# Decision Procedures

Jochen Hoenicke

Software Engineering
Albert-Ludwigs-University Freiburg

Summer 2012

# Foundations: Propositional Logic

# Syntax of Propositional Logic

| | |
|---|---|
| Atom | truth symbols $\top$ ("true") and $\bot$ ("false") |
| | propositional variables $P, Q, R, P_1, Q_1, R_1, \cdots$ |
| Literal | atom $\alpha$ or its negation $\neg\alpha$ |
| Formula | literal or application of a |
| | logical connective to formulae $F, F_1, F_2$ |

| | | |
|---|---|---|
| $\neg F$ | "not" | (negation) |
| $(F_1 \land F_2)$ | "and" | (conjunction) |
| $(F_1 \lor F_2)$ | "or" | (disjunction) |
| $(F_1 \rightarrow F_2)$ | "implies" | (implication) |
| $(F_1 \leftrightarrow F_2)$ | "if and only if" | (iff) |

# Example: Syntax

formula $F$ : $((P \land Q) \to (\top \lor \neg Q))$
atoms: $P, Q, \top$
literal: $\neg Q$
subformulas: $(P \land Q)$, $(\top \lor \neg Q)$
abbreviation

$$F : P \land Q \to \top \lor \neg Q$$

# Semantics (meaning) of PL

Formula $F$ and Interpretation $I$ is evaluated to a truth value 0/1
where 0 corresponds to value false
1 true

Interpretation $I$ : $\{P \mapsto 1, Q \mapsto 0, \cdots\}$

Evaluation of logical operators:

| $F_1$ | $F_2$ | $\neg F_1$ | $F_1 \wedge F_2$ | $F_1 \vee F_2$ | $F_1 \rightarrow F_2$ | $F_1 \leftrightarrow F_2$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | | 1 | 1 | 1 | 1 |

$F : P \land Q \to P \lor \neg Q$
$I : \{P \mapsto 1, Q \mapsto 0\}$

| $P$ | $Q$ | $\neg Q$ | $P \land Q$ | $P \lor \neg Q$ | $F$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 1 |

$1 = \text{true}$ $\quad$ $0 = \text{false}$

# Example: Semantics

$F : P \land Q \rightarrow P \lor \neg Q$
$I : \{P \mapsto 1, Q \mapsto 0\}$

| $P$ | $Q$ | $\neg Q$ | $P \land Q$ | $P \lor \neg Q$ | $F$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 1 |

$1 =$ true $\qquad 0 =$ false

$F$ evaluates to true under $I$

# Inductive Definition of PL's Semantics

| | | | |
|---|---|---|---|
| $I \models F$ | if $F$ evaluates to | 1 / true | under $I$ |
| $I \not\models F$ | | 0 / false | |

# Inductive Definition of PL's Semantics

$$\begin{array}{lll} I \models F & \text{if } F \text{ evaluates to} & 1 \ / \ \text{true} \quad \text{under } I \\ I \not\models F & & 0 \ / \ \text{false} \end{array}$$

Base Case:

$$I \models \top$$

$$I \not\models \bot$$

$$I \models P \quad \text{iff} \quad I[P] = 1$$

$$I \not\models P \quad \text{iff} \quad I[P] = 0$$

# Inductive Definition of PL's Semantics

$I \models F$ if $F$ evaluates to 1 / true under $I$
$I \not\models F$ 0 / false

Base Case:

$$I \models \top$$
$$I \not\models \bot$$
$$I \models P \quad \text{iff} \quad I[P] = 1$$
$$I \not\models P \quad \text{iff} \quad I[P] = 0$$

Inductive Case:

$I \models \neg F$ iff $I \not\models F$
$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$
$I \models F_1 \rightarrow F_2$ iff, if $I \models F_1$ then $I \models F_2$
$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$, or $I \not\models F_1$ and $I \not\models F_2$

# Example: Inductive Reasoning

$$F : P \wedge Q \rightarrow P \vee \neg Q$$

$$I : \{P \mapsto 1,\ Q \mapsto 0\}$$

# Example: Inductive Reasoning

$$F : P \wedge Q \rightarrow P \vee \neg Q$$

$$I : \{P \mapsto 1,\ Q \mapsto 0\}$$

1. $I \models P$ since $I[P] = 1$

# Example: Inductive Reasoning

$$F : P \wedge Q \rightarrow P \vee \neg Q$$

$$I : \{P \mapsto 1, \; Q \mapsto 0\}$$

1. $I \models P$ since $I[P] = 1$
2. $I \not\models Q$ since $I[Q] = 0$

# Example: Inductive Reasoning

$$F : P \wedge Q \rightarrow P \vee \neg Q$$

$$I : \{P \mapsto 1,\ Q \mapsto 0\}$$

| | | |
|---|---|---|
| 1. | $I \models P$ | since $I[P] = 1$ |
| 2. | $I \not\models Q$ | since $I[Q] = 0$ |
| 3. | $I \models \neg Q$ | by 2, $\neg$ |

# Example: Inductive Reasoning

$$F \; : \; P \wedge Q \rightarrow P \vee \neg Q$$

$$I \; : \; \{P \;\mapsto\; 1, \; Q \;\mapsto\; 0\}$$

| | | |
|---|---|---|
| 1. | $I \models P$ | since $I[P] = 1$ |
| 2. | $I \not\models Q$ | since $I[Q] = 0$ |
| 3. | $I \models \neg Q$ | by 2, $\neg$ |
| 4. | $I \not\models P \wedge Q$ | by 2, $\wedge$ |

# Example: Inductive Reasoning

$$F \;:\; P \wedge Q \rightarrow P \vee \neg Q$$

$$I \;:\; \{P \;\mapsto\; 1,\; Q \;\mapsto\; 0\}$$

| | | |
|---|---|---|
| 1. | $I \models P$ | since $I[P] = 1$ |
| 2. | $I \not\models Q$ | since $I[Q] = 0$ |
| 3. | $I \models \neg Q$ | by 2, $\neg$ |
| 4. | $I \not\models P \wedge Q$ | by 2, $\wedge$ |
| 5. | $I \models P \vee \neg Q$ | by 1, $\vee$ |

$$F : P \land Q \rightarrow P \lor \neg Q$$

$$I : \{P \mapsto 1,\ Q \mapsto 0\}$$

| | | |
|---|---|---|
| 1. $I \models P$ | since $I[P] = 1$ | |
| 2. $I \not\models Q$ | since $I[Q] = 0$ | |
| 3. $I \models \neg Q$ | by 2, $\neg$ | |
| 4. $I \not\models P \land Q$ | by 2, $\land$ | |
| 5. $I \models P \lor \neg Q$ | by 1, $\lor$ | |
| 6. $I \models F$ | by 4, $\rightarrow$ | Why? |

# Example: Inductive Reasoning

$$F \, : \, P \land Q \to P \lor \neg Q$$

$$I \, : \, \{P \;\mapsto\; 1, \; Q \;\mapsto\; 0\}$$

| | | | |
|---|---|---|---|
| 1. | $I \models P$ | since $I[P] = 1$ | |
| 2. | $I \not\models Q$ | since $I[Q] = 0$ | |
| 3. | $I \models \neg Q$ | by 2, $\neg$ | |
| 4. | $I \not\models P \land Q$ | by 2, $\land$ | |
| 5. | $I \models P \lor \neg Q$ | by 1, $\lor$ | |
| 6. | $I \models F$ | by 4, $\to$ | Why? |

Thus, $F$ is true under $I$.

# Satisfiability and Validity

**Definition (Satisfiability)**

$F$ is satisfiable iff there exists an interpretation $I$ such that $I \models F$.

**Definition (Validity)**

$F$ is valid iff for all interpretations $I$, $I \models F$.

# Satisfiability and Validity

**Definition (Satisfiability)**

$F$ is satisfiable iff there exists an interpretation $I$ such that $I \models F$.

**Definition (Validity)**

$F$ is valid iff for all interpretations $I$, $I \models F$.

**Note**

$F$ is valid iff $\neg F$ is unsatisfiable

# Satisfiability and Validity

**Definition (Satisfiability)**

$F$ is satisfiable iff there exists an interpretation $I$ such that $I \models F$.

**Definition (Validity)**

$F$ is valid iff for all interpretations $I$, $I \models F$.

**Note**

$F$ is valid iff $\neg F$ is unsatisfiable

**Proof.**

$F$ is valid iff $\forall I : I \models F$ iff $\neg \exists I : I \not\models F$ iff $\neg F$ is unsatisfiable. □

# Satisfiability and Validity

## Definition (Satisfiability)

$F$ is satisfiable iff there exists an interpretation $I$ such that $I \models F$.

## Definition (Validity)

$F$ is valid iff for all interpretations $I$, $I \models F$.

## Note

$F$ is valid iff $\neg F$ is unsatisfiable

## Proof.

$F$ is valid iff $\forall I : I \models F$ iff $\neg \exists I : I \not\models F$ iff $\neg F$ is unsatisfiable. □

Decision Procedure: An algorithm for deciding validity or satisfiability.

Now assume, you are a decision procedure.

Which of the following formulae is satisfiable, which is valid?

- $F_1 : P \land Q$
- $F_2 : \neg(P \land Q)$
- $F_3 : P \lor \neg P$
- $F_4 : \neg(P \lor \neg P)$
- $F_5 : (P \to Q) \land (P \lor Q) \land \neg Q$

# Examples: Satisfiability and Validity

Now assume, you are a decision procedure.

Which of the following formulae is satisfiable, which is valid?

- $F_1 \ : \ P \wedge Q$
  satisfiable, not valid
- $F_2 \ : \ \neg(P \wedge Q)$
- $F_3 \ : \ P \vee \neg P$
- $F_4 \ : \ \neg(P \vee \neg P)$
- $F_5 \ : \ (P \rightarrow Q) \wedge (P \vee Q) \wedge \neg Q$

# Examples: Satisfiability and Validity

Now assume, you are a decision procedure.

Which of the following formulae is satisfiable, which is valid?

- $F_1 : P \land Q$
  satisfiable, not valid
- $F_2 : \neg(P \land Q)$
  satisfiable, not valid
- $F_3 : P \lor \neg P$
- $F_4 : \neg(P \lor \neg P)$
- $F_5 : (P \rightarrow Q) \land (P \lor Q) \land \neg Q$

# Examples: Satisfiability and Validity

Now assume, you are a decision procedure.

Which of the following formulae is satisfiable, which is valid?

- $F_1 : P \land Q$
  satisfiable, not valid
- $F_2 : \neg(P \land Q)$
  satisfiable, not valid
- $F_3 : P \lor \neg P$
  satisfiable, valid
- $F_4 : \neg(P \lor \neg P)$
- $F_5 : (P \rightarrow Q) \land (P \lor Q) \land \neg Q$

Now assume, you are a decision procedure.

Which of the following formulae is satisfiable, which is valid?

- $F_1 : P \wedge Q$
  satisfiable, not valid
- $F_2 : \neg(P \wedge Q)$
  satisfiable, not valid
- $F_3 : P \vee \neg P$
  satisfiable, valid
- $F_4 : \neg(P \vee \neg P)$
  unsatisfiable, not valid
- $F_5 : (P \rightarrow Q) \wedge (P \vee Q) \wedge \neg Q$

# Examples: Satisfiability and Validity

Now assume, you are a decision procedure.

Which of the following formulae is satisfiable, which is valid?

- $F_1 : P \land Q$
  satisfiable, not valid
- $F_2 : \neg(P \land Q)$
  satisfiable, not valid
- $F_3 : P \lor \neg P$
  satisfiable, valid
- $F_4 : \neg(P \lor \neg P)$
  unsatisfiable, not valid
- $F_5 : (P \rightarrow Q) \land (P \lor Q) \land \neg Q$
  unsatisfiable, not valid

Now assume, you are a decision procedure.

Which of the following formulae is satisfiable, which is valid?

- $F_1 : P \wedge Q$
  satisfiable, not valid
- $F_2 : \neg(P \wedge Q)$
  satisfiable, not valid
- $F_3 : P \vee \neg P$
  satisfiable, valid
- $F_4 : \neg(P \vee \neg P)$
  unsatisfiable, not valid
- $F_5 : (P \rightarrow Q) \wedge (P \vee Q) \wedge \neg Q$
  unsatisfiable, not valid

Is there a formula that is unsatisfiable and valid?

# Method 1: Truth Tables

$F : P \land Q \rightarrow P \lor \neg Q$

| $P$ | $Q$ | $P \land Q$ | $\neg Q$ | $P \lor \neg Q$ | $F$ |
|---|---|---|---|---|---|
| 0 | 0 | | | | |
| 0 | 1 | | | | |
| 1 | 0 | | | | |
| 1 | 1 | | | | |

$F : P \wedge Q \rightarrow P \vee \neg Q$

| $P$ $Q$ | $P \wedge Q$ | $\neg Q$ | $P \vee \neg Q$ | $F$ |
|---|---|---|---|---|
| 0 0 | 0 | 1 | 1 | 1 |
| 0 1 | 0 | 0 | 0 | 1 |
| 1 0 | 0 | 1 | 1 | 1 |
| 1 1 | 1 | 0 | 1 | 1 |

Thus $F$ is valid.

$F : P \wedge Q \rightarrow P \vee \neg Q$

| $P$ | $Q$ | $P \wedge Q$ | $\neg Q$ | $P \vee \neg Q$ | $F$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 |

Thus $F$ is valid.

$F : P \vee Q \rightarrow P \wedge Q$

| $P$ | $Q$ | $P \vee Q$ | $P \wedge Q$ | $F$ |
|---|---|---|---|---|
| 0 | 0 | | | |
| 0 | 1 | | | |
| 1 | 0 | | | |
| 1 | 1 | | | |

$F : P \land Q \rightarrow P \lor \neg Q$

| $P$ | $Q$ | $P \land Q$ | $\neg Q$ | $P \lor \neg Q$ | $F$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 |

Thus $F$ is valid.

$F : P \lor Q \rightarrow P \land Q$

| $P$ | $Q$ | $P \lor Q$ | $P \land Q$ | $F$ | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | $\leftarrow$ satisfying $I$ |
| 0 | 1 | 1 | 0 | 0 | $\leftarrow$ falsifying $I$ |
| 1 | 0 | 1 | 0 | 0 | |
| 1 | 1 | 1 | 1 | 1 | |

Thus $F$ is satisfiable, but invalid.

# Method 2: Semantic Argument (Semantic Tableaux)

- Assume $F$ is not valid and $I$ a falsifying interpretation: $I \not\models F$
- Apply proof rules.
- If no contradiction reached and no more rules applicable, $F$ is invalid.
- If in every branch of proof a contradiction reached, $F$ is valid.

# Semantic Argument: Proof rules

$$\frac{I \models \neg F}{I \not\models F} \qquad \frac{I \not\models \neg F}{I \models F}$$

$$\frac{I \models F \land G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \leftarrow \text{and} \qquad \frac{I \not\models F \land G}{I \not\models F \quad | \quad I \not\models G} \quad (| \text{ — or})$$

$$\frac{I \models F \lor G}{I \models F \quad | \quad I \models G} \qquad \frac{I \not\models F \lor G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \rightarrow G}{I \not\models F \quad | \quad I \models G} \qquad \frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{I \models F \land G \quad | \quad I \not\models F \lor G} \qquad \frac{I \not\models F \leftrightarrow G}{I \models F \land \neg G \quad | \quad I \models \neg F \land G}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \bot}$$

Prove $F : P \wedge Q \rightarrow P \vee \neg Q$ is valid.

Let's assume that $F$ is not valid and that $I$ is a falsifying interpretation.

Prove $F : P \wedge Q \rightarrow P \vee \neg Q$ is valid.

Let's assume that $F$ is not valid and that $I$ is a falsifying interpretation.

1. $I \not\models P \wedge Q \rightarrow P \vee \neg Q$ assumption

# Example

Prove $F : P \wedge Q \rightarrow P \vee \neg Q$ is valid.

Let's assume that $F$ is not valid and that $I$ is a falsifying interpretation.

| | | |
|---|---|---|
| 1. | $I \not\models P \wedge Q \rightarrow P \vee \neg Q$ | assumption |
| 2. | $I \models P \wedge Q$ | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \vee \neg Q$ | 1, Rule $\rightarrow$ |

# Example

Prove $F : P \wedge Q \rightarrow P \vee \neg Q$ is valid.

Let's assume that $F$ is not valid and that $I$ is a falsifying interpretation.

| | | |
|---|---|---|
| 1. | $I \not\models P \wedge Q \rightarrow P \vee \neg Q$ | assumption |
| 2. | $I \models P \wedge Q$ | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \vee \neg Q$ | 1, Rule $\rightarrow$ |
| 4. | $I \models P$ | 2, Rule $\wedge$ |

# Example

Prove $F : P \land Q \to P \lor \neg Q$ is valid.

Let's assume that $F$ is not valid and that $I$ is a falsifying interpretation.

| | | |
|---|---|---|
| 1. | $I \not\models P \land Q \to P \lor \neg Q$ | assumption |
| 2. | $I \models P \land Q$ | 1, Rule $\to$ |
| 3. | $I \not\models P \lor \neg Q$ | 1, Rule $\to$ |
| 4. | $I \models P$ | 2, Rule $\land$ |
| 5. | $I \not\models P$ | 3, Rule $\lor$ |

# Example

Prove $F : P \wedge Q \rightarrow P \vee \neg Q$ is valid.

Let's assume that $F$ is not valid and that $I$ is a falsifying interpretation.

| | | |
|---|---|---|
| 1. | $I \not\models P \wedge Q \rightarrow P \vee \neg Q$ | assumption |
| 2. | $I \models P \wedge Q$ | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \vee \neg Q$ | 1, Rule $\rightarrow$ |
| 4. | $I \models P$ | 2, Rule $\wedge$ |
| 5. | $I \not\models P$ | 3, Rule $\vee$ |
| 6. | $I \models \bot$ | 4 and 5 are contradictory |

# Example

Prove $F : P \land Q \rightarrow P \lor \neg Q$ is valid.

Let's assume that $F$ is not valid and that $I$ is a falsifying interpretation.

| | | |
|---|---|---|
| 1. | $I \not\models P \land Q \rightarrow P \lor \neg Q$ | assumption |
| 2. | $I \models P \land Q$ | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \lor \neg Q$ | 1, Rule $\rightarrow$ |
| 4. | $I \models P$ | 2, Rule $\land$ |
| 5. | $I \not\models P$ | 3, Rule $\lor$ |
| 6. | $I \models \bot$ | 4 and 5 are contradictory |

Thus $F$ is valid.

# Example 2

Prove $F : (P \to Q) \land (Q \to R) \to (P \to R)$ is valid.

Let's assume that $F$ is not valid.

# Example 2

Prove $F : (P \to Q) \land (Q \to R) \to (P \to R)$ is valid.

Let's assume that $F$ is not valid.

1. $I \not\models F$ assumption

# Example 2

Prove $F : (P \to Q) \land (Q \to R) \to (P \to R)$ is valid.

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \to Q) \land (Q \to R)$ | 1, Rule $\to$ |
| 3. | $I \not\models P \to R$ | 1, Rule $\to$ |

# Example 2

Prove $F : (P \to Q) \land (Q \to R) \to (P \to R)$ is valid.

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \to Q) \land (Q \to R)$ | 1, Rule $\to$ |
| 3. | $I \not\models P \to R$ | 1, Rule $\to$ |
| 4. | $I \models P$ | 3, Rule $\to$ |
| 5. | $I \not\models R$ | 3, Rule $\to$ |

# Example 2

Prove $F : (P \to Q) \wedge (Q \to R) \to (P \to R)$ is valid.

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \to Q) \wedge (Q \to R)$ | 1, Rule $\to$ |
| 3. | $I \not\models P \to R$ | 1, Rule $\to$ |
| 4. | $I \models P$ | 3, Rule $\to$ |
| 5. | $I \not\models R$ | 3, Rule $\to$ |
| 6. | $I \models P \to Q$ | 2, Rule $\wedge$ |
| 7. | $I \models Q \to R$ | 2, Rule $\wedge$ |

# Example 2

Prove $F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid.

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$ | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \rightarrow R$ | 1, Rule $\rightarrow$ |
| 4. | $I \models P$ | 3, Rule $\rightarrow$ |
| 5. | $I \not\models R$ | 3, Rule $\rightarrow$ |
| 6. | $I \models P \rightarrow Q$ | 2, Rule $\wedge$ |
| 7. | $I \models Q \rightarrow R$ | 2, Rule $\wedge$ |

$8a.\quad I \not\models P$ | $8b.\quad I \models Q \quad 6 \rightarrow$

# Example 2

Prove $F : (P \rightarrow Q) \land (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid.

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \rightarrow Q) \land (Q \rightarrow R)$ | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \rightarrow R$ | 1, Rule $\rightarrow$ |
| 4. | $I \models P$ | 3, Rule $\rightarrow$ |
| 5. | $I \not\models R$ | 3, Rule $\rightarrow$ |
| 6. | $I \models P \rightarrow Q$ | 2, Rule $\land$ |
| 7. | $I \models Q \rightarrow R$ | 2, Rule $\land$ |

| | | | | |
|---|---|---|---|---|
| $8a.$ $I \not\models P$ | | $8b.$ $I \models Q$ | $6 \rightarrow$ |
| $9a.$ $I \models \bot$ | | | |

Prove $F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid.

Let's assume that $F$ is not valid.

| | | | |
|---|---|---|---|
| 1. | $I \not\models F$ | | assumption |
| 2. | $I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$ | | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \rightarrow R$ | | 1, Rule $\rightarrow$ |
| 4. | $I \models P$ | | 3, Rule $\rightarrow$ |
| 5. | $I \not\models R$ | | 3, Rule $\rightarrow$ |
| 6. | $I \models P \rightarrow Q$ | | 2, Rule $\wedge$ |
| 7. | $I \models Q \rightarrow R$ | | 2, Rule $\wedge$ |

$8a.\quad I \not\models P$ | $8b.\quad I \models Q \qquad 6 \rightarrow$

$9a.\quad I \models \bot$ | $9ba.\quad I \not\models Q$ | $9bb.\quad I \models R$

# Example 2

Prove $F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid.

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$ | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \rightarrow R$ | 1, Rule $\rightarrow$ |
| 4. | $I \models P$ | 3, Rule $\rightarrow$ |
| 5. | $I \not\models R$ | 3, Rule $\rightarrow$ |
| 6. | $I \models P \rightarrow Q$ | 2, Rule $\wedge$ |
| 7. | $I \models Q \rightarrow R$ | 2, Rule $\wedge$ |

| | | |
|---|---|---|
| $8a.\quad I \not\models P$ | $8b.\quad I \models Q \qquad 6 \rightarrow$ | |
| $9a.\quad I \models \bot$ | $9ba.\quad I \not\models Q$ | $9bb.\quad I \models R$ |
| | $10ba.\quad I \models \bot$ | |

# Example 2

Prove $F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid.

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$ | 1, Rule $\rightarrow$ |
| 3. | $I \not\models P \rightarrow R$ | 1, Rule $\rightarrow$ |
| 4. | $I \models P$ | 3, Rule $\rightarrow$ |
| 5. | $I \not\models R$ | 3, Rule $\rightarrow$ |
| 6. | $I \models P \rightarrow Q$ | 2, Rule $\wedge$ |
| 7. | $I \models Q \rightarrow R$ | 2, Rule $\wedge$ |

| Branch a | Branch b | |
|---|---|---|
| $8a.$ $I \not\models P$ | $8b.$ $I \models Q$   $6 \rightarrow$ | |
| $9a.$ $I \models \bot$ | $9ba.$ $I \not\models Q$ | $9bb.$ $I \models R$ |
| | $10ba.$ $I \models \bot$ | $10bb.$ $I \models \bot$ |

# Example 2

Prove $F : (P \to Q) \wedge (Q \to R) \to (P \to R)$ is valid.

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \to Q) \wedge (Q \to R)$ | 1, Rule $\to$ |
| 3. | $I \not\models P \to R$ | 1, Rule $\to$ |
| 4. | $I \models P$ | 3, Rule $\to$ |
| 5. | $I \not\models R$ | 3, Rule $\to$ |
| 6. | $I \models P \to Q$ | 2, Rule $\wedge$ |
| 7. | $I \models Q \to R$ | 2, Rule $\wedge$ |

| | |
|---|---|
| $8a.\quad I \not\models P$ | $8b.\quad I \models Q \quad 6 \to$ |
| $9a.\quad I \models \bot$ | $9ba.\quad I \not\models Q$ \| $9bb.\quad I \models R$ |
| | $10ba.\quad I \models \bot$ \| $10bb.\quad I \models \bot$ |

Our assumption is incorrect in all cases — $F$ is valid.

# Example 3

Is $\quad F : P \vee Q \rightarrow P \wedge Q \quad$ valid?

Let's assume that $F$ is not valid.

# Example 3

Is $F : P \vee Q \rightarrow P \wedge Q$ valid?

Let's assume that $F$ is not valid.

1. $I \not\models P \vee Q \rightarrow P \wedge Q$ assumption

# Example 3

Is $F : P \vee Q \rightarrow P \wedge Q$ valid?

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models P \vee Q \rightarrow P \wedge Q$ | assumption |
| 2. | $I \models P \vee Q$ | 1 and $\rightarrow$ |
| 3. | $I \not\models P \wedge Q$ | 1 and $\rightarrow$ |

# Example 3

Is $F : P \vee Q \rightarrow P \wedge Q$ valid?

Let's assume that $F$ is not valid.

$$
\begin{array}{lll}
1. & I \not\models P \vee Q \rightarrow P \wedge Q & \text{assumption} \\
2. & I \models P \vee Q & 1 \text{ and } \rightarrow \\
3. & I \not\models P \wedge Q & 1 \text{ and } \rightarrow \\
4a. & I \models P & 2 \text{ and } \vee
\end{array}
$$

# Example 3

Is $F : P \vee Q \rightarrow P \wedge Q$ valid?

Let's assume that $F$ is not valid.

$$
\begin{array}{lll}
1. & I \not\models P \vee Q \rightarrow P \wedge Q & \text{assumption} \\
2. & I \models P \vee Q & 1 \text{ and } \rightarrow \\
3. & I \not\models P \wedge Q & 1 \text{ and } \rightarrow
\end{array}
$$

$4a.\quad I \models P \qquad 2 \text{ and } \vee$

$$
\begin{array}{l|l}
5aa.\ I \not\models P & 5ab.\ I \not\models Q \\
6aa.\ I \models \bot &
\end{array}
$$

# Example 3

Is $F : P \vee Q \rightarrow P \wedge Q$ valid?

Let's assume that $F$ is not valid.

$$
\begin{array}{lll}
1. & I \not\models P \vee Q \rightarrow P \wedge Q & \text{assumption} \\
2. & I \models P \vee Q & 1 \text{ and } \rightarrow \\
3. & I \not\models P \wedge Q & 1 \text{ and } \rightarrow
\end{array}
$$

$$
\begin{array}{l|l}
4a. \quad I \models P \qquad 2 \text{ and } \vee & 4b. \quad I \models Q \qquad 2 \text{ and } \vee \\
\begin{array}{l|l}
5aa. \; I \not\models P & 5ab. \; I \not\models Q \\
6aa. \; I \models \bot &
\end{array} &
\end{array}
$$

# Example 3

Is $\quad F : P \lor Q \rightarrow P \land Q \quad$ valid?

Let's assume that $F$ is not valid.

$$
\begin{array}{lll}
1. & I \not\models P \lor Q \rightarrow P \land Q & \text{assumption} \\
2. & I \models P \lor Q & 1 \text{ and } \rightarrow \\
3. & I \not\models P \land Q & 1 \text{ and } \rightarrow
\end{array}
$$

$$
\begin{array}{ll|ll}
4a. \; I \models P & 2 \text{ and } \lor & 4b. \; I \models Q & 2 \text{ and } \lor \\
\begin{array}{l} 5aa. \; I \not\models P \\ 6aa. \; I \models \bot \end{array} & \begin{array}{|l} 5ab. \; I \not\models Q \\ \; \end{array} & \begin{array}{l} 5ba. \; I \not\models P \\ \; \end{array} & \begin{array}{|l} 5bb. \; I \not\models Q \\ 6bb. \; I \models \bot \end{array}
\end{array}
$$

# Example 3

Is $\quad F : P \lor Q \rightarrow P \land Q \quad$ valid?

Let's assume that $F$ is not valid.

$$
\begin{array}{lll}
1. & I \not\models P \lor Q \rightarrow P \land Q & \text{assumption} \\
2. & I \models P \lor Q & 1 \text{ and } \rightarrow \\
3. & I \not\models P \land Q & 1 \text{ and } \rightarrow
\end{array}
$$

$$
\begin{array}{c|c}
4a. \quad I \models P \qquad 2 \text{ and } \lor & 4b. \quad I \models Q \qquad 2 \text{ and } \lor \\
\begin{array}{l|l} 5aa. \; I \not\models P & 5ab. \; I \not\models Q \\ 6aa. \; I \models \bot & \end{array} & \begin{array}{l|l} 5ba. \; I \not\models P & 5bb. \; I \not\models Q \\ & 6bb. \; I \models \bot \end{array}
\end{array}
$$

We cannot always derive a contradiction. $F$ is not valid.

Is $F : P \vee Q \rightarrow P \wedge Q$ valid?

Let's assume that $F$ is not valid.

$$
\begin{array}{lll}
1. & I \not\models P \vee Q \rightarrow P \wedge Q & \text{assumption} \\
2. & I \models P \vee Q & 1 \text{ and } \rightarrow \\
3. & I \not\models P \wedge Q & 1 \text{ and } \rightarrow
\end{array}
$$

$$
\begin{array}{c|c}
4a.\ I \models P \quad 2 \text{ and } \vee & 4b.\ I \models Q \quad 2 \text{ and } \vee \\
\begin{array}{l|l} 5aa.\ I \not\models P & 5ab.\ I \not\models Q \\ 6aa.\ I \models \bot & \end{array} & \begin{array}{l|l} 5ba.\ I \not\models P & 5bb.\ I \not\models Q \\ & 6bb.\ I \models \bot \end{array}
\end{array}
$$

We cannot always derive a contradiction. $F$ is not valid.

Falsifying interpretation:

$I_1 : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ $\quad$ $I_2 : \{Q \mapsto \text{true}, P \mapsto \text{false}\}$

# Example 3

Is $F : P \vee Q \rightarrow P \wedge Q$ valid?

Let's assume that $F$ is not valid.

$$
\begin{array}{lll}
1. & I \not\models P \vee Q \rightarrow P \wedge Q & \text{assumption} \\
2. & I \models P \vee Q & 1 \text{ and } \rightarrow \\
3. & I \not\models P \wedge Q & 1 \text{ and } \rightarrow
\end{array}
$$

$$
\begin{array}{c|c}
4a.\ I \models P \quad 2 \text{ and } \vee & 4b.\ I \models Q \quad 2 \text{ and } \vee \\
\begin{array}{l|l} 5aa.\ I \not\models P & 5ab.\ I \not\models Q \\ 6aa.\ I \models \bot & \end{array} & \begin{array}{l|l} 5ba.\ I \not\models P & 5bb.\ I \not\models Q \\ & 6bb.\ I \models \bot \end{array}
\end{array}
$$

We cannot always derive a contradiction. $F$ is not valid.

Falsifying interpretation:

$I_1 : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ $\quad$ $I_2 : \{Q \mapsto \text{true}, P \mapsto \text{false}\}$

We have to derive a contradiction in all cases for $F$ to be valid.

# Normal Forms

Idea: Simplify decision procedure, by simplifying the formula first.
Convert it into a simpler normal form, e.g.:

- Negation Normal Form: No $\rightarrow$ and no $\leftrightarrow$; negation only before atoms.
- Conjunctive Normal Form: Negation normal form, where conjunction is outside, disjunction is inside.
- Disjunctive Normal Form: Negation normal form, where disjunction is outside, conjunction is inside.

The formula in normal form should be equivalent to the original input.

# Equivalence

$F_1$ and $F_2$ are equivalent ($F_1 \Leftrightarrow F_2$)
iff for all interpretations $I$, $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$ show $F_1 \leftrightarrow F_2$ is valid.

$F_1$ implies $F_2$ ($F_1 \Rightarrow F_2$)
iff for all interpretations $I$, $I \models F_1 \rightarrow F_2$

$F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!

# Equivalence is a Congruence relation

If $F_1 \Leftrightarrow F_1'$ and $F_2 \Leftrightarrow F_2'$, then

- $\neg F_1 \Leftrightarrow \neg F_1'$
- $F_1 \vee F_2 \Leftrightarrow F_1' \vee F_2'$
- $F_1 \wedge F_2 \Leftrightarrow F_1' \wedge F_2'$
- $F_1 \rightarrow F_2 \Leftrightarrow F_1' \rightarrow F_2'$
- $F_1 \leftrightarrow F_2 \Leftrightarrow F_1' \leftrightarrow F_2'$

- if we replace in a formula $F$ a subformula $F_1$ by $F_1'$ and obtain $F'$, then $F \Leftrightarrow F'$.

Negations appear only in literals. (only $\neg, \wedge, \vee$)

To transform $F$ to equivalent $F'$ in NNF use recursively the following template equivalences (left-to-right):

$$\neg\neg F_1 \Leftrightarrow F_1 \qquad \neg\top \Leftrightarrow \bot \qquad \neg\bot \Leftrightarrow \top$$

$$\left.\begin{array}{l}\neg(F_1 \wedge F_2) \Leftrightarrow \neg F_1 \vee \neg F_2 \\ \neg(F_1 \vee F_2) \Leftrightarrow \neg F_1 \wedge \neg F_2\end{array}\right\}\text{De Morgan's Law}$$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$$

$$F_1 \leftrightarrow F_2 \Leftrightarrow (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$$

# Example: Negation Normal Form

Convert $F : (Q_1 \lor \neg\neg R_1) \land (\neg Q_2 \rightarrow R_2)$ into NNF

# Example: Negation Normal Form

Convert $F : (Q_1 \lor \neg\neg R_1) \land (\neg Q_2 \rightarrow R_2)$ into NNF

$$
\begin{array}{rl}
 & (Q_1 \lor \neg\neg R_1) \land (\neg Q_2 \rightarrow R_2) \\
\Leftrightarrow & (Q_1 \lor R_1) \land (\neg Q_2 \rightarrow R_2) \\
\Leftrightarrow & (Q_1 \lor R_1) \land (\neg\neg Q_2 \lor R_2) \\
\Leftrightarrow & (Q_1 \lor R_1) \land (Q_2 \lor R_2)
\end{array}
$$

The last formula is equivalent to $F$ and is in NNF.

# Disjunctive Normal Form (DNF)

Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert $F$ into equivalent $F'$ in DNF,
transform $F$ into NNF and then
use the following template equivalences (left-to-right):

$$\left.\begin{array}{l} (F_1 \vee F_2) \wedge F_3 \Leftrightarrow (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) \Leftrightarrow (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \end{array}\right\} \mathit{dist}$$

# Example

Convert $F : (Q_1 \vee \neg\neg R_1) \wedge (\neg Q_2 \rightarrow R_2)$ into DNF

# Example

Convert $F : (Q_1 \lor \neg\neg R_1) \land (\neg Q_2 \to R_2)$ into DNF

$$\begin{array}{lll} & (Q_1 \lor \neg\neg R_1) \land (\neg Q_2 \to R_2) & \\ \Leftrightarrow & (Q_1 \lor R_1) \land (Q_2 \lor R_2) & \text{in NNF} \\ \Leftrightarrow & (Q_1 \land (Q_2 \lor R_2)) \lor (R_1 \land (Q_2 \lor R_2)) & \text{dist} \\ \Leftrightarrow & (Q_1 \land Q_2) \lor (Q_1 \land R_2) \lor (R_1 \land Q_2) \lor (R_1 \land R_2) & \text{dist} \end{array}$$

The last formula is equivalent to $F$ and is in DNF. Note that formulas can grow exponentially.

# Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert $F$ into equivalent $F'$ in CNF,
transform $F$ into NNF and then
use the following template equivalences (left-to-right):

$$(F_1 \wedge F_2) \vee F_3 \Leftrightarrow (F_1 \vee F_3) \wedge (F_2 \vee F_3)$$
$$F_1 \vee (F_2 \wedge F_3) \Leftrightarrow (F_1 \vee F_2) \wedge (F_1 \vee F_3)$$

# Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert $F$ into equivalent $F'$ in CNF,
transform $F$ into NNF and then
use the following template equivalences (left-to-right):

$$(F_1 \wedge F_2) \vee F_3 \Leftrightarrow (F_1 \vee F_3) \wedge (F_2 \vee F_3)$$
$$F_1 \vee (F_2 \wedge F_3) \Leftrightarrow (F_1 \vee F_2) \wedge (F_1 \vee F_3)$$

A disjunction of literals $P_1 \vee P_2 \vee \neg P_3$ is called a clause.

# Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert $F$ into equivalent $F'$ in CNF,
transform $F$ into NNF and then
use the following template equivalences (left-to-right):

$$\begin{aligned}(F_1 \wedge F_2) \vee F_3 &\Leftrightarrow (F_1 \vee F_3) \wedge (F_2 \vee F_3)\\ F_1 \vee (F_2 \wedge F_3) &\Leftrightarrow (F_1 \vee F_2) \wedge (F_1 \vee F_3)\end{aligned}$$

A disjunction of literals $P_1 \vee P_2 \vee \neg P_3$ is called a clause.
For brevity we write it as set: $\{P_1, P_2, \overline{P_3}\}$.
A formula in CNF is a set of clauses (a set of sets of literals).

# Equisatisfiability

UNI FREIBURG

**Definition (Equisatisfiability)**

$F$ and $F'$ are equisatisfiable, iff

$$F \text{ is satisfiable if and only if } F' \text{ is satisfiable}$$

# Equisatisfiability

**Definition (Equisatisfiability)**

$F$ and $F'$ are equisatisfiable, iff

$$F \text{ is satisfiable if and only if } F' \text{ is satisfiable}$$

Every formula is equisatifiable to either $\top$ or $\bot$.

# Equisatisfiability

**Definition (Equisatisfiability)**

$F$ and $F'$ are equisatisfiable, iff

$$F \text{ is satisfiable if and only if } F' \text{ is satisfiable}$$

Every formula is equisatifiable to either $\top$ or $\bot$.
There is a efficient conversion of $F$ to $F'$ where

- $F'$ is in CNF and
- $F$ and $F'$ are equisatisfiable

**Definition (Equisatisfiability)**

$F$ and $F'$ are equisatisfiable, iff

$$F \text{ is satisfiable if and only if } F' \text{ is satisfiable}$$

Every formula is equisatifiable to either $\top$ or $\bot$.

There is a efficient conversion of $F$ to $F'$ where

- $F'$ is in CNF and
- $F$ and $F'$ are equisatisfiable

Note: efficient means polynomial in the size of $F$.

# Conversion to CNF

Basic Idea:

- Introduce a new variable $P_G$ for every subformula $G$; unless $G$ is already an atom.

# Conversion to CNF

Basic Idea:

- Introduce a new variable $P_G$ for every subformula $G$; unless $G$ is already an atom.
- For each subformula $G : G_1 \circ G_2$ produce a small formula $P_G \leftrightarrow P_{G_1} \circ P_{G_2}$.

# Conversion to CNF

Basic Idea:

- Introduce a new variable $P_G$ for every subformula $G$; unless $G$ is already an atom.
- For each subformula $G : G_1 \circ G_2$ produce a small formula $P_G \leftrightarrow P_{G_1} \circ P_{G_2}$.
- encode each of these (small) formulae separately to CNF.

# Conversion to CNF

Basic Idea:

- Introduce a new variable $P_G$ for every subformula $G$; unless $G$ is already an atom.
- For each subformula $G : G_1 \circ G_2$ produce a small formula $P_G \leftrightarrow P_{G_1} \circ P_{G_2}$.
- encode each of these (small) formulae separately to CNF.

The formula

$$P_F \wedge \bigwedge_G CNF(P_G \leftrightarrow P_{G_1} \circ P_{G_2})$$

is equisatisfiable to $F$.

# Conversion to CNF

Basic Idea:

- Introduce a new variable $P_G$ for every subformula $G$; unless $G$ is already an atom.
- For each subformula $G : G_1 \circ G_2$ produce a small formula $P_G \leftrightarrow P_{G_1} \circ P_{G_2}$.
- encode each of these (small) formulae separately to CNF.

The formula

$$P_F \wedge \bigwedge_G CNF(P_G \leftrightarrow P_{G_1} \circ P_{G_2})$$

is equisatisfiable to $F$.
The number of subformulae is linear in the size of $F$.
The time to convert one small formula is constant!

# Example: CNF

Convert $F : P \vee Q \rightarrow P \wedge \neg R$ to CNF.

# Example: CNF

Convert $F : P \lor Q \to P \land \neg R$ to CNF.

Introduce new variables: $P_F$, $P_{P \lor Q}$, $P_{P \land \neg R}$, $P_{\neg R}$.

# Example: CNF

Convert $F : P \lor Q \to P \land \neg R$ to CNF.
Introduce new variables: $P_F$, $P_{P\lor Q}$, $P_{P\land\neg R}$, $P_{\neg R}$. Create new formulae and convert them to CNF separately:

- $P_F \leftrightarrow (P_{P\lor Q} \to P_{P\land\neg R})$ in CNF:

$$F_1 : \{\{\overline{P_F}, \overline{P_{P\lor Q}}, P_{P\land\neg R}\}, \{P_F, P_{P\lor Q}\}, \{P_F, \overline{P_{P\land\neg R}}\}\}$$

- $P_{P\lor Q} \leftrightarrow P \lor Q$ in CNF:

$$F_2 : \{\{\overline{P_{P\lor Q}}, P \lor Q\}, \{P_{P\lor Q}, \overline{P}\}, \{P_{P\lor Q}, \overline{Q}\}\}$$

- $P_{P\land\neg R} \leftrightarrow P \land P_{\neg R}$ in CNF:

$$F_3 : \{\{\overline{P_{P\land\neg R}} \lor P\}, \{\overline{P_{P\land\neg R}}, P_{\neg R}\}, \{P_{P\land\neg R}, \overline{P}, \overline{P_{\neg R}}\}\}$$

- $P_{\neg R} \leftrightarrow \neg R$ in CNF: $F_4 : \{\{\overline{P_{\neg R}}, \overline{R}\}, \{P_{\neg R}, R\}\}$

$\{\{P_F\}\} \cup F_1 \cup F_2 \cup F_3 \cup F_4$ is in CNF and equisatisfiable to $F$.

# Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

- Algorithm to decide PL formulae in CNF.
- Published by Davis, Logemann, Loveland (1962).
- Often miscited as Davis, Putnam (1960), which describes a different algorithm.

Decides the satisfiability of PL formulae in CNF

Decision Procedure DPLL: Given $F$ in CNF

```
let rec DPLL F =
  let F' = PROP F in
  let F'' = PLP F' in
  if F'' = ⊤ then true
  else if F'' = ⊥ then false
  else
     let P = CHOOSE vars(F'') in
     (DPLL F''{P ↦ ⊤}) ∨ (DPLL F''{P ↦ ⊥})
```

# Unit Propagagion

Unit Propagation (PROP)

If a clause contains one literal $\ell$,

- Set $\ell$ to $\top$.
- Remove all clauses containing $\ell$.
- Remove $\neg\ell$ in all clauses.

# Unit Propagagion

Unit Propagation (PROP)

If a clause contains one literal $\ell$,

- Set $\ell$ to $\top$.
- Remove all clauses containing $\ell$.
- Remove $\neg\ell$ in all clauses.

Based on resolution

$$\frac{\ell \qquad \neg\ell \vee C}{C} \leftarrow \text{clause}$$

# Pure Literal Propagagion

Pure Literal Propagation (PLP)

If $P$ occurs only positive (without negation), set it to $\top$.

# Pure Literal Propagagion

Pure Literal Propagation (PLP)

If $P$ occurs only positive (without negation), set it to $\top$.
If $P$ occurs only negative set it to $\bot$.

# Example

$$F : (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$

# Example

$$F : (\neg P \lor Q \lor R) \land (\neg Q \lor R) \land (\neg Q \lor \neg R) \land (P \lor \neg Q \lor \neg R)$$

Branching on $Q$

# Example

$$F : (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$

Branching on $Q$

$$F\{Q \mapsto \top\} : (R) \wedge (\neg R) \wedge (P \vee \neg R)$$

By unit resolution

$$\frac{R \qquad (\neg R)}{\bot}$$

$F\{Q \mapsto \top\} = \bot \Rightarrow$ false

On the other branch

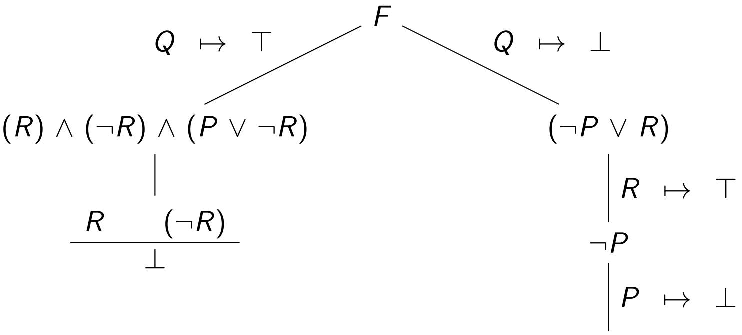
$F\{Q \mapsto \bot\} : (\neg P \vee R)$

$F\{Q \mapsto \bot, R \mapsto \top, P \mapsto \bot\} = \top \Rightarrow$ true

$F$ is satisfiable with satisfying interpretation

$$I : \{P \mapsto \text{false}, Q \mapsto \text{false}, R \mapsto \text{true}\}$$

$$F : (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$




$$I : \{P \mapsto \text{false},\ Q \mapsto \text{false},\ R \mapsto \text{true}\}$$

# Knight and Knaves

A island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet four inhabitants: Alice, Bob, Charles and Doris.

- Alice says that Doris is a knave.
- Bob tells you that Alice is a knave.
- Charles claims that Alice is a knave.
- Doris tells you, ‘Of Charles and Bob, exactly one is a knight.’

# Knight and Knaves

Let $A$ denote that Alice is a Knight, etc. Then:

- $A \leftrightarrow \neg D$
- $B \leftrightarrow \neg A$
- $C \leftrightarrow \neg A$
- $D \leftrightarrow \neg(C \leftrightarrow B)$

# Knight and Knaves

Let $A$ denote that Alice is a Knight, etc. Then:

- $A \leftrightarrow \neg D$
- $B \leftrightarrow \neg A$
- $C \leftrightarrow \neg A$
- $D \leftrightarrow \neg(C \leftrightarrow B)$

In CNF:

- $\{\overline{A}, \overline{D}\}, \{A, D\}$
- $\{\overline{B}, \overline{A}\}, \{B, A\}$
- $\{\overline{C}, \overline{A}\}, \{C, A\}$
- $\{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}$

# Solving Knights and Knaves

$$F : \{\{\overline{A}, \overline{D}\}, \{A, D\}, \{\overline{B}, \overline{A}\}, \{B, A\}, \{\overline{C}, \overline{A}\}, \{C, A\}, \{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}\}$$

# Solving Knights and Knaves

$$F : \{\{\overline{A}, \overline{D}\}, \{A, D\}, \{\overline{B}, \overline{A}\}, \{B, A\}, \{\overline{C}, \overline{A}\}, \{C, A\}, \{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}\}$$

PROP and PLP are not applicable. Decide on $A$:

$F\{A \mapsto \bot\} : \{\{D\}, \{B\}, \{C\}, \{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}\}$

By PROP we get:

$$F\{A \mapsto \bot, D \mapsto \top, B \mapsto \top, C \mapsto \top\} : \bot$$

Unsatisfiable! Now set $A$ to $\top$:

$F\{A \mapsto \top\} : \{\{\overline{D}\}, \{\overline{B}\}, \{\overline{C}\}, \{\overline{D}, \overline{C}, \overline{B}\}, \{\overline{D}, C, B\}, \{D, \overline{C}, B\}, \{D, C, \overline{B}\}\}$

By PROP we get:

$$F\{A \mapsto \top, D \mapsto \bot, B \mapsto \bot, C \mapsto \bot\} : \top$$

Satisfying assignment!

# Learning is Useful

Consider the following problem:

$$\{\{A_1, B_1\}, \{\overline{P_0}, \overline{A_1}, P_1\}, \{\overline{P_0}, \overline{B_1}, P_1\}, \{A_2, B_2\}, \{\overline{P_1}, \overline{A_2}, P_2\}, \{\overline{P_1}, \overline{B_2}, P_2\}, \\ \dots, \{A_n, B_n\}, \{\overline{P_{n-1}}, \overline{A_n}, P_n\}, \{\overline{P_{n-1}}, \overline{B_n}, P_n\}, \{P_0\}, \{\overline{P_n}\}\}$$

Consider the following problem:

$$\{\{A_1, B_1\}, \{\overline{P_0}, \overline{A_1}, P_1\}, \{\overline{P_0}, \overline{B_1}, P_1\}, \{A_2, B_2\}, \{\overline{P_1}, \overline{A_2}, P_2\}, \{\overline{P_1}, \overline{B_2}, P_2\},$$
$$\ldots, \{A_n, B_n\}, \{\overline{P_{n-1}}, \overline{A_n}, P_n\}, \{\overline{P_{n-1}}, \overline{B_n}, P_n\}, \{P_0\}, \{\overline{P_n}\}\}$$

For some literal orderings, we need exponentially many steps.
Note, that

$$\{\{A_i, B_i\}, \{\overline{P_{i-1}}, \overline{A_i}, P_i\}, \{\overline{P_{i-1}}, \overline{B_i}, P_i\}\} \Rightarrow \{\{\overline{P_{i-1}}, P_i\}\}$$

If we learn the right clauses, unit propagation will immediately give unsatisfiable.

# Partial Assignments and Unit/Conflict Clauses

Do not change the clause set, but only assign literals (as global variables).
When you assign true to a literal $\ell$,also assign false to $\overline{\ell}$.

# Partial Assignments and Unit/Conflict Clauses

Do not change the clause set, but only assign literals (as global variables). When you assign true to a literal $\ell$,also assign false to $\bar{\ell}$.
For a partial assignment

- A clause is true if one of its literals is assigned true.
- A clause is a conflict clause if all its literals are assigned false.
- A clause is a unit clause if all but one literals are assigned false and the last literal is unassigned.

# Partial Assignments and Unit/Conflict Clauses

Do not change the clause set, but only assign literals (as global variables).
When you assign true to a literal $\ell$,also assign false to $\bar{\ell}$.
For a partial assignment

- A clause is true if one of its literals is assigned true.
- A clause is a conflict clause if all its literals are assigned false.
- A clause is a unit clause if all but one literals are assigned false and the last literal is unassigned.

If the assignment of a literal from a conflict clause is removed we get a unit clause.

Do not change the clause set, but only assign literals (as global variables).
When you assign true to a literal $\ell$,also assign false to $\bar{\ell}$.
For a partial assignment

- A clause is true if one of its literals is assigned true.
- A clause is a conflict clause if all its literals are assigned false.
- A clause is a unit clause if all but one literals are assigned false and the last literal is unassigned.

If the assignment of a literal from a conflict clause is removed we get a unit clause.
Explain unsatisfiability of partial assignment by conflict clause and learn it!

# Conflict Driven Clause Learning (CDCL)

Idea: Explain unsatisfiability of partial assignment by conflict clause and learn it!

- If a conflict is found we return the conflict clause.
- If variable in conflict were derived by unit propagation use resolution rule to generate a new conflict clause.
- If variable in conflict was derived by decision, use learned conflict as unit clause

# DPLL with CDCL

The functions DPLL and PROP return a conflict clause or satisfiable.

```
let rec DPLL =
  let PROP U =
    ...
  if conflictclauses ≠ ∅
    CHOOSE conflictclauses
  else if unitclauses ≠ ∅
    PROP (CHOOSE unitclauses)
  else if coreclauses ≠ ∅
     let ℓ = CHOOSE (⋃coreclauses) ∩ unassigned in
     val[ℓ] := ⊤
     let C = DPLL in
     if (C = satisfiable) satisfiable
     else
        val[ℓ] := undef
        if (ℓ̄ ∉ C) C
        else LEARN C; PROP C
  else satisfiable
```

# Unit propagation

The function PROP takes a unit clause and does unit propagation. It calls DPLL recursively and returns a conflict clause or satisfiable. recursively:

```
let PROP U =
  let ℓ = CHOOSE U ∩ unassigned in
  val[ℓ] := ⊤
  let C = DPLL in
  if (C = satisfiable)
    satisfiable
  else
    val[ℓ] := undef
    if (ℓ̄ ∉ C) C
    else  U \ {ℓ} ∪ C \ {ℓ̄}
```

The last line does resolution:

$$\frac{\ell \vee C_1 \qquad \neg\ell \vee C_2}{C_1 \vee C_2}$$

# Example

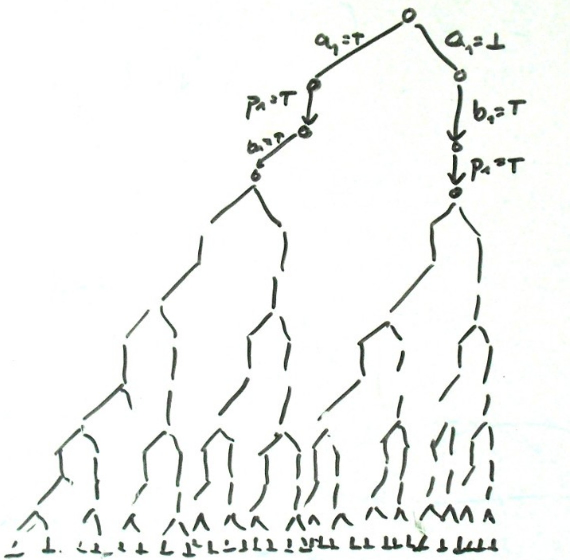
$$\{\{A_1, B_1\}, \{\overline{P_0}, \overline{A_1}, P_1\}, \{\overline{P_0}, \overline{B_1}, P_1\}, \{A_2, B_2\}, \{\overline{P_1}, \overline{A_2}, P_2\}, \{\overline{P_1}, \overline{B_2}, P_2\},$$
$$\dots, \{A_n, B_n\}, \{\overline{P_{n-1}}, \overline{A_n}, P_n\}, \{\overline{P_{n-1}}, \overline{B_n}, P_n\}, \{P_0\}, \{\overline{P_n}\}\}$$

# Example

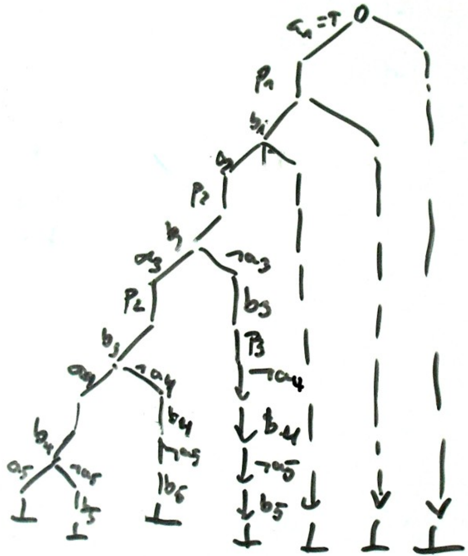
$$\{\{A_1, B_1\}, \{\overline{P_0}, \overline{A_1}, P_1\}, \{\overline{P_0}, \overline{B_1}, P_1\}, \{A_2, B_2\}, \{\overline{P_1}, \overline{A_2}, P_2\}, \{\overline{P_1}, \overline{B_2}, P_2\}, \\ \dots, \{A_n, B_n\}, \{\overline{P_{n-1}}, \overline{A_n}, P_n\}, \{\overline{P_{n-1}}, \overline{B_n}, P_n\}, \{P_0\}, \{\overline{P_n}\}\}$$

- Unit propagation (PROP) sets $P_0$ and $\overline{P_n}$ to true.
- Decide, e.g. $A_1$, PROP sets $\overline{P_1}$
- Continue until $A_{n-1}$, PROP sets $\overline{P_{n-1}}$,$\overline{A_n}$ and $\overline{B_n}$
- Conflict clause computed: $\{\overline{A_{n-1}}, \overline{P_{n-2}}, P_n\}$.
- Conflict clause does not depend on $A_1, \dots, A_{n-2}$ and can be used again.

# DPLL (without Learning)



# DPLL with CDCL



# Some Notes about DPLL with Learning

- Pure Literal Propagation is unnecessary:
  A pure literal is always chosen right and never causes a conflict.

# Some Notes about DPLL with Learning

- Pure Literal Propagation is unnecessary:
  A pure literal is always chosen right and never causes a conflict.
- Modern SAT-solvers use this procedure but differ in

# Some Notes about DPLL with Learning

- Pure Literal Propagation is unnecessary:
  A pure literal is always chosen right and never causes a conflict.
- Modern SAT-solvers use this procedure but differ in
  - heuristics to choose literals/clauses.

# Some Notes about DPLL with Learning

- Pure Literal Propagation is unnecessary:
  A pure literal is always chosen right and never causes a conflict.
- Modern SAT-solvers use this procedure but differ in
  - heuristics to choose literals/clauses.
  - efficient data structures to find unit clauses.

# Some Notes about DPLL with Learning

- Pure Literal Propagation is unnecessary:
  A pure literal is always chosen right and never causes a conflict.
- Modern SAT-solvers use this procedure but differ in
  - heuristics to choose literals/clauses.
  - efficient data structures to find unit clauses.
  - better conflict resolution to minimize learned clauses.

# Some Notes about DPLL with Learning

- Pure Literal Propagation is unnecessary:
  A pure literal is always chosen right and never causes a conflict.
- Modern SAT-solvers use this procedure but differ in
  - heuristics to choose literals/clauses.
  - efficient data structures to find unit clauses.
  - better conflict resolution to minimize learned clauses.
  - restarts (without forgetting learned clauses).

# Some Notes about DPLL with Learning

- Pure Literal Propagation is unnecessary:
  A pure literal is always chosen right and never causes a conflict.
- Modern SAT-solvers use this procedure but differ in
  - heuristics to choose literals/clauses.
  - efficient data structures to find unit clauses.
  - better conflict resolution to minimize learned clauses.
  - restarts (without forgetting learned clauses).
- Even with the optimal heuristics DPLL is still exponential:
  The Pidgeon-Hole problem requires exponential resolution proofs.

# Summary

- Syntax and Semantics of Propositional Logic

# Summary

- Syntax and Semantics of Propositional Logic
- Methods to decide satisfiability/validity of formulae:

# Summary

- Syntax and Semantics of Propositional Logic
- Methods to decide satisfiability/validity of formulae:
  - Truth table
  - Semantic Tableaux
  - DPLL

# Summary

- Syntax and Semantics of Propositional Logic
- Methods to decide satisfiability/validity of formulae:
  - Truth table
  - Semantic Tableaux
  - DPLL
- Run-time of all algorithm is worst-case exponential in length of formula.

# Summary

- Syntax and Semantics of Propositional Logic
- Methods to decide satisfiability/validity of formulae:
  - Truth table
  - Semantic Tableaux
  - DPLL
- Run-time of all algorithm is worst-case exponential in length of formula.
- Deciding satisfiability is NP-complete.