

Decision Procedures

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Theories

In first-order logic function symbols have no predefined meaning:

The formula $1 + 1 = 3$ is satisfiable.

We want to fix the meaning for some function symbols.

Examples:

- Equality theory
- Theory of natural numbers
- Theory of rational numbers
- Theory of arrays or lists

Definition (First-order theory)

A **First-order theory** T consists of

- A **Signature** Σ - set of constant, function, and predicate symbols
- A set of **axioms** A_T - set of **closed** (no free variables) Σ -formulae

A **Σ -formula** is a formula constructed of constants, functions, and predicate symbols from Σ , and variables, logical connectives, and quantifiers

- The symbols of Σ are **just symbols** without prior meaning
- The axioms of T provide their meaning

Signature $\Sigma_{=} : \{=, a, b, c, \dots, f, g, h, \dots, p, q, r, \dots\}$

- $=$, a binary predicate, **interpreted** by axioms.
- all constant, function, and predicate symbols.

Axioms of T_E :

- 1 $\forall x. x = x$ (reflexivity)
- 2 $\forall x, y. x = y \rightarrow y = x$ (symmetry)
- 3 $\forall x, y, z. x = y \wedge y = z \rightarrow x = z$ (transitivity)
- 4 for each positive integer n and n -ary function symbol f ,
 $\forall x_1, \dots, x_n, y_1, \dots, y_n. \bigwedge_i x_i = y_i \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$
(congruence)
- 5 for each positive integer n and n -ary predicate symbol p ,
 $\forall x_1, \dots, x_n, y_1, \dots, y_n. \bigwedge_i x_i = y_i \rightarrow (p(x_1, \dots, x_n) \leftrightarrow p(y_1, \dots, y_n))$
(equivalence)

Congruence and Equivalence are **axiom schemata**.

- 4 for each positive integer n and n -ary function symbol f ,

$$\forall x_1, \dots, x_n, y_1, \dots, y_n. \bigwedge_i x_i = y_i \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$$
(congruence)
- 5 for each positive integer n and n -ary predicate symbol p ,

$$\forall x_1, \dots, x_n, y_1, \dots, y_n. \bigwedge_i x_i = y_i \rightarrow (p(x_1, \dots, x_n) \leftrightarrow p(y_1, \dots, y_n))$$
(equivalence)

For every function symbol there is an instance of the congruence axiom schemata.

Example: Congruence axiom for binary function f_2 :

$$\forall x_1, x_2, y_1, y_2. x_1 = y_1 \wedge x_2 = y_2 \rightarrow f_2(x_1, x_2) = f_2(y_1, y_2)$$

A_{T_E} contains an infinite number of these axioms.

Definition (T -interpretation)

An interpretation I is a T -interpretation, if it satisfies all the axioms of T .

Definition (T -valid)

A Σ -formula F is **valid in theory T** (T -valid, also $T \models F$), if every T -interpretation satisfies F .

Definition (T -satisfiable)

A Σ -formula F is **satisfiable in T** (T -satisfiable), if there is a T -interpretation that satisfies F .

Definition (T -equivalent)

Two Σ -formulae F_1 and F_2 are **equivalent in T** (T -equivalent), if $F_1 \leftrightarrow F_2$ is T -valid,

Example: T_E -validity

Semantic argument method can be used for T_E

Prove

$$F : a = b \wedge b = c \rightarrow g(f(a), b) = g(f(c), a) \quad T_E\text{-valid.}$$

Suppose not; then there exists a T_E -interpretation I such that $I \not\models F$.

Then,

1.	$I \not\models F$	assumption
2.	$I \models a = b \wedge b = c$	1, \rightarrow
3.	$I \not\models g(f(a), b) = g(f(c), a)$	1, \rightarrow
4.	$I \models \forall x, y, z. x = y \wedge y = z \rightarrow x = z$	transitivity
5.	$I \models a = b \wedge b = c \rightarrow a = c$	4, $3 \times \forall\{x \mapsto a, y \mapsto b, z \mapsto c\}$
6a	$I \not\models a = b \wedge b = c$	5, \rightarrow
7a	$I \models \perp$	2 and 6a contradictory
6b.	$I \models a = c$	4, 5, (5, \rightarrow)
7b.	$I \models a = c \rightarrow f(a) = f(c)$	(congruence), $2 \times \forall$
8ba.	$I \not\models a = c \quad \dots I \models \perp$	
8bb.	$I \models f(a) = f(c)$	7b, \rightarrow
9bb.	$I \models a = b$	2, \wedge
10bb.	$I \models a = b \rightarrow b = a$	(symmetry), $2 \times \forall$
11bba.	$I \not\models a = b \quad \dots I \models \perp$	
11bbb.	$I \models b = a$	10bb, \rightarrow
12bbb.	$I \models f(a) = f(c) \wedge b = a \rightarrow g(f(a), b) = g(f(c), a)$	(congruence), $4 \times \forall$
... 13	$I \models g(f(a), b) = g(f(c), a)$	8bb, 11bbb, 12bbb

3 and 13 are contradictory. Thus, F is T_E -valid.

Is it possible to decide T_E -validity?

T_E -validity is undecidable.

If we restrict ourselves to quantifier-free formulae we get decidability:

For a quantifier-free formula T_E -validity is decidable.

A **fragment of theory** T is a syntactically-restricted subset of formulae of the theory.

Example: **quantifier-free fragment** of theory T is the set of quantifier-free formulae in T .

A theory T is **decidable** if $T \models F$ (T -validity) is decidable for every Σ -formula F ,

i.e., there is an algorithm that always terminate with “yes”, if F is T -valid, and “no”, if F is T -invalid.

A fragment of T is **decidable** if $T \models F$ is decidable for every Σ -formula F in the fragment.

Natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$

Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Three variations:

- Peano arithmetic T_{PA} : natural numbers with addition and multiplication
- Presburger arithmetic $T_{\mathbb{N}}$: natural numbers with addition
- Theory of integers $T_{\mathbb{Z}}$: integers with $+$, $-$, $>$

Signature: $\Sigma_{PA} : \{0, 1, +, \cdot, =\}$

Axioms of T_{PA} : axioms of T_E ,

- 1 $\forall x. \neg(x + 1 = 0)$ (zero)
- 2 $\forall x, y. x + 1 = y + 1 \rightarrow x = y$ (successor)
- 3 $F[0] \wedge (\forall x. F[x] \rightarrow F[x + 1]) \rightarrow \forall x. F[x]$ (induction)
- 4 $\forall x. x + 0 = x$ (plus zero)
- 5 $\forall x, y. x + (y + 1) = (x + y) + 1$ (plus successor)
- 6 $\forall x. x \cdot 0 = 0$ (times zero)
- 7 $\forall x, y. x \cdot (y + 1) = x \cdot y + x$ (times successor)

Line 3 is an axiom schema.

$3x + 5 = 2y$ can be written using Σ_{PA} as

$$x + x + x + 1 + 1 + 1 + 1 + 1 = y + y$$

We can define $>$ and \geq : $3x + 5 > 2y$ write as

$$\exists z. z \neq 0 \wedge 3x + 5 = 2y + z$$

$$3x + 5 \geq 2y \quad \text{write as} \quad \exists z. 3x + 5 = 2y + z$$

Examples for valid formulae:

- Pythagorean Theorem is T_{PA} -valid

$$\exists x, y, z. x \neq 0 \wedge y \neq 0 \wedge z \neq 0 \wedge xx + yy = zz$$

- Fermat's Last Theorem is T_{PA} -valid (Andrew Wiles, 1994)

$$\forall n. n > 2 \rightarrow \neg \exists x, y, z. x \neq 0 \wedge y \neq 0 \wedge z \neq 0 \wedge x^n + y^n = z^n$$

In Fermat's theorem we used x^n , which is not a valid term in Σ_{PA} . However, there is the Σ_{PA} -formula $EXP[x, n, r]$ with

- 1 $EXP[x, 0, r] \leftrightarrow r = 1$
- 2 $EXP[x, i + 1, r] \leftrightarrow \exists r_1. EXP[x, i, r_1] \wedge r = r_1 \cdot x$

$$\begin{aligned} EXP[x, n, r] : & \exists d, m. (\exists z. d = (m + 1)z + 1) \wedge \\ & (\forall i, r_1. i < n \wedge r_1 < m \wedge (\exists z. d = ((i + 1)m + 1)z + r_1) \rightarrow \\ & r_1 x < m \wedge (\exists z. d = ((i + 2)m + 1)z + r_1 \cdot x)) \wedge \\ & r < m \wedge (\exists z. d = ((n + 1)m + 1)z + r) \end{aligned}$$

Fermat's theorem can be stated as:

$$\begin{aligned} \forall n. n > 2 \rightarrow \neg \exists x, y, z, rx, ry. x \neq 0 \wedge y \neq 0 \wedge z \neq 0 \wedge \\ EXP[x, n, rx] \wedge EXP[y, n, ry] \wedge EXP[z, n, rx + ry] \end{aligned}$$

Decidability of Peano Arithmetic

Gödel showed that for every **recursive** function $f : \mathbb{N}^n \rightarrow \mathbb{N}$ there is a Σ_{PA} -formula $F[x_1, \dots, x_n, r]$ with

$$F[x_1, \dots, x_n, r] \leftrightarrow r = f(x_1, \dots, x_n)$$

T_{PA} is undecidable. (Gödel, Turing, Post, Church)

The quantifier-free fragment of T_{PA} is undecidable. (Matiyasevich, 1970)

Remark: Gödel's first incompleteness theorem

Peano arithmetic T_{PA} does not capture true arithmetic:

There exist closed Σ_{PA} -formulae representing valid propositions of number theory that are not T_{PA} -valid.

The reason: T_{PA} actually admits **nonstandard interpretations**

For decidability: no multiplication

Signature: $\Sigma_{\mathbb{N}} : \{0, 1, +, =\}$ no multiplication!

Axioms of $T_{\mathbb{N}}$: axioms of T_E ,

- 1 $\forall x. \neg(x + 1 = 0)$ (zero)
- 2 $\forall x, y. x + 1 = y + 1 \rightarrow x = y$ (successor)
- 3 $F[0] \wedge (\forall x. F[x] \rightarrow F[x + 1]) \rightarrow \forall x. F[x]$ (induction)
- 4 $\forall x. x + 0 = x$ (plus zero)
- 5 $\forall x, y. x + (y + 1) = (x + y) + 1$ (plus successor)

3 is an axiom schema.

$T_{\mathbb{N}}$ -satisfiability and $T_{\mathbb{N}}$ -validity are decidable. (Presburger 1929)

Signature:

$\Sigma_{\mathbb{Z}} : \{ \dots, -2, -1, 0, 1, 2, \dots, -3\cdot, -2\cdot, 2\cdot, 3\cdot, \dots, +, -, =, > \}$

where

- $\dots, -2, -1, 0, 1, 2, \dots$ are constants
- $\dots, -3\cdot, -2\cdot, 2\cdot, 3\cdot, \dots$ are unary functions
(intended meaning: $2 \cdot x$ is $x + x$)
- $+, -, =, >$ have the usual meanings.

Relation between $T_{\mathbb{Z}}$ and $T_{\mathbb{N}}$

$T_{\mathbb{Z}}$ and $T_{\mathbb{N}}$ have the same expressiveness:

- For every $\Sigma_{\mathbb{Z}}$ -formula there is an equisatisfiable $\Sigma_{\mathbb{N}}$ -formula.
- For every $\Sigma_{\mathbb{N}}$ -formula there is an equisatisfiable $\Sigma_{\mathbb{Z}}$ -formula.

$\Sigma_{\mathbb{Z}}$ -formula F and $\Sigma_{\mathbb{N}}$ -formula G are **equisatisfiable** iff:

F is $T_{\mathbb{Z}}$ -satisfiable iff G is $T_{\mathbb{N}}$ -satisfiable

Example: $\Sigma_{\mathbb{Z}}$ -formula to $\Sigma_{\mathbb{N}}$ -formula

Consider the $\Sigma_{\mathbb{Z}}$ -formula

$$F_0 : \forall w, x. \exists y, z. x + 2y - z - 7 > -3w + 4$$

Introduce two variables, v_p and v_n (range over the nonnegative integers) for each variable v (range over the integers) of F_0

$$F_1 : \forall w_p, w_n, x_p, x_n. \exists y_p, y_n, z_p, z_n. \\ (x_p - x_n) + 2(y_p - y_n) - (z_p - z_n) - 7 > -3(w_p - w_n) + 4$$

Eliminate $-$ by moving to the other side of $>$

$$F_2 : \forall w_p, w_n, x_p, x_n. \exists y_p, y_n, z_p, z_n. \\ x_p + 2y_p + z_n + 3w_p > x_n + 2y_n + z_p + 7 + 3w_n + 4$$

Eliminate $>$ and numbers:

$$F_3 : \forall w_p, w_n, x_p, x_n. \exists y_p, y_n, z_p, z_n. \exists u. \\ \neg(u = 0) \wedge x_p + y_p + y_p + z_n + w_p + w_p + w_p \\ = x_n + y_n + y_n + z_p + w_n + w_n + w_n + u \\ + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

which is a $\Sigma_{\mathbb{N}}$ -formula equisatisfiable to F_0 .

Example: $\Sigma_{\mathbb{N}}$ -formula to $\Sigma_{\mathbb{Z}}$ -formula.

Example: The $\Sigma_{\mathbb{N}}$ -formula

$$\forall x. \exists y. x = y + 1$$

is equisatisfiable to the $\Sigma_{\mathbb{Z}}$ -formula:

$$\forall x. x > -1 \rightarrow \exists y. y > -1 \wedge x = y + 1.$$

To decide $T_{\mathbb{Z}}$ -validity for a $\Sigma_{\mathbb{Z}}$ -formula F :

- transform $\neg F$ to an equisatisfiable $\Sigma_{\mathbb{N}}$ -formula $\neg G$,
- decide $T_{\mathbb{N}}$ -validity of G .

$$\Sigma = \{0, 1, +, -, \cdot, =, \geq\}$$

- Theory of Reals $T_{\mathbb{R}}$ (with multiplication)

$$x \cdot x = 2 \quad \Rightarrow \quad x = \pm\sqrt{2}$$

- Theory of Rationals $T_{\mathbb{Q}}$ (no multiplication)

$$\underbrace{2x}_{x+x} = 7 \quad \Rightarrow \quad x = \frac{2}{7}$$

Note: Strict inequality

$$\forall x, y. \exists z. x + y > z$$

can be expressed as

$$\forall x, y. \exists z. \neg(x + y = z) \wedge x + y \geq z$$

Signature: $\Sigma_{\mathbb{R}} : \{0, 1, +, -, \cdot, =, \geq\}$ with multiplication.

Axioms of $T_{\mathbb{R}}$: axioms of T_E ,

- | | | |
|----|---|--------------------------|
| 1 | $\forall x, y, z. (x + y) + z = x + (y + z)$ | (+ associativity) |
| 2 | $\forall x, y. x + y = y + x$ | (+ commutativity) |
| 3 | $\forall x. x + 0 = x$ | (+ identity) |
| 4 | $\forall x. x + (-x) = 0$ | (+ inverse) |
| 5 | $\forall x, y, z. (x \cdot y) \cdot z = x \cdot (y \cdot z)$ | (\cdot associativity) |
| 6 | $\forall x, y. x \cdot y = y \cdot x$ | (\cdot commutativity) |
| 7 | $\forall x. x \cdot 1 = x$ | (\cdot identity) |
| 8 | $\forall x. x \neq 0 \rightarrow \exists y. x \cdot y = 1$ | (\cdot inverse) |
| 9 | $\forall x, y, z. x \cdot (y + z) = x \cdot y + x \cdot z$ | (distributivity) |
| 10 | $0 \neq 1$ | (separate identities) |
| 11 | $\forall x, y. x \geq y \wedge y \geq x \rightarrow x = y$ | (antisymmetry) |
| 12 | $\forall x, y, z. x \geq y \wedge y \geq z \rightarrow x \geq z$ | (transitivity) |
| 13 | $\forall x, y. x \geq y \vee y \geq x$ | (totality) |
| 14 | $\forall x, y, z. x \geq y \rightarrow x + z \geq y + z$ | (+ ordered) |
| 15 | $\forall x, y. x \geq 0 \wedge y \geq 0 \rightarrow x \cdot y \geq 0$ | (\cdot ordered) |
| 16 | $\forall x. \exists y. x = y \cdot y \vee x = -y \cdot y$ | (square root) |
| 17 | for each odd integer n ,
$\forall x_0, \dots, x_{n-1}. \exists y. y^n + x_{n-1}y^{n-1} \dots + x_1y + x_0 = 0$ | (at least one root) |

Example

$F: \forall a, b, c. b^2 - 4ac \geq 0 \leftrightarrow \exists x. ax^2 + bx + c = 0$ is $T_{\mathbb{R}}$ -valid.

As usual: x^2 abbreviates $x \cdot x$, we omit \cdot , e.g. in $4ac$,

4 abbreviate $1 + 1 + 1 + 1$ and $a - b$ abbreviates $a + (-b)$.

- | | | |
|------|--|--------------------------------------|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models \exists y. bb - 4ac = y^2 \vee bb - 4ac = -y^2$ | square root, \forall |
| 3. | $I \models d^2 = bb - 4ac \vee d^2 = -(bb - 4ac)$ | 2, \exists |
| 4. | $I \models d \geq 0 \vee 0 \geq d$ | \geq total |
| 5. | $I \models d^2 \geq 0$ | 4, case distinction, \cdot ordered |
| 6. | $I \models 2a \cdot e = 1$ | \cdot inverse, \forall, \exists |
| 7a. | $I \models bb - 4ac \geq 0$ | 1, \leftrightarrow |
| 8a. | $I \not\models \exists x. axx + bx + c = 0$ | 1, \leftrightarrow |
| 9a. | $I \not\models a((-b + d)e)^2 + b(-b + d)e + c = 0$ | 8a, \exists |
| 10a. | $I \not\models ab^2e^2 - 2abde^2 + ad^2e^2 - b^2e + bde + c = 0$ | distributivity |
| 11a. | $I \models dd = bb - 4ac$ | 3, 5, 7a |
| 12a. | $I \not\models ab^2e^2 - bde + a(b^2 - 4ac)e^2 - b^2e + bde + c = 0$ | 6, 11a, congruence |
| 13a. | $I \not\models 0 = 0$ | 3, distributivity, inverse |
| 14a. | $I \models \perp$ | 13a, reflexivity |

Example

$F: \forall a, b, c. bb - 4ac \geq 0 \leftrightarrow \exists x. axx + bx + c = 0$ is $T_{\mathbb{R}}$ -valid.

As usual: x^2 abbreviates $x \cdot x$, we omit \cdot , e.g., in $4ac$,

4 abbreviate $1 + 1 + 1 + 1$ and $a - b$ abbreviates $a + (-b)$.

1.	$I \not\models F$	assumption
2.	$I \models \exists y. bb - 4ac = y^2 \vee bb - 4ac = -y^2$	square root, \forall
3.	$I \models d^2 = bb - 4ac \vee d^2 = -(bb - 4ac)$	2, \exists
4.	$I \models d \geq 0 \vee 0 \geq d$	\geq total
5.	$I \models d^2 \geq 0$	4, case distinction, \cdot ordered
6.	$I \models 2a \cdot e = 1$	\cdot inverse, \forall, \exists
7b.	$I \not\models bb - 4ac \geq 0$	1, \leftrightarrow
8b.	$I \models \exists x. axx + bx + c = 0$	1, \leftrightarrow
9b.	$I \models aff + bf + c = 0$	8b, \exists
10b.	$I \models (2af + b)^2 = bb - 4ac$	field axioms, T_E
11b.	$I \models (2af + b)^2 \geq 0$	analogous to 5
12b.	$I \models bb - 4ac \geq 0$	10b, 11b, equivalence
13b.	$I \models \perp$	12b, 7b

$T_{\mathbb{R}}$ is decidable (Tarski, 1930)
High time complexity: $O(2^{2^{kn}})$

Signature: $\Sigma_{\mathbb{Q}} : \{0, 1, +, -, =, \geq\}$ no multiplication!

Axioms of $T_{\mathbb{Q}}$: axioms of T_E ,

- 1 $\forall x, y, z. (x + y) + z = x + (y + z)$ (+ associativity)
- 2 $\forall x, y. x + y = y + x$ (+ commutativity)
- 3 $\forall x. x + 0 = x$ (+ identity)
- 4 $\forall x. x + (-x) = 0$ (+ inverse)
- 5 $1 \geq 0 \wedge 1 \neq 0$ (one)
- 6 $\forall x, y. x \geq y \wedge y \geq x \rightarrow x = y$ (antisymmetry)
- 7 $\forall x, y, z. x \geq y \wedge y \geq z \rightarrow x \geq z$ (transitivity)
- 8 $\forall x, y. x \geq y \vee y \geq x$ (totality)
- 9 $\forall x, y, z. x \geq y \rightarrow x + z \geq y + z$ (+ ordered)
- 10 For every positive integer n :
 $\forall x. \exists y. x = \underbrace{y + \dots + y}_n$ (divisible)

Rational coefficients are simple to express in $T_{\mathbb{Q}}$

Example: Rewrite

$$\frac{1}{2}x + \frac{2}{3}y \geq 4$$

as the $\Sigma_{\mathbb{Q}}$ -formula

$$x + x + x + y + y + y + y \geq \underbrace{1 + 1 + \dots + 1}_{24}$$

$T_{\mathbb{Q}}$ is decidable

Efficient algorithm for quantifier free fragment

- Data Structures are tuples of variables.
Like `struct` in C, `record` in Pascal.
- In Recursive Data Structures, one of the tuple elements can be the data structure again.
Linked lists or trees.

$$\Sigma_{\text{cons}} : \{\text{cons}, \text{car}, \text{cdr}, \text{atom}, =\}$$

where

$\text{cons}(a, b)$ – list constructed by adding a in front of list b

$\text{car}(x)$ – left projector of x : $\text{car}(\text{cons}(a, b)) = a$

$\text{cdr}(x)$ – right projector of x : $\text{cdr}(\text{cons}(a, b)) = b$

$\text{atom}(x)$ – true iff x is a single-element list

Axioms: The axioms of A_{T_E} plus

- $\forall x, y. \text{car}(\text{cons}(x, y)) = x$ (left projection)
- $\forall x, y. \text{cdr}(\text{cons}(x, y)) = y$ (right projection)
- $\forall x. \neg \text{atom}(x) \rightarrow \text{cons}(\text{car}(x), \text{cdr}(x)) = x$ (construction)
- $\forall x, y. \neg \text{atom}(\text{cons}(x, y))$ (atom)

- 1 The axioms of **reflexivity**, **symmetry**, and **transitivity** of =
- 2 **Congruence** axioms

$$\forall x_1, x_2, y_1, y_2. x_1 = x_2 \wedge y_1 = y_2 \rightarrow \text{cons}(x_1, y_1) = \text{cons}(x_2, y_2)$$

$$\forall x, y. x = y \rightarrow \text{car}(x) = \text{car}(y)$$

$$\forall x, y. x = y \rightarrow \text{cdr}(x) = \text{cdr}(y)$$

- 3 **Equivalence** axiom

$$\forall x, y. x = y \rightarrow (\text{atom}(x) \leftrightarrow \text{atom}(y))$$

- 4 $\forall x, y. \text{car}(\text{cons}(x, y)) = x$ (left projection)
- 5 $\forall x, y. \text{cdr}(\text{cons}(x, y)) = y$ (right projection)
- 6 $\forall x. \neg \text{atom}(x) \rightarrow \text{cons}(\text{car}(x), \text{cdr}(x)) = x$ (construction)
- 7 $\forall x, y. \neg \text{atom}(\text{cons}(x, y))$ (atom)

T_{cons} is undecidable

Quantifier-free fragment of T_{cons} is efficiently decidable

Example: T_{CONS} -Validity

We argue that the following Σ_{CONS} -formula F is T_{CONS} -valid:

$$F : \quad \text{car}(a) = \text{car}(b) \wedge \text{cdr}(a) = \text{cdr}(b) \wedge \neg \text{atom}(a) \wedge \neg \text{atom}(b) \\ \rightarrow a = b$$

- | | | |
|-----|---|-----------------------------|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models \text{car}(a) = \text{car}(b)$ | 1, \rightarrow , \wedge |
| 3. | $I \models \text{cdr}(a) = \text{cdr}(b)$ | 1, \rightarrow , \wedge |
| 4. | $I \models \neg \text{atom}(a)$ | 1, \rightarrow , \wedge |
| 5. | $I \models \neg \text{atom}(b)$ | 1, \rightarrow , \wedge |
| 6. | $I \not\models a = b$ | 1, \rightarrow |
| 7. | $I \models \text{cons}(\text{car}(a), \text{cdr}(a)) = \text{cons}(\text{car}(b), \text{cdr}(b))$ | 2, 3, (congruence) |
| 8. | $I \models \text{cons}(\text{car}(a), \text{cdr}(a)) = a$ | 4, (construction) |
| 9. | $I \models \text{cons}(\text{car}(b), \text{cdr}(b)) = b$ | 5, (construction) |
| 10. | $I \models a = b$ | 7, 8, 9, (transitivity) |

Lines 6 and 10 are contradictory. Therefore, F is T_{CONS} -valid.

Signature: $\Sigma_A : \{ \cdot[\cdot], \cdot\langle \cdot \triangleleft \cdot \rangle, = \},$

where

- $a[i]$ binary function –
read array a at index i (“read(a, i)”)
- $a\langle i \triangleleft v \rangle$ ternary function –
write value v to index i of array a (“write(a, i, v)”)

Axioms

- 1 the axioms of (reflexivity), (symmetry), and (transitivity) of T_E
- 2 $\forall a, i, j. i = j \rightarrow a[i] = a[j]$ (array congruence)
- 3 $\forall a, v, i, j. i = j \rightarrow a\langle i \triangleleft v \rangle[j] = v$ (read-over-write 1)
- 4 $\forall a, v, i, j. i \neq j \rightarrow a\langle i \triangleleft v \rangle[j] = a[j]$ (read-over-write 2)

Note: $=$ is only defined for array elements

$$a[i] = e \rightarrow a\langle i \triangleleft e \rangle = a$$

not T_A -valid, but

$$a[i] = e \rightarrow \forall j. a\langle i \triangleleft e \rangle[j] = a[j] ,$$

is T_A -valid.

Also

$$a = b \rightarrow a[i] = b[i]$$

is not T_A -valid: We only axiomatized a restricted congruence.

T_A is undecidable

Quantifier-free fragment of T_A is decidable

Signature and axioms of $T_A^=$ are the same as T_A , with one additional axiom

$$\forall a, b. (\forall i. a[i] = b[i]) \leftrightarrow a = b \quad (\text{extensionality})$$

Example:

$$F : a[i] = e \rightarrow a\langle i \triangleleft e \rangle = a$$

is $T_A^=$ -valid.

$T_A^=$ is undecidable

Quantifier-free fragment of $T_A^=$ is decidable

How do we show that

$$1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$$

is $(T_E \cup T_{\mathbb{Z}})$ -unsatisfiable?

Or how do we prove properties about
an array of integers, or
a list of reals ... ?

Given theories T_1 and T_2 such that

$$\Sigma_1 \cap \Sigma_2 = \{=\}$$

The **combined theory** $T_1 \cup T_2$ has

- signature $\Sigma_1 \cup \Sigma_2$
- axioms $A_1 \cup A_2$

qff = quantifier-free fragment

Nelson & Oppen showed that

if satisfiability of qff of T_1 is decidable,
satisfiability of qff of T_2 is decidable, and
certain technical requirements are met
then satisfiability of qff of $T_1 \cup T_2$ is decidable.

$$T_{\text{cons}}^= : T_E \cup T_{\text{cons}}$$

Signature: $\Sigma_E \cup \Sigma_{\text{cons}}$

(this includes uninterpreted constants, functions, and predicates)

Axioms: union of the axioms of T_E and T_{cons}

$T_{\text{cons}}^=$ is undecidable

Quantifier-free fragment of $T_{\text{cons}}^=$ is efficiently decidable

We argue that the following $\Sigma_{\text{cons}}^=$ -formula F is $T_{\text{cons}}^=$ -valid:

$$F : \quad \text{car}(a) = \text{car}(b) \wedge \text{cdr}(a) = \text{cdr}(b) \wedge \neg \text{atom}(a) \wedge \neg \text{atom}(b) \\ \rightarrow f(a) = f(b)$$

1. $I \not\models F$ assumption
2. $I \models \text{car}(a) = \text{car}(b)$ 1, \rightarrow , \wedge
3. $I \models \text{cdr}(a) = \text{cdr}(b)$ 1, \rightarrow , \wedge
4. $I \models \neg \text{atom}(a)$ 1, \rightarrow , \wedge
5. $I \models \neg \text{atom}(b)$ 1, \rightarrow , \wedge
6. $I \not\models f(a) = f(b)$ 1, \rightarrow
7. $I \models \text{cons}(\text{car}(a), \text{cdr}(a)) = \text{cons}(\text{car}(b), \text{cdr}(b))$
2, 3, (congruence)
8. $I \models \text{cons}(\text{car}(a), \text{cdr}(a)) = a$ 4, (construction)
9. $I \models \text{cons}(\text{car}(b), \text{cdr}(b)) = b$ 5, (construction)
10. $I \models a = b$ 7, 8, 9, (transitivity)
11. $I \models f(a) = f(b)$ 10, (congruence)

Lines 6 and 11 are contradictory. Therefore, F is $T_{\text{cons}}^=$ -valid.

	Theory	Decidable	QFF Dec.
T_E	Equality	—	✓
T_{PA}	Peano Arithmetic	—	—
$T_{\mathbb{N}}$	Presburger Arithmetic	✓	✓
$T_{\mathbb{Z}}$	Linear Integer Arithmetic	✓	✓
$T_{\mathbb{R}}$	Real Arithmetic	✓	✓
$T_{\mathbb{Q}}$	Linear Rationals	✓	✓
T_{cons}	Lists	—	✓
$T_{\text{cons}}^=$	Lists with Equality	—	✓
T_A	Arrays	—	✓
$T_A^=$	Arrays with Extensionality	—	✓