\]

where

- \( V_{\mathcal{A}} := Q \times \mathbb{N} \) (decision nodes of player \( A \))
- \( V_P := 2^Q \times \mathbb{N} \) (decision nodes of player \( P \))

(Thomas and L"oding, \( \sim 2000 \))
**Infinite Parity Game (2)**

**Definition (Game)**

A game for a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0a_1a_2\ldots \in \Sigma^\omega$ is a directed graph

$$G_{\mathcal{A}, w} := \langle V_A \cup V_P, E \rangle$$

where

- $V_A := Q \times \mathbb{N}$ (decision nodes of player $A$)
- $V_P := 2^Q \times \mathbb{N}$ (decision nodes of player $P$)
- $E \subseteq (V_A \times V_P) \cup (V_P \times V_A)$

(Thomas and Löding, ~2000)
**Definition (Game)**

A game for a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed graph

$$G_{\mathcal{A}, w} := \langle V_A \cup V_P, E \rangle$$

where

- $V_A := Q \times \mathbb{N}$ (decision nodes of player $A$)
- $V_P := 2^Q \times \mathbb{N}$ (decision nodes of player $P$)
- $E \subseteq (V_A \times V_P) \cup (V_P \times V_A)$

s.t. the only contained edges are

- $\langle \langle q, i \rangle, \langle M, i \rangle \rangle$ \iff $M \in \text{Mod}_\downarrow(\delta(q, a_i))$

for $q \in Q$, $M \subseteq Q$, $i \in \mathbb{N}$

(Thomas and Löding, $\sim$ 2000)
**Infinite Parity Game (2)**

**Definition (Game)**

A game for a WAPA $A = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0a_1a_2\ldots \in \Sigma^\omega$ is a directed graph

$$G_{A,w} := \langle V_A \cup V_P, E \rangle$$

where

- $V_A := Q \times \mathbb{N}$ (decision nodes of player $A$)
- $V_P := 2^Q \times \mathbb{N}$ (decision nodes of player $P$)
- $E \subseteq (V_A \times V_P) \cup (V_P \times V_A)$

s.t. the only contained edges are

- $\langle \langle q, i \rangle, \langle M, i \rangle \rangle$ iff $M \in \text{Mod}_\downarrow(\delta(q, a_i))$
- $\langle \langle M, i \rangle, \langle q, i + 1 \rangle \rangle$ iff $q \in M$

for $q \in Q$, $M \subseteq Q$, $i \in \mathbb{N}$

(Thomas and Löding, ~2000)
**Definition (Play)**

A play $\gamma$ in a game $G_{A,w}$ is an infinite path starting with $\langle q_{in}, 0 \rangle$. 

**Example**

```
q_0, 0  ->  \{q_0\}, 0  ->  q_0, 1  \{q_0\}, 1  ->  q_0, 2  \ldots
\{q_1, q_2\}, 0  ->  q_1, 1  \{q_1, q_2\}, 1  ->  q_1, 2  \ldots
q_1, 1  ->  \{q_0, q_1\}, 1
q_2, 1  \{q_2\}, 1  ->  q_2, 2  \ldots
```
Playing a Game

**Definition (Play)**

A **play** $\gamma$ in a game $G_{A,w}$ is an infinite path starting with $\langle q_{\text{in}}, 0 \rangle$.

**Definition (Winner)**

The **winner** of a play $\gamma$ is

- player $A$ iff the smallest parity of occurring $V_A$-nodes is **even**
- player $P$ **odd**

**Example**

```
q_0, 0 → \{q_0\}, 0 → q_0, 1 → \{q_0\}, 1 → q_0, 2 → \ldots
q_1, 1 → \{q_0, q_1\}, 1 → q_1, 2 → \ldots
q_1, 1 → \{q_1, q_2\}, 0 → q_2, 1 → \{q_2\}, 1 → q_2, 2 → \ldots
```
**Playing a Game**

**Definition (Play)**

A play $\gamma$ in a game $G_{A,w}$ is an infinite path starting with $\langle q_{\text{in}}, 0 \rangle$.

**Definition (Winner)**

The winner of a play $\gamma$ is

- player $A$ iff the smallest parity of occurring $V_A$-nodes is even
- player $P$ odd

$X \in \{A, P\}$: a player, $\overline{X}$: its opponent

**Definition (Strategy)**

A strategy $f_X : V_X \rightarrow V_{\overline{X}}$ for player $X$ selects for every decision node of player $X$ one of its successor nodes in $G_{A,w}$.
**Playing a Game**

**Definition (Play)**

A play $\gamma$ in a game $G_{A,w}$ is an infinite path starting with $\langle q_{in}, 0 \rangle$.

**Definition (Winner)**

The winner of a play $\gamma$ is

- player $A$ iff the smallest parity of occurring $V_A$-nodes is even
- player $P$ iff the smallest parity of occurring $V_A$-nodes is odd

$X \in \{A, P\}$: a player, $\overline{X}$: its opponent

**Definition (Strategy)**

- A strategy $f_X : V_X \rightarrow V_{\overline{X}}$ for player $X$ selects for every decision node of player $X$ one of its successor nodes in $G_{A,w}$.
- $f_X$ is a winning strategy iff player $X$ wins every play $\gamma$ that is played according to $f_X$. 
Strategies

Example

Winning strategy for player A (so far):

q_0, 0 \rightarrow \{q_0\}, 0 \rightarrow q_0, 1 \rightarrow \{q_0\}, 1 \rightarrow q_0, 2 \rightarrow \ldots

\{q_1, q_2\}, 0 \rightarrow q_1, 1 \rightarrow \{q_1, q_2\}, 1 \rightarrow \{q_0, q_1\}, 1 \rightarrow q_1, 2 \rightarrow \ldots

q_2, 1 \rightarrow \{q_2\}, 1 \rightarrow q_2, 2 \rightarrow \ldots

parities

\ldots \quad q_0 \mapsto 2

\ldots \quad q_1 \mapsto 1

\ldots \quad q_2 \mapsto 0
Strategies

Example

Winning strategy for player A (so far):

- \( q_0, 0 \) \rightarrow \{ q_0 \}, 0 \rightarrow \{ q_0 \}, 1 \rightarrow q_0, 2 \rightarrow \cdots \rightarrow q_0 \leftarrow 2

- \{ q_1, q_2 \}, 0 \rightarrow q_1, 1 \rightarrow \{ q_0, q_1 \}, 1 \rightarrow q_1, 2 \rightarrow \cdots \rightarrow q_1 \leftarrow 1

- q_2, 1 \rightarrow \{ q_2 \}, 1 \rightarrow q_2, 2 \rightarrow \cdots \rightarrow q_2 \leftarrow 0

Parities:

- \( q_0 \leftarrow 2 \)
- \( q_1 \leftarrow 1 \)
- \( q_2 \leftarrow 0 \)
Example

Winning strategy for player A (so far):

\[
q_0, 0 \rightarrow \{q_0\}, 0 \rightarrow q_0, 1 \rightarrow \{q_0\}, 1 \rightarrow q_0, 2 \rightarrow \ldots \rightarrow q_1, 1 \rightarrow \{q_0, q_1\}, 1 \rightarrow q_0, 2 \rightarrow \ldots
\]

Not a winning strategy for player A:

\[
q_0, 0 \rightarrow \{q_0\}, 0 \rightarrow q_0, 1 \rightarrow \{q_0\}, 1 \rightarrow q_0, 2 \rightarrow \ldots \rightarrow q_1, 1 \rightarrow \{q_0, q_1\}, 1 \rightarrow q_0, 2 \rightarrow \ldots
\]
Outline

1. Weak Alternating Parity Automata
2. Infinite Parity Games
3. Proof of the Complementation Theorem
   - Lemma 1
   - Lemma 2
   - Lemma 3
     - Sublemma
   - Putting it All Together
4. Büchi Complementation Algorithm
Lemma 1

Let $\mathcal{A}$ be a WAPA and $w \in \Sigma^\omega$.

Player $A$ has a winning strategy in $G_{A,w}$ iff $\mathcal{A}$ accepts $w$. 

Explanation (oral):
Player $A$ wins every play $\gamma$ played according to $f_A$.

There is a run graph $R$ in which every path $\rho$ is accepting.
Lemma 1

Let $A$ be a WAPA and $w \in \Sigma^\omega$.

**Lemma 1**

Player $A$ has a winning strategy in $G_{A,w}$ iff $A$ accepts $w$.

**Explanation (oral):**

Player $A$ wins every play $\gamma$ played according to $f_A$.

$$G_{A,w}:$$

- $p, i$ 
- $\{q, q', q''\}, i$ 
- $q, i + 1$
- $q', i + 1$
- $q'', i + 1$
- \(\ldots\)
Lemma 1

Let $A$ be a WAPA and $w \in \Sigma^\omega$.

**Lemma 1**

Player $A$ has a winning strategy in $G_{A,w}$ iff $A$ accepts $w$.

**Explanation (oral):**

Player $A$ wins every play $\gamma$ played according to $f_A$.

There is a run graph $R$ in which every path $\rho$ is accepting.
Lemma 2

Let $A$ be a WAPA and $w \in \Sigma^\omega$.

**Lemma 2**

Player $P$ has a winning strategy in $G_{A,w}$ iff $A$ does not accept $w$.

(pointed out by Jan Leike)
Lemma 2

Let $\mathcal{A}$ be a WAPA and $w \in \Sigma^\omega$.

Lemma 2
Player $P$ has a winning strategy in $G_{\mathcal{A},w}$ iff $\mathcal{A}$ does not accept $w$.

(POINTED OUT BY JAN LEIKE)

EXPLANATION (oral):

Player $P$ wins every play $\gamma$ played according to $f_P$.

$G_{\mathcal{A},w}$: 

$p, i$ 

$\{.., q', ..\}, i$ $\rightarrow$ $q', i + 1$ 

$\ldots$ 

$\{.., q'', ..\}, i$ $\rightarrow$ $q'', i + 1$ 

$\ldots$ 

$p, i$ 

$\{.., q, ..\}, i$ $\rightarrow$ $q, i + 1$ 

$\ldots$
Lemma 2

Let $A$ be a WAPA and $w \in \Sigma^\omega$.

**Lemma 2**

Player $P$ has a winning strategy in $G_{A,w}$ iff $A$ does not accept $w$.

(pointed out by Jan Leike)

**Explanation (oral):**

Player $P$ wins every play $\gamma$ played according to $f_P$.

Every run graph $R$ contains a rejecting path $\rho$.

$G_{A,w}$:

- $p, i$  $\{\ldots, q, \ldots\}, i$  $q, i + 1$
- $\ldots$

$R$:

- $p, i$  $q, i + 1$
- $\ldots$

$R'$:

- $p, i$  $q', i + 1$
- $\ldots$

$R''$:

- $p, i$  $q'', i + 1$
- $\ldots$
Let $\theta \in B^+(Q)$ be a formula over $Q$. 

**Sublemma**

$S \subseteq Q$ is a model of $\overline{\theta}$ iff for all $M \in \text{Mod}\downarrow(\theta)$: $S \cap M \neq \emptyset$. 

Proof:

W.l.o.g. $\theta$ is in DNF, i.e. $\theta = \bigvee_{M \in \text{Mod}\downarrow(\theta)} \bigwedge q \in M q$

Then $\theta$ is in CNF, i.e. $\theta = \bigwedge_{M \in \text{Mod}\downarrow(\theta)} \bigvee q \in M q$

Thus $S \subseteq Q$ is a model of $\theta$ iff it contains at least one element from each disjunct of $\theta$. 

Let $\theta \in \mathbb{B}^+(Q)$ be a formula over $Q$.

**Sublemma**

$S \subseteq Q$ is a model of $\overline{\theta}$ iff for all $M \in \text{Mod}_{\downarrow}(\theta)$: $S \cap M \neq \emptyset$.

**Proof:**

- W.l.o.g. $\theta$ is in DNF, i.e.

$$\theta = \bigvee_{M \in \text{Mod}_{\downarrow}(\theta)} \bigwedge_{q \in M} q$$
Sublemma

Let $\theta \in \mathbb{B}^+(Q)$ be a formula over $Q$.

Sublemma

$S \subseteq Q$ is a model of $\bar{\theta}$ iff for all $M \in \text{Mod}_\downarrow(\theta)$: $S \cap M \neq \emptyset$.

Proof:

- W.l.o.g. $\theta$ is in DNF, i.e.

$$\theta = \bigvee_{M \in \text{Mod}_\downarrow(\theta)} \bigwedge_{q \in M} q$$

- Then $\bar{\theta}$ is in CNF, i.e.

$$\bar{\theta} = \bigwedge_{M \in \text{Mod}_\downarrow(\theta)} \bigvee_{q \in M} q$$
**Sublemma**

Let \( \theta \in \mathbb{B}^+(Q) \) be a formula over \( Q \).

\[ S \subseteq Q \text{ is a model of } \overline{\theta} \iff \text{ for all } M \in \text{Mod}_\downarrow(\theta): S \cap M \neq \emptyset. \]

**Proof:**

- **W.l.o.g.** \( \theta \) is in DNF, i.e.
  \[
  \theta = \bigvee_{M \in \text{Mod}_\downarrow(\theta)} \bigwedge q_{q \in M}
  \]

- **Then** \( \overline{\theta} \) is in CNF, i.e.
  \[
  \overline{\theta} = \bigwedge_{M \in \text{Mod}_\downarrow(\theta)} \bigvee q_{q \in M}
  \]

- **Thus** \( S \subseteq Q \) is a model of \( \overline{\theta} \) iff it contains at least one element from each disjunct of \( \theta \).
Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w = a_0a_1a_2 \ldots \in \Sigma^\omega$.

**Lemma 3**

Player $A$ has a winning strategy in $G_{\mathcal{A},w}$

iff player $P$ has a winning strategy in $G_{\overline{\mathcal{A}},w}$. 
Lemma 3

Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w = a_0a_1a_2\ldots \in \Sigma^\omega$.

**Lemma 3**

Player $A$ has a winning strategy in $G_{\mathcal{A},w}$ if and only if player $P$ has a winning strategy in $G_{\overline{\mathcal{A}},w}$.

**Proof:**

$\Rightarrow$ Construct a winning strategy $\overline{f_P}$ for player $P$ in $G_{\overline{\mathcal{A}},w}$.

$\ldots$

$\Leftarrow$ Construct a winning strategy $f_A$ for player $A$ in $G_{\mathcal{A},w}$.

$\ldots$
Lemma 3 (2)

⇒ Construct a winning strategy $\overline{f_P}$ for player $P$ in $G_{\overline{A},w}$.

At position $\langle S, i \rangle \in V_P$

in $G_{\overline{A},w}$:
Lemma 3 (2)

⇒ Construct a winning strategy $f_P$ for player $P$ in $G_{A,w}$.

At position $\langle S, i \rangle \in V_P$

in $G_{A,w}$:

- $f_A$: winning strategy for player $A$ in $G_{A,w}$
Lemma 3 (2)

⇒ Construct a winning strategy $\overline{f}_P$ for player $P$ in $G_{A,w}$.

- $f_A$: winning strategy for player $A$ in $G_{A,w}$
- Assume there is $\langle p, i \rangle \in V_A$ occurring in a play $\gamma$ in $G_{A,w}$ played according to $f_A$ s.t.
  $S \in \text{Mod}_{\downarrow}(\delta(p, a_i))$ (otherwise don’t care).

At position $\langle S, i \rangle \in V_P$ in $G_{\overline{A},w}$:

- $\ldots \quad \ldots \quad \ldots$

\[ p, i \rightarrow S, i \rightarrow \ldots \]

\[ \ldots \quad \ldots \quad \ldots \]

in $G_{\overline{A},w}$:

- $\ldots$

\[ p, i \rightarrow \ldots \]

\[ \ldots \quad \ldots \quad \ldots \]

in $G_{A,w}$:

- $\ldots$

\[ p, i \rightarrow \ldots \]

\[ \ldots \quad \ldots \quad \ldots \]
Lemma 3 (2)

⇒ Construct a winning strategy $\overline{f_P}$ for player $P$ in $G_{A,w}$.

- $f_A$: winning strategy for player $A$ in $G_{A,w}$
- Assume there is $\langle p, i \rangle \in V_A$ occurring in a play $\gamma$ in $G_{A,w}$ played according to $f_A$ s.t. $S \in \text{Mod}_{\downarrow}(\delta(p, a_i))$ (otherwise don’t care).
- $f_A(\langle p, i \rangle) = \langle M, i \rangle \Rightarrow M \in \text{Mod}_{\downarrow}(\delta(p, a_i))$
Lemma 3 (2)

⇒ Construct a winning strategy \( \bar{f}_P \) for player \( P \) in \( G_{A,w} \).

At position \( \langle S, i \rangle \in V_P \)

- \( f_A \): winning strategy for player \( A \) in \( G_{A,w} \)

- Assume there is \( \langle p, i \rangle \in V_A \) occurring in a play \( \gamma \) in \( G_{A,w} \) played according to \( f_A \) s.t. \( S \in \text{Mod}_\downarrow(\delta(p, a_i)) \) (otherwise don’t care).

- \( f_A(\langle p, i \rangle) = \langle M, i \rangle \Rightarrow M \in \text{Mod}_\downarrow(\delta(p, a_i)) \) (sublemma)

- \( \quad \Rightarrow \quad \) There exists a \( q \in S \cap M \).
Lemma 3 (2)

Construct a winning strategy $\overline{f}_P$ for player $P$ in $G_{A,w}$.

- $f_A$: winning strategy for player $A$ in $G_{A,w}$
- Assume there is $\langle p, i \rangle \in V_A$ occurring in a play $\gamma$ in $G_{A,w}$ played according to $f_A$ s.t. $S \in \text{Mod}_\downarrow(\delta(p, a_i))$ (otherwise don’t care).
- $f_A(\langle p, i \rangle) = \langle M, i \rangle \Rightarrow M \in \text{Mod}_\downarrow(\delta(p, a_i))$
  (sublemma)
- $\Rightarrow$ There exists a $q \in S \cap M$.
- Define $\overline{f}_P(\langle S, i \rangle) := \langle q, i + 1 \rangle$
Lemma 3 (2)

⇒ Construct a winning strategy $\overline{f_P}$ for player $P$ in $G_{A,w}$.

- $f_A$: winning strategy for player $A$ in $G_{A,w}$.
- Assume there is $\langle p, i \rangle \in V_A$ occurring in a play $\gamma$ in $G_{A,w}$ played according to $f_A$ s.t. $S \in \text{Mod}_\downarrow(\delta(p, a_i))$ (otherwise don’t care).
- $f_A(\langle p, i \rangle) = \langle M, i \rangle \Rightarrow M \in \text{Mod}_\downarrow(\delta(p, a_i))$ (sublemma)
- $\Rightarrow$ There exists a $q \in S \cap M$.
- Define $\overline{f_P}(\langle S, i \rangle) := \langle q, i + 1 \rangle$

- $\forall \overline{\gamma}$: play in $G_{A,w}$ played according to $\overline{f_P}$
- $\exists \gamma$: play in $G_{A,w}$ played according to $f_A$ s.t. $\overline{\gamma}$ and $\gamma$ contain the same $V_A$-nodes.
Lemma 3 (2)

⇒ Construct a winning strategy $\overline{f}_P$ for player $P$ in $G_{\overline{A},w}$.

- $f_A$: winning strategy for player $A$ in $G_{A,w}$
- Assume there is $\langle p, i \rangle \in V_A$ occurring in a play $\gamma$ in $G_{A,w}$ played according to $f_A$ s.t. $S \in \text{Mod}_{\downarrow}(\delta(p, a_i))$ (otherwise don’t care).
- $f_A(\langle p, i \rangle) = \langle M, i \rangle \Rightarrow M \in \text{Mod}_{\downarrow}(\delta(p, a_i))$
  (sublemma)
- $\Rightarrow$ There exists a $q \in S \cap M$.
- Define $\overline{f}_P(\langle S, i \rangle) := \langle q, i + 1 \rangle$
- $\forall \overline{\gamma}$: play in $G_{\overline{A},w}$ played according to $\overline{f}_P$
- $\exists \gamma$: play in $G_{A,w}$ played according to $f_A$ s.t. $\overline{\gamma}$ and $\gamma$ contain the same $V_A$-nodes.
  - Player $A$ wins $\gamma$ in $G_{A,w}$.
Lemma 3 (2)

Construct a winning strategy \( f_P \) for player \( P \) in \( G_{\overline{A},w} \).

At position \( \langle S, i \rangle \in V_P \) in \( G_{\overline{A},w} \):

- Assume there is \( \langle p, i \rangle \in V_A \) occurring in a play \( \gamma \) in \( G_{\overline{A},w} \) played according to \( f_A \) s.t. \( S \in \text{Mod}_\downarrow(\delta(p, a_i)) \) (otherwise don’t care).

- Define \( f_A(\langle p, i \rangle) = \langle M, i \rangle \Rightarrow M \in \text{Mod}_\downarrow(\delta(p, a_i)) \) (sublemma)

- There exists a \( q \in S \cap M \).

- Define \( \overline{f_P}(\langle S, i \rangle) := \langle q, i + 1 \rangle \)

- \( \forall \overline{\gamma} \): play in \( G_{\overline{A},w} \) played according to \( \overline{f_P} \)

- \( \exists \gamma \): play in \( G_{\overline{A},w} \) played according to \( f_A \) s.t. \( \overline{\gamma} \) and \( \gamma \) contain the same \( V_A \)-nodes.

  - Player \( A \) wins \( \gamma \) in \( G_{\overline{A},w} \).
  - \( \forall q \in Q : \overline{\pi}(q) = \pi(q) + 1 \)
Lemma 3 (2)

⇒ Construct a winning strategy \( \overline{f}_P \) for player \( P \) in \( G_{A,w} \).

- \( f_A \): winning strategy for player \( A \) in \( G_{A,w} \).
- Assume there is \( \langle p, i \rangle \in V_A \) occurring in a play \( \gamma \) in \( G_{A,w} \) played according to \( f_A \) s.t. \( S \in \text{Mod}_{\downarrow}(\delta(p, a_i)) \) (otherwise don’t care).

- Define \( \overline{f}_P(\langle S, i \rangle) := \langle q, i + 1 \rangle \)

- \( \forall \overline{\gamma} \): play in \( G_{A,w} \) played according to \( \overline{f}_P \)

- \( \exists \gamma \): play in \( G_{A,w} \) played according to \( f_A \) s.t. \( \overline{\gamma} \) and \( \gamma \) contain the same \( V_A \)-nodes.
  - Player \( A \) wins \( \gamma \) in \( G_{A,w} \).
  - \( \forall q \in Q : \overline{\pi}(q) = \pi(q) + 1 \)

\[ \Rightarrow \] Player \( P \) wins \( \overline{\gamma} \) in \( G_{A,w} \).

\[ \Rightarrow \] Assume there is \( \langle p, i \rangle \in V_A \) occurring in a play \( \gamma \) in \( G_{A,w} \) played according to \( f_A \) s.t. \( S \in \text{Mod}_{\downarrow}(\delta(p, a_i)) \) (otherwise don’t care).

- Define \( \overline{f}_P(\langle S, i \rangle) := \langle q, i + 1 \rangle \)

- \( \forall \overline{\gamma} \): play in \( G_{A,w} \) played according to \( \overline{f}_P \)

- \( \exists \gamma \): play in \( G_{A,w} \) played according to \( f_A \) s.t. \( \overline{\gamma} \) and \( \gamma \) contain the same \( V_A \)-nodes.
  - Player \( A \) wins \( \gamma \) in \( G_{A,w} \).
  - \( \forall q \in Q : \overline{\pi}(q) = \pi(q) + 1 \)

\[ \Rightarrow \] Player \( P \) wins \( \overline{\gamma} \) in \( G_{A,w} \).
Lemma 3 (3)

Construct a winning strategy \( f_A \) for player A in \( G_{A,w} \).

At position \( \langle p, i \rangle \in V_A \)

\[
\begin{array}{c}
\vdots \\
p, i \\
\vdots \\
\end{array}
\]

in \( G_{A,w} \):

\[
\begin{array}{c}
\vdots \\
\vdots \\
\end{array}
\]

\( \pi(q) = \pi(q) - 1 \)
Lemma 3 (3)

Construct a winning strategy $f_A$ for player $A$ in $G_{A, w}$.

At position $\langle p, i \rangle \in V_A$

- $f_A$: winning strategy for player $A$ in $G_{A, w}$
- $f_P$: winning strategy for player $P$ in $G_{\overline{A}, w}$
Lemma 3 (3)

Construct a winning strategy $f_A$ for player $A$ in $G_{A, w}$.

At position $\langle p, i \rangle \in V_A$ in $G_{A, w}$:

- $\overline{f_P}$: winning strategy for player $P$ in $G_{A, w}$
- $M^* := \{ q \in Q \mid \exists S \in \text{Mod}_\downarrow (\overline{\delta}(p, a_i)) : \overline{f_P}(\langle S, i \rangle) = \langle q, i+1 \rangle \}$
Lemma 3 (3)

⇐ Construct a winning strategy \( f_A \) for player \( A \) in \( G_{A,w} \).

At position \( \langle p, i \rangle \in V_A \) in \( G_{A,w} \):

- In \( G_{A,w} \):
  - \( p, i \)
  - \( S, i \)
  - \( S', i \)
  - \( S'', i \)
  - \( M^* \)

- In \( G_{\overline{A},w} \):
  - \( \ldots \)
  - \( q, i + 1 \)
  - \( q', i + 1 \)
  - \( q'', i + 1 \)

\( f_P \): winning strategy for player \( P \) in \( G_{\overline{A},w} \).

\( M^* := \{ q \in Q \mid \exists S \in \text{Mod}\downarrow(\overline{\delta}(p, a_i)) : f_P(\langle S, i \rangle) = \langle q, i + 1 \rangle \} \)

(sublemma) \( M^* \) is a model of \( \delta(p, a_i) \).
Lemma 3 (3)

⇐ Construct a winning strategy $f_A$ for player $A$ in $G_A,w$.

At position $\langle p, i \rangle \in V_A$

<table>
<thead>
<tr>
<th>in $G_A,w$:</th>
<th>[ q, i + 1 ]</th>
<th>[ q', i + 1 ]</th>
<th>[ q'', i + 1 ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p, i$</td>
<td>$M, i$</td>
<td>$q', i + 1$</td>
<td>$q'', i + 1$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
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<tbody>
<tr>
<td>$p, i$</td>
<td>$S', i$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$S'', i$</td>
<td>$q'', i + 1$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

- $\overline{f}_P$: winning strategy for player $P$ in $G_A,w$
- $M^* := \{ q \in Q \mid \exists S \in \text{Mod}_\downarrow(\overline{\delta}(p, a_i)) : \overline{f}_P(\langle S, i \rangle) = \langle q, i + 1 \rangle \}$
  (sublemma) $\implies M^*$ is a model of $\delta(p, a_i)$.
- $M$: subset of $M^*$ that is a minimal model $M \subseteq M^*$, $M \in \text{Mod}_\downarrow(\delta(p, a_i))$

$\ldots$
Lemma 3 (3)

Construct a winning strategy $f_A$ for player $A$ in $G_{A,w}$.

At position $\langle p, i \rangle \in V_A$ in $G_{A,w}$:

- $\overline{f_P}$: winning strategy for player $P$ in $G_{A,w}$
- $M^* := \{ q \in Q \mid \exists S \in \text{Mod}_\downarrow(\overline{\delta}(p, a_i)) : \overline{f_P}(\langle S, i \rangle) = \langle q, i + 1 \rangle \}$
- $(\text{sublemma}) \implies M^*$ is a model of $\delta(p, a_i)$.
- $M$: subset of $M^*$ that is a minimal model $M \subseteq M^*$, $M \in \text{Mod}_\downarrow(\delta(p, a_i))$
- Define $f_A(\langle p, i \rangle) := \langle M, i \rangle$
Lemma 3 (3)

⇐ Construct a winning strategy $f_A$ for player $A$ in $G_{A,w}$.

At position $⟨p, i⟩ ∈ V_A$

in $G_{A,w}$:

- $p, i$ → $M, i$ → $q, i + 1$
- $p, i$ → $M, i$ → $q', i + 1$
- $p, i$ → $M, i$ → $q'', i + 1$

in $G_{A,w}$:

- $p, i$ → $S, i$ → $q, i + 1$
- $p, i$ → $S', i$ → $q', i + 1$
- $p, i$ → $S'', i$ → $q'', i + 1$

- $M^∗$ is a model of $δ(p, a_i)$.

- $M$ is subset of $M^∗$ that is a minimal model $M ⊆ M^∗$, $M ∈ Mod(δ(p, a_i))$

- Define $f_A(⟨p, i⟩) := ⟨M, i⟩$

- $∀ γ$: play in $G_{A,w}$ played according to $f_A$
- $∃ \overline{γ}$: play in $G_{A,w}$ played according to $f_P$
  s.t. $γ$ and $\overline{γ}$ contain the same $V_A$-nodes.
\[\text{Lemma 3 (3)}\]

Construct a winning strategy \(f_A\) for player \(A\) in \(G_{A,w}\).

At position \(\langle p, i \rangle \in V_A\) in \(G_{A,w}\):

- \(\bar{f}_P\): winning strategy for player \(P\) in \(G_{A,w}\)
- \(M^* := \{ q \in Q | \exists S \in \text{Mod}_{\downarrow}(\delta(p, a_i)) : \bar{f}_P(\langle S, i \rangle) = \langle q, i + 1 \rangle \}\)
  
  \[\text{(sublemma)}\]
  \[\implies M^* \text{ is a model of } \delta(p, a_i).\]

- \(M\): subset of \(M^*\) that is a minimal model
  \[M \subseteq M^*, \quad M \in \text{Mod}_{\downarrow}(\delta(p, a_i))\]

- Define \(f_A(\langle p, i \rangle) := \langle M, i \rangle\)

- \(\forall \gamma\): play in \(G_{A,w}\) played according to \(f_A\)
- \(\exists \bar{\gamma}\): play in \(G_{A,w}\) played according to \(\bar{f}_P\)
  
  s.t. \(\gamma\) and \(\bar{\gamma}\) contain the same \(V_A\)-nodes.
  
  - Player \(P\) wins \(\bar{\gamma}\) in \(G_{A,w}\).
Lemma 3 (3)\[\Leftarrow\]

Construct a winning strategy $f_A$ for player $A$ in $G_{A,w}$.

At position $\langle p, i \rangle \in V_A$ in $G_{A,w}$:
- $\overline{f}_P$: winning strategy for player $P$ in $G_{A,w}$
- $M^* := \{ q \in Q \mid \exists S \in \text{Mod}_\downarrow(\overline{\delta}(p, a_i)) : \overline{f}_P(\langle S, i \rangle) = \langle q, i + 1 \rangle \}$
  \[(\text{sublemma})\]
  \[\implies M^* \text{ is a model of } \delta(p, a_i).\]
- $M$: subset of $M^*$ that is a minimal model
  $M \subseteq M^*$, $M \in \text{Mod}_\downarrow(\delta(p, a_i))$
- Define $f_A(\langle p, i \rangle) := \langle M, i \rangle$

- $\forall \gamma$: play in $G_{A,w}$ played according to $f_A$
- $\exists \overline{\gamma}$: play in $G_{A,w}$ played according to $\overline{f}_P$
  s.t. $\gamma$ and $\overline{\gamma}$ contain the same $V_A$-nodes.
  - Player $P$ wins $\overline{\gamma}$ in $G_{A,w}$.
  - $\forall q \in Q : \pi(q) = \overline{\pi}(q) - 1$
Construct a winning strategy $f_A$ for player $A$ in $G_{A,w}$.

At position $\langle p, i \rangle \in V_A$ in $G_{A,w}$:

- $f_P$: winning strategy for player $P$ in $G_{A,w}$

- $M^* := \{ q \in Q \mid \exists S \in \text{Mod}_\downarrow(\delta(p, a_i)) : f_P(\langle S, i \rangle) = \langle q, i + 1 \rangle \}$

  (sublemma) $M^*$ is a model of $\delta(p, a_i)$.

- $M$: subset of $M^*$ that is a minimal model $M \subseteq M^*$, $M \in \text{Mod}_\downarrow(\delta(p, a_i))$

- Define $f_A(\langle p, i \rangle) := \langle M, i \rangle$

- $\forall \gamma$: play in $G_{A,w}$ played according to $f_A$
- $\exists \overline{\gamma}$: play in $G_{A,w}$ played according to $f_P$

  s.t. $\gamma$ and $\overline{\gamma}$ contain the same $V_A$-nodes.

  - Player $P$ wins $\overline{\gamma}$ in $G_{A,w}$.
  - $\forall q \in Q : \pi(q) = \overline{\pi(q)} - 1$

  $\implies$ Player $A$ wins $\gamma$ in $G_{A,w}$.
Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w \in \Sigma^\omega$.

**Lemma 1**

Player $A$ has a winning strategy in $G_{A,w}$ iff $\mathcal{A}$ accepts $w$.

**Lemma 2**

Player $P$ has a winning strategy in $G_{A,w}$ iff $\mathcal{A}$ does not accept $w$.

**Lemma 3**

Player $A$ has a winning strategy in $G_{A,w}$ iff player $P$ has a winning strategy in $G_{A,w}$.
**Complementation Theorem**

**Theorem (Complementation)**

The dual $\bar{A}$ of a WAPA $A$ accepts its complement, i.e.

$$\mathcal{L}(\bar{A}) = \Sigma^\omega \setminus \mathcal{L}(A)$$

(Thomas and Löding, \sim 2000)
Theorem (Complementation)

The dual $\overline{A}$ of a WAPA $A$ accepts its complement, i.e.

$$\mathcal{L}(\overline{A}) = \Sigma^\omega \setminus \mathcal{L}(A)$$

(Thomas and Löding, ∼2000)

Proof:

$A$ accepts $w$ \iff (lemma 1) player $A$ has a winning strategy in $G_{A,w}$
Theorem (Complementation)

The dual $\overline{A}$ of a WAPA $A$ accepts its complement, i.e.

$$L(\overline{A}) = \Sigma^\omega \setminus L(A)$$

(Thomas and Löding, ~2000)

Proof:

$A$ accepts $w$ (lemma 1) $\iff$ player $A$ has a winning strategy in $G_{A,w}$

(lemma 3) $\iff$ player $P$ has a winning strategy in $G_{\overline{A},w}$
Complementation Theorem

Theorem (Complementation)

The dual $\overline{A}$ of a WAPA $A$ accepts its complement, i.e.

$$\mathcal{L}(\overline{A}) = \Sigma^\omega \setminus \mathcal{L}(A)$$

(Thomas and Löding, ~2000)

Proof:

$A$ accepts $w$ $\iff$ player $A$ has a winning strategy in $G_{A,w}$

(lemma 1)

$\iff$ player $P$ has a winning strategy in $G_{\overline{A},w}$

(lemma 3)

$\iff$ $\overline{A}$ does not accept $w$

(lemma 2)
1 Weak Alternating Parity Automata

2 Infinite Parity Games

3 Proof of the Complementation Theorem

4 Büchi Complementation Algorithm
B"uchi Complementation Algorithm

\[
\begin{align*}
B \xrightarrow{2^\Omega(n \log n)} \bar{B} \\
\bar{A} \xleftarrow{O(1)} \bar{A}
\end{align*}
\]

Total complexity: \(O(n^2)\)

Can reach \(O(n \log n)\) (lower bound) by improving \(A \rightarrow B\).
Büchi Complementation Algorithm

Total complexity: $2\Omega(n \log n)$. Can reach $2\Omega(n \log n)$ (lower bound) by improving $A \rightarrow B$. 
**Büchi Complementation Algorithm**

\[
\begin{align*}
\mathcal{B} & \quad \xrightarrow{2 \Omega(n \log n)} \quad \mathcal{B} \\
\mathcal{A} & \quad \xleftarrow{O(1)} \quad \mathcal{A}
\end{align*}
\]

Total complexity: \(2^{O(n)}\)
Büchi Complementation Algorithm

- Total complexity: $2^\Omega(n^2)$
**Büchi Complementation Algorithm**

- Total complexity: $2^{O(n^2)}$
- Can reach $2^{O(n \log n)}$ (lower bound) by improving $\overline{A} \rightarrow \overline{B}$. 

![Diagram showing Büchi Complementation Algorithm]


Appendix
From BA to WAPA

Given:

- $B = \langle Q, \Sigma, \delta, q_{in}, F \rangle$: BA
- $n = |Q|$
**From BA to WAPA**

**Given:**
- \( \mathcal{B} = \langle Q, \Sigma, \delta, q_{\text{in}}, F \rangle \): BA
- \( n = |Q| \)

**Construction (BA → WAPA)**

\[ \mathcal{A} := \langle Q \times \{0, \ldots, 2n\}, \Sigma, \delta', \langle q_{\text{in}}, 2n \rangle, \pi \rangle \]

\( \mathcal{O}(n^2) \)

(Thomas and Löding, ~2000)
From BA to WAPA

Given:
- $B = \langle Q, \Sigma, \delta, q_{in}, F \rangle$: BA
- $n = |Q|

Construction ($BA \rightarrow WAPA$)

$$A := \langle Q \times \{0, \ldots, 2n\}, \Sigma, \delta', \langle q_{in}, 2n \rangle, \pi \rangle$$

where

- $\pi(\langle p, i \rangle) := i$

for $p \in Q$, $a \in \Sigma$, $i \in \{0, \ldots, 2n\}$

(Thomas and Löding, ~2000)
**From BA to WAPA**

**Given:**
- $\mathcal{B} = \langle Q, \Sigma, \delta, q_{in}, F \rangle$: BA
- $n = |Q|$

**Construction ($\text{BA} \rightarrow \text{WAPA}$)**

$\mathcal{A} := \langle Q \times \{0, \ldots, 2n\}, \Sigma, \delta', \langle q_{in}, 2n \rangle, \pi \rangle$

where

\[ \delta'(\langle p, i \rangle, a) := \begin{cases} \bigvee_{q \in \delta(p, a)} \langle q, 0 \rangle & \text{if } i = 0 \\ \bigvee_{q \in \delta(p, a)} \langle q, i \rangle \land \langle q, i - 1 \rangle & \text{if } i \text{ even, } i > 0 \end{cases} \]

$\pi(\langle p, i \rangle) := i$

for $p \in Q$, $a \in \Sigma$, $i \in \{0, \ldots, 2n\}$

(Thomas and Löding, \sim 2000)
From BA to WAPA

Given:
- $\mathcal{B} = \langle Q, \Sigma, \delta, q_{\text{in}}, F \rangle$: BA
- $n = |Q|$

Construction ($\mathcal{BA} \to \mathcal{WAPA}$)

$\mathcal{A} := \langle Q \times \{0, \ldots, 2n\}, \Sigma, \delta', \langle q_{\text{in}}, 2n \rangle, \pi \rangle$

where

- $\delta'(\langle p, i \rangle, a) :=$
  \[
  \begin{cases}
    \bigvee_{q \in \delta(p, a)} \langle q, 0 \rangle & \text{if } i = 0 \\
    \bigvee_{q \in \delta(p, a)} \langle q, i \rangle \land \langle q, i - 1 \rangle & \text{if } i \text{ even}, i > 0 \\
    \bigvee_{q \in \delta(p, a)} \langle q, i \rangle & \text{if } i \text{ odd}, p \notin F \\
    \bigvee_{q \in \delta(p, a)} \langle q, i - 1 \rangle & \text{if } i \text{ odd}, p \in F
  \end{cases}
  \]

- $\pi(\langle p, i \rangle) := i$

for $p \in Q$, $a \in \Sigma$, $i \in \{0, \ldots, 2n\}$

(Thomas and Löding, ~2000)
Given:

\[ \mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle: \text{stratified WAPA, i.e.} \]

\[ \forall p \in Q \ \forall a \in \Sigma : \delta(p, a) \in B^+ \left( \{ q \in Q \mid \pi(p) \geq \pi(q) \} \right) \]
From WAPA to BA

**Given:**

- $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$: stratified WAPA, i.e.
  - $\forall p \in Q \ \forall a \in \Sigma : \delta(p, a) \in B^+ \{ q \in Q \mid \pi(p) \geq \pi(q) \}$
- $E \subseteq Q$: all states with even parity
From WAPA to BA

**Given:**

- $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$: stratified WAPA, i.e.
  \[ \forall p \in Q \ \forall a \in \Sigma : \delta(p, a) \in \mathbb{B}^+ \left( \{ q \in Q \mid \pi(p) \geq \pi(q) \} \right) \]
- $E \subseteq Q$: all states with even parity

**Construction (WAPA $\to$ BA)**

\[ \mathcal{B} := \langle 2^Q \times 2^Q, \Sigma, \delta', \langle \{ q_{in} \}, \emptyset \rangle, 2^Q \times \{ \emptyset \} \rangle \]

(Miyano and Hayashi, 1984)
From WAPA to BA

Given:
- \( \mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle \): stratified WAPA, i.e.
  \[
  \forall p \in Q \ \forall a \in \Sigma : \ \delta(p, a) \in \mathbb{B}^+(\{q \in Q \mid \pi(p) \geq \pi(q)\})
  \]
- \( E \subseteq Q \): all states with even parity

Construction (WAPA → BA)

\( \mathcal{B} := \langle 2^Q \times 2^Q, \Sigma, \delta', \langle \{q_{in}\}, \emptyset \rangle, 2^Q \times \{\emptyset\} \rangle \)

where
- \( \delta'(\langle M, \emptyset \rangle, a) := \left\{ \langle M', M' \setminus E \rangle \mid M' \in \text{Mod}_{\downarrow}(\land_{q \in M} \delta(q, a)) \right\} \)

for \( a \in \Sigma, \ M, O \subseteq Q, \ O \neq \emptyset \)

(Miyano and Hayashi, 1984)
FROM WAPA TO BA

Given:
- \( A = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle \): stratified WAPA, i.e.
  \[ \forall p \in Q \ \forall a \in \Sigma : \delta(p, a) \in B^+ \left( \{ q \in Q \mid \pi(p) \geq \pi(q) \} \right) \]
- \( E \subseteq Q \): all states with even parity

Construction (WAPA \( \rightarrow \) BA)

\[ B := \langle 2^Q \times 2^Q, \Sigma, \delta', \langle \{q_{in}\}, \emptyset \rangle, 2^Q \times \{\emptyset\} \rangle \]

where

- \( \delta'(\langle M, \emptyset \rangle, a) := \left\{ \langle M', M' \setminus E \rangle \mid M' \in \text{Mod}_{\downarrow}(\bigwedge_{q \in M} \delta(q, a)) \right\} \)
- \( \delta'(\langle M, O \rangle, a) := \left\{ \langle M', O' \setminus E \rangle \mid M' \in \text{Mod}_{\downarrow}(\bigwedge_{q \in M} \delta(q, a)), O' \subseteq M', O' \in \text{Mod}_{\downarrow}(\bigwedge_{q \in O} \delta(q, a)) \right\} \)

for \( a \in \Sigma, M, O \subseteq Q, O \neq \emptyset \)

(Miyano and Hayashi, 1984)