Decision Procedures

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Nelson-Oppen Theory Combination

Motivation: How do we show that

$$F\ :\ 1 \leq x \ \wedge\ x \leq 2 \ \wedge\ f(x) \neq f(1) \ \wedge\ f(x) \neq f(2)$$

is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable?

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Given

Multiple Theories T_i over signatures Σ_i
(constants, functions, predicates)
with corresponding decision procedures P_i for T_i-satisfiability.

Goal

Decide satisfiability of a sentence in theory $\cup_i T_i$.

Nelson-Oppen Combination Method (N-O Method)

$$\Sigma_1 \cap \Sigma_2 = \{=\}$$

Σ_1-theory T_1

$\boxed{P_1}$ for T_1-satisfiability
of quantifier-free Σ_1-formulae

Σ_2-theory T_2

$\boxed{P_2}$ for T_2-satisfiability
of quantifier-free Σ_2-formulae

$\boxed{P}$ for $(T_1 \cup T_2)$-satisfiability
of quantifier-free $(\Sigma_1 \cup \Sigma_2)$-formulae

We show how to get Procedure P from Procdures P_1 and P_2.

Nelson-Oppen: Limitations

Given formula F in theory $T_1 \cup T_2$.

1. F must be quantifier-free.
2. Signatures Σ_i of the combined theory only share $=$, i.e.,

$$\Sigma_1 \cap \Sigma_2 = \{=\}$$

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Note:

- Algorithm can be extended to combine arbitrary number of theories T_i — combine two, then combine with another, and so on.
- We restrict F to be conjunctive formula — otherwise convert to DNF and check each disjunct.

Stably Infinite Theories

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Definition (stably infinite)

A Σ-theory T is stably infinite iff
for every quantifier-free Σ-formula F:
if F is T-satisfiable
then there exists some infinite T-interpretation that satisfies F
with infinite cardinality.

Example: Stably Infinite

- $T_{\mathbb{Z}}$:

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- $T_{\mathbb{Z}}$: stably infinite (all T-interpretations are infinite).
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- T_{E}: stably infinite (one can add infinitely many fresh and distinct values).
- Σ-theory T with $\Sigma : \{a, b, =\}$ and axiom $\forall x.\ x = a \ \vee \ x = b$: not stable infinite,
  since every T-interpretation has at most two elements.

Example: Σ_E and $\Sigma_{\mathbb{Z}}$

Consider quantifier-free conjunctive $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \; 1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2) \; .$$

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The signatures of T_E and $T_{\mathbb{Z}}$ only share $=$. Also, both theories are stably infinite. Hence, the NO combination of the decision procedures for T_E and $T_{\mathbb{Z}}$ decides the $(T_E \cup T_{\mathbb{Z}})$-satisfiability of F.

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F is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable:
The first two literals imply $x = 1 \vee x = 2$ so that
$f(x) = f(1) \vee f(x) = f(2)$. This contradicts last two literals.

N-O Overview

Phase 1: Variable Abstraction

- Given conjunction Γ in theory $T_1 \cup T_2$.
- Convert to conjunction $\Gamma_1 \cup \Gamma_2$ s.t.
  - Γ_i in theory T_i
  - $\Gamma_1 \cup \Gamma_2$ satisfiable iff Γ satisfiable.

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Phase 2: Check

- If there is some set S of equalities and disequalities between the shared variables of Γ_1 and Γ_2
  $\text{shared}(\Gamma_1, \Gamma_2) = \text{free}(\Gamma_1) \cap \text{free}(\Gamma_2)$
  s.t. $S \cup \Gamma_i$ are T_i-satisfiable for all i,
  then Γ is **satisfiable**.
- Otherwise, **unsatisfiable**.

Nelson-Oppen Method: Overview

Consider quantifier-free conjunctive $(\Sigma_1 \cup \Sigma_2)$-formula F.

Two versions:

- nondeterministic — simple to present, but high complexity
- deterministic — efficient

Nelson-Oppen (N-O) method proceeds in two steps:

- Phase 1 (variable abstraction)
  — same for both versions
- Phase 2
  nondeterministic: guess equalities/disequalities and check
  deterministic: generate equalities/disequalities by equality propagation

Given quantifier-free conjunctive $(\Sigma_1 \cup \Sigma_2)$-formula F.
Transform F into two quantifier-free conjunctive formulae

$$\Sigma_1\text{-formula } F_1 \qquad \text{and} \qquad \Sigma_2\text{-formula } F_2$$

s.t. F is $(T_1 \cup T_2)$-satisfiable iff $F_1 \wedge F_2$ is $(T_1 \cup T_2)$-satisfiable

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F_1 and F_2 are linked via a set of shared variables.

For term t, let $\mathrm{hd}(t)$ be the root symbol, e.g. $\mathrm{hd}(f(x)) = f$.

For $i, j \in \{1, 2\}$ and $i \neq j$, repeat the transformations

1. if function $f \in \Sigma_i$ and $\text{hd}(t) \in \Sigma_j$,

   $$F[f(t_1, \ldots, t, \ldots, t_n)] \quad \textit{eqsat}.$$

2. if predicate $p \in \Sigma_i$ and $\text{hd}(t) \in \Sigma_j$,

   $$F[p(t_1, \ldots, t, \ldots, t_n)] \quad \textit{eqsat}.$$

3. if $\text{hd}(s) \in \Sigma_i$ and $\text{hd}(t) \in \Sigma_j$,

   $$F[s = t] \quad \textit{eqsat}.$$

4. if $\text{hd}(s) \in \Sigma_i$ and $\text{hd}(t) \in \Sigma_j$,

   $$F[s \neq t] \quad \textit{eqsat}.$$

where w, w_1, and w_2 are fresh variables.

Generation of F_1 and F_2

For $i, j \in \{1, 2\}$ and $i \neq j$, repeat the transformations

1. if function $f \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[f(t_1, \dots, t, \dots, t_n)] \quad eqsat. \quad F[f(t_1, \dots, w, \dots, t_n)] \land w = t$$

2. if predicate $p \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[p(t_1, \dots, t, \dots, t_n)] \quad eqsat. \quad F[p(t_1, \dots, w, \dots, t_n)] \land w = t$$

3. if $\mathrm{hd}(s) \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[s = t] \quad eqsat. \quad F[\top] \land w = s \land w = t$$

4. if $\mathrm{hd}(s) \in \Sigma_i$ and $\mathrm{hd}(t) \in \Sigma_j$,

$$F[s \neq t] \quad eqsat. \quad F[w_1 \neq w_2] \land w_1 = s \land w_2 = t$$

where w, w_1, and w_2 are fresh variables.

Example: Phase 1

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \ 1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2) \ .$$

Example: Phase 1

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : 1 \leq x \land x \leq 2 \land f(x) \neq f(1) \land f(x) \neq f(2) .$$

According to transformation 1, since $f \in \Sigma_E$ and $1 \in \Sigma_{\mathbb{Z}}$, replace $f(1)$ by $f(w_1)$ and add $w_1 = 1$. Similarly, replace $f(2)$ by $f(w_2)$ and add $w_2 = 2$.

Example: Phase 1

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

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According to transformation 1, since $f \in \Sigma_E$ and $1 \in \Sigma_{\mathbb{Z}}$, replace $f(1)$ by $f(w_1)$ and add $w_1 = 1$. Similarly, replace $f(2)$ by $f(w_2)$ and add $w_2 = 2$. Now, the literals

$$\Gamma_{\mathbb{Z}} : \; \{1 \leq x, \; x \leq 2, \; w_1 = 1, \; w_2 = 2\}$$

are $T_{\mathbb{Z}}$-literals, while the literals

$$\Gamma_E : \; \{f(x) \neq f(w_1), \; f(x) \neq f(w_2)\}$$

are T_E-literals. Hence, construct the $\Sigma_{\mathbb{Z}}$-formula

$$F_1 : \; 1 \leq x \wedge x \leq 2 \wedge w_1 = 1 \wedge w_2 = 2$$

and the Σ_E-formula

$$F_2 : \; f(x) \neq f(w_1) \wedge f(x) \neq f(w_2) \; .$$

F_1 and F_2 share the variables $\{x, w_1, w_2\}$.
$F_1 \wedge F_2$ is $(T_E \cup T_{\mathbb{Z}})$-equisatisfiable to F.

Example: Phase 1

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \; f(x) = x + y \wedge x \leq y + z \wedge x + z \leq y \wedge y = 1 \wedge f(x) \neq f(2) \, .$$

Example: Phase 1

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \; f(x) = x + y \land x \leq y + z \land x + z \leq y \land y = 1 \land f(x) \neq f(2) \; .$$

In the first literal, $\mathrm{hd}(f(x)) = f \in \Sigma_{\mathrm{E}}$ and $\mathrm{hd}(x + y) = + \in \Sigma_{\mathbb{Z}}$; thus, by (3), replace the literal with

$$w_1 = f(x) \land w_1 = x + y \; .$$

In the final literal, $f \in \Sigma_E$ but $2 \in \Sigma_{\mathbb{Z}}$, so by (1), replace it with

$$f(x) \neq f(w_2) \land w_2 = 2 \; .$$

Now, separating the literals results in two formulae:

$$F_1 : \; w_1 = x + y \land x \leq y + z \land x + z \leq y \land y = 1 \land w_2 = 2$$

is a $\Sigma_{\mathbb{Z}}$-formula, and

$$F_2 : \; w_1 = f(x) \land f(x) \neq f(w_2)$$

is a Σ_E-formula.
The conjunction $F_1 \land F_2$ is $(T_E \cup T_{\mathbb{Z}})$-equisatisfiable to F.

Phase 2: Guess and Check (Nondeterministic)

- Phase 1 separated $(\Sigma_1 \cup \Sigma_2)$-formula F into two formulae:
  Σ_1-formula F_1 and Σ_2-formula F_2
- F_1 and F_2 are linked by a set of shared variables:
  $V = \mathrm{shared}(F_1, F_2) = \mathrm{free}(F_1) \cap \mathrm{free}(F_2)$
- Let E be an equivalence relation over V.
- The arrangement $\alpha(V, E)$ of V induced by E is:

$$\alpha(V, E) : \bigwedge_{u,v \,\in\, V.\ uEv} u = v \wedge \bigwedge_{u,v \,\in\, V.\ \neg(uEv)} u \neq v$$

Correctness of Phase 2

Lemma

The original formula F is $(T_1 \cup T_2)$-satisfiable iff
there exists an equivalence relation E of V s.t.
(1) $F_1 \wedge \alpha(V, E)$ is T_1-satisfiable, and
(2) $F_2 \wedge \alpha(V, E)$ is T_2-satisfiable.

Lemma

The original formula F is $(T_1 \cup T_2)$-satisfiable iff there exists an equivalence relation E of V s.t.
(1) $F_1 \wedge \alpha(V, E)$ is T_1-satisfiable, and
(2) $F_2 \wedge \alpha(V, E)$ is T_2-satisfiable.

Proof:

$\Rightarrow$ If F is $(T_1 \cup T_2)$-satisfiable, then $F_1 \wedge F_2$ is $(T_1 \cup T_2)$-satisfiable, hence there is a $T_1 \cup T_2$-Interpretation I with $I \models F_1 \wedge F_2$.

Define $E \subseteq V \times V$ with $u \ E \ v$ iff $I \models u = v$.
Then E is a equivalence relation.
By definition of E and $\alpha(V, E)$, $I \models \alpha(V, E)$.
Hence $I \models F_1 \wedge \alpha(V, E)$ and $I \models F_2 \wedge \alpha(V, E)$.
Thus, these formulae are T_1- and T_2-satisfiable, respectively.

$\Leftarrow$ Let I_1 and I_2 be T_1- and T_2-interpretations, respectively, with

$$I_1 \models F_1 \wedge \alpha(V, E) \text{ and } I_2 \models F_2 \wedge \alpha(V, E).$$

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W.l.o.g. assume that $\alpha_{I_1}[=](v, w)$ iff $v = w$ iff $\alpha_{I_2}[=](v, w)$.
(Otherwise, replace D_{I_i} with $D_{I_i}/\alpha_{I_i}[=]$)

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Since $I_1 \models \alpha(V, E)$ and $I_2 \models \alpha(V, E)$, for $x, y \in V$:

$$\alpha_{I_1}[x] = \alpha_{I_1}[y] \text{ iff } \alpha_{I_2}[x] = \alpha_{I_2}[y].$$

$\Leftarrow$ Let I_1 and I_2 be T_1- and T_2-interpretations, respectively, with

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Since $I_1 \models \alpha(V, E)$ and $I_2 \models \alpha(V, E)$, for $x, y \in V$:

$$\alpha_{I_1}[x] = \alpha_{I_1}[y] \text{ iff } \alpha_{I_2}[x] = \alpha_{I_2}[y].$$

Construct bijective function $g : D_{I_1} \to D_{I_2}$ with $g(\alpha_{I_1}[x]) = \alpha_{I_2}[x]$ for all $x \in V$. Define I as follows: $D_I = D_{I_2}$,
$\alpha_I[x] = \alpha_{I_2}[x] (= g(\alpha_{I_1}[x]))$ for $x \in V$,
$\alpha_I[=](v, w)$ iff $v = w$,
$\alpha_I[f_2] = \alpha_{I_2}[f_2]$ for $f_2 \in \Sigma_2$,
$\alpha_I[f_1](v_1, \ldots, v_n) = g(\alpha_{I_1}[f_1](g^{-1}(v_1), \ldots, g^{-1}(v_n)))$ for $f_1 \in \Sigma_1$.

$\Leftarrow$ Let I_1 and I_2 be T_1- and T_2-interpretations, respectively, with

$$I_1 \models F_1 \wedge \alpha(V, E) \text{ and } I_2 \models F_2 \wedge \alpha(V, E).$$

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Since T_1 and T_2 are stably infinite, we can assume that D_{I_1} and D_{I_2} are of the same cardinality.

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$$\alpha_{I_1}[x] = \alpha_{I_1}[y] \text{ iff } \alpha_{I_2}[x] = \alpha_{I_2}[y].$$

Construct bijective function $g : D_{I_1} \to D_{I_2}$ with $g(\alpha_{I_1}[x]) = \alpha_{I_2}[x]$ for all $x \in V$. Define I as follows: $D_I = D_{I_2}$,
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Then I is a $T_1 \cup T_2$-interpretation, and satisfies $F_1 \wedge F_2$.
Hence F is $T_1 \cup T_2$-satisfiable.

Example: Phase 2

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \; 1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$$

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Phase 1 separates this formula into the $\Sigma_{\mathbb{Z}}$-formula

$$F_1 : \ 1 \leq x \wedge x \leq 2 \wedge w_1 = 1 \wedge w_2 = 2$$

and the Σ_E-formula

$$F_2 : \ f(x) \neq f(w_1) \wedge f(x) \neq f(w_2)$$

with

$$V = \text{shared}(F_1, F_2) = \{x, w_1, w_2\}$$

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There are 5 equivalence relations to consider, which we list by stating the partitions:

Example: Phase 2 (cont)

1. $\{\{x, w_1, w_2\}\}$, *i.e.*, $x = w_1 = w_2$:
   $x = w_1$ and $f(x) \neq f(w_1) \Rightarrow F_2 \wedge \alpha(V, E)$ is T_E-unsatisfiable.
2. $\{\{x, w_1\}, \{w_2\}\}$, *i.e.*, $x = w_1$, $x \neq w_2$:
   $x = w_1$ and $f(x) \neq f(w_1) \Rightarrow F_2 \wedge \alpha(V, E)$ is T_E-unsatisfiable.
3. $\{\{x, w_2\}, \{w_1\}\}$, *i.e.*, $x = w_2$, $x \neq w_1$:
   $x = w_2$ and $f(x) \neq f(w_2) \Rightarrow F_2 \wedge \alpha(V, E)$ is T_E-unsatisfiable.
4. $\{\{x\}, \{w_1, w_2\}\}$, *i.e.*, $x \neq w_1$, $w_1 = w_2$:
   $w_1 = w_2$ and $w_1 = 1 \wedge w_2 = 2$
   $\Rightarrow F_1 \wedge \alpha(V, E)$ is $T_{\mathbb{Z}}$-unsatisfiable.
5. $\{\{x\}, \{w_1\}, \{w_2\}\}$, *i.e.*, $x \neq w_1$, $x \neq w_2$, $w_1 \neq w_2$:
   $x \neq w_1 \wedge x \neq w_2$ and $x = w_1 = 1 \vee x = w_2 = 2$
   (since $1 \leq x \leq 2$ implies that $x = 1 \vee x = 2$ in $T_{\mathbb{Z}}$)
   $\Rightarrow F_1 \wedge \alpha(V, E)$ is $T_{\mathbb{Z}}$-unsatisfiable.

Hence, F is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.

Example: Phase 2 (cont)

Consider the $(\Sigma_{\text{cons}} \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \ \text{car}(x) + \text{car}(y) = z \wedge \text{cons}(x, z) \neq \text{cons}(y, z) \ .$$

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$$F : \ \text{car}(x) + \text{car}(y) = z \wedge \text{cons}(x, z) \neq \text{cons}(y, z) \ .$$

After two applications of (1), Phase 1 separates F into the Σ_{cons}-formula

$$F_1 : \ w_1 = \text{car}(x) \wedge w_2 = \text{car}(y) \wedge \text{cons}(x, z) \neq \text{cons}(y, z)$$

and the $\Sigma_{\mathbb{Z}}$-formula

$$F_2 : \ w_1 + w_2 = z \ ,$$

with

$$V = \text{shared}(F_1, F_2) = \{z, w_1, w_2\} \ .$$

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and the $\Sigma_{\mathbb{Z}}$-formula

$$F_2 \;:\; w_1 + w_2 = z \;,$$

with

$$V = \mathsf{shared}(F_1, F_2) = \{z, w_1, w_2\} \;.$$

Consider the equivalence relation E given by the partition

$$\{\{z\}, \{w_1\}, \{w_2\}\} \;.$$

Example: Phase 2 (cont)

Consider the $(\Sigma_{\mathrm{cons}} \cup \Sigma_{\mathbb{Z}})$-formula

$$F\ :\ \mathrm{car}(x) + \mathrm{car}(y) = z \wedge \mathrm{cons}(x, z) \neq \mathrm{cons}(y, z)\ .$$

After two applications of (1), Phase 1 separates F into the Σ_{cons}-formula

$$F_1\ :\ w_1 = \mathrm{car}(x) \wedge w_2 = \mathrm{car}(y) \wedge \mathrm{cons}(x, z) \neq \mathrm{cons}(y, z)$$

and the $\Sigma_{\mathbb{Z}}$-formula

$$F_2\ :\ w_1 + w_2 = z\ ,$$

with

$$V = \mathrm{shared}(F_1, F_2) = \{z, w_1, w_2\}\ .$$

Consider the equivalence relation E given by the partition

$$\{\{z\}, \{w_1\}, \{w_2\}\}\ .$$

The arrangement

$$\alpha(V, E)\ :\ z \neq w_1 \wedge z \neq w_2 \wedge w_1 \neq w_2$$

satisfies both F_1 and F_2: $F_1 \wedge \alpha(V, E)$ is T_{cons}-satisfiable, and $F_2 \wedge \alpha(V, E)$ is $T_{\mathbb{Z}}$-satisfiable.

Hence, F is $(T_{\mathrm{cons}} \cup T_{\mathbb{Z}})$-satisfiable.

Practical Efficiency

Phase 2 was formulated as "guess and check":
First, guess an equivalence relation E,
then check the induced arrangement.

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e.g., 12 shared variables $\Rightarrow$ over four million equivalence relations.

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Solution: Deterministic Version

Phase 1 as before

Phase 2 asks the decision procedures P_1 and P_2 to propagate new equalities.

Example 1:

Real linear arithmetic $T_{\mathbb{R}}$	Theory of equality T_E
$\boxed{P_{\mathbb{R}}}$	$\boxed{P_E}$

$$F : \quad f(f(x)-f(y)) \neq f(z) \ \wedge \ x \leq y \ \wedge \ y + z \leq x \ \wedge \ 0 \leq z$$

Phase 1: Variable Abstraction

$$F : \; f(f(x) - f(y)) \neq f(z) \;\wedge\; x \leq y \;\wedge\; y + z \leq x \;\wedge\; 0 \leq z$$

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$$f(x) \;\Rightarrow\; u \qquad f(y) \;\Rightarrow\; v \qquad u - v \;\Rightarrow\; w$$

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$$f(x)\ \Rightarrow\ u \qquad f(y)\ \Rightarrow\ v \qquad u - v\ \Rightarrow\ w$$

$\Gamma_E\ :\quad \{f(w) \neq f(z),\ u = f(x),\ v = f(y)\}$ $\quad \ldots T_E$-formula

$\Gamma_{\mathbb{R}}\ :\quad \{x \leq y,\ y + z \leq x,\ 0 \leq z,\ w = u - v\}$ $\quad \ldots T_{\mathbb{R}}$-formula

$$\text{shared}(\Gamma_{\mathbb{R}}, \Gamma_E) = \{x, y, z, u, v, w\}$$

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Nondeterministic version — over 200 Es!
Let's try the deterministic version.

Phase 2: Equality Propagation

$\boxed{P_{\mathbb{R}}}$ $\quad s_0 : \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{\} \rangle \quad$ $\boxed{P_E}$

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$\Gamma_{\mathbb{R}} \models x = y$

$$s_1 : \langle \Gamma_{\mathbb{R}}, \Gamma_E, \{x = y\} \rangle$$

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$\Gamma_E \cup \{z = w\} \models \text{false}$

$$s_4 : \text{false}$$

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Contradiction. Thus, F is $(T_{\mathbb{R}} \cup T_E)$-unsatisfiable.

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Contradiction. Thus, F is $(T_{\mathbb{R}} \cup T_E)$-unsatisfiable.

If there were no contradiction, F would be $(T_{\mathbb{R}} \cup T_E)$-satisfiable.

Convex Theories

Definition (convex theory)

A Σ-theory T is convex iff
for every quantifier-free conjunction Σ-formula F
and for every disjunction $\bigvee_{i=1}^{n}(u_i = v_i)$

if $F \models \bigvee_{i=1}^{n}(u_i = v_i)$
then $F \models u_i = v_i$, for some $i \in \{1, \ldots, n\}$

Claim

Equality propagation is a decision procedure for convex theories.

Convex Theories

- T_E, $T_{\mathbb{R}}$, $T_{\mathbb{Q}}$, T_{cons} are convex

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Example: $T_{\mathbb{Z}}$ is not convex

Consider quantifier-free conjunctive

$$F : \quad 1 \leq z \ \wedge \ z \leq 2 \ \wedge \ u = 1 \ \wedge \ v = 2$$

- T_E, $T_{\mathbb{R}}$, $T_{\mathbb{Q}}$, T_{cons} are convex
- $T_{\mathbb{Z}}$, T_{A} are not convex

Example: $T_{\mathbb{Z}}$ is not convex

Consider quantifier-free conjunctive

$$F: \quad 1 \leq z \ \wedge \ z \leq 2 \ \wedge \ u = 1 \ \wedge \ v = 2$$

Then

$$F \models z = u \vee z = v$$

but

$$F \not\models z = u$$
$$F \not\models z = v$$

Example:

The theory of arrays T_{A} is not convex.

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Consider the quantifier-free conjunctive Σ_A-formula

$$F : \; a\langle i \triangleleft v\rangle[j] = v \; .$$

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Consider the quantifier-free conjunctive Σ_A-formula

$$F \;:\; a\langle i \triangleleft v\rangle[j] = v \;.$$

Then

$$F \Rightarrow i = j \vee a[j] = v \;,$$

but

$$\begin{aligned} F &\not\Rightarrow i = j \\ F &\not\Rightarrow a[j] = v \;. \end{aligned}$$

What if T is Not Convex?

Case split when:

$$\Gamma \models \bigvee_{i=1}^{n} (u_i = v_i)$$

but

$$\Gamma \not\models u_i = v_i \qquad \text{for all } i = 1, \ldots, n$$

What if T is Not Convex?

Case split when:

$$\Gamma \models \bigvee_{i=1}^{n} (u_i = v_i)$$

but

$$\Gamma \not\models u_i = v_i \qquad \text{for all } i = 1, \dots, n$$

- For each $i = 1, \dots, n$, construct a branch on which $u_i = v_i$ is assumed.
- If all branches are contradictory, then **unsatisfiable**. Otherwise, **satisfiable**.

Example 2: Non-Convex Theory

$T_{\mathbb{Z}}$ not convex! $\boxed{P_{\mathbb{Z}}}$

T_E convex $\boxed{P_E}$

$$\Gamma : \left\{ \begin{array}{ll} 1 \leq x, & x \leq 2, \\ f(x) \neq f(1), & f(x) \neq f(2) \end{array} \right\} \quad \text{in } T_{\mathbb{Z}} \cup T_E$$

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- Replace $f(1)$ by $f(w_1)$, and add $w_1 = 1$.
- Replace $f(2)$ by $f(w_2)$, and add $w_2 = 2$.

Result:

$$\Gamma_{\mathbb{Z}} = \left\{ \begin{array}{l} 1 \leq x, \\ x \leq 2, \\ w_1 = 1, \\ w_2 = 2 \end{array} \right\} \quad \text{and} \quad \Gamma_E = \left\{ \begin{array}{l} f(x) \neq f(w_1), \\ f(x) \neq f(w_2) \end{array} \right\}$$

$$\text{shared}(\Gamma_{\mathbb{Z}}, \Gamma_E) = \{x, w_1, w_2\}$$

Example 2: Non-Convex Theory

$$s_0 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{\} \rangle$$
$$\Gamma_{\mathbb{Z}} \models x = w_1 \ \vee \ x = w_2$$

$x = w_1$ $\qquad\qquad$ $x = w_2$

$$s_1 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_1\} \rangle \qquad s_3 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_2\} \rangle$$
$$\Gamma_E \cup \{x = w_1\} \models \bot \qquad \Gamma_E \cup \{x = w_2\} \models \bot$$

$$s_2 : \bot \qquad\qquad s_4 : \bot$$

All leaves are labeled with $\bot \Rightarrow \Gamma$ is $(T_{\mathbb{Z}} \cup T_E)$-unsatisfiable.

Example 3: Non-Convex Theory

$$\Gamma : \left\{ \begin{array}{c} 1 \leq x, \quad x \leq 3, \\ f(x) \neq f(1),\ f(x) \neq f(3),\ f(1) \neq f(2) \end{array} \right\} \quad \text{in } T_{\mathbb{Z}} \cup T_E$$

Example 3: Non-Convex Theory

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- Replace $f(1)$ by $f(w_1)$, and add $w_1 = 1$.
- Replace $f(2)$ by $f(w_2)$, and add $w_2 = 2$.
- Replace $f(3)$ by $f(w_3)$, and add $w_3 = 3$.

Result:

$$\Gamma_{\mathbb{Z}} = \left\{ \begin{array}{l} 1 \leq x, \\ x \leq 3, \\ w_1 = 1, \\ w_2 = 2, \\ w_3 = 3 \end{array} \right\} \quad \text{and} \quad \Gamma_E = \left\{ \begin{array}{l} f(x) \neq f(w_1), \\ f(x) \neq f(w_3), \\ f(w_1) \neq f(w_2) \end{array} \right\}$$

$$\text{shared}(\Gamma_{\mathbb{Z}}, \Gamma_E) = \{x, w_1, w_2, w_3\}$$

Example 3: Non-Convex Theory

$$s_0 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{\} \rangle$$
$$\Gamma_{\mathbb{Z}} \models x = w_1 \lor x = w_2 \lor x = w_3$$

$x = w_1$ $\qquad$ $x = w_2$ $\qquad$ $x = w_3$

$s_1 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_1\} \rangle$ $\quad$ $s_3 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_2\} \rangle$ $\quad$ $s_4 : \langle \Gamma_{\mathbb{Z}}, \Gamma_E, \{x = w_3\} \rangle$

$\Gamma_E \cup \{x = w_1\} \models \bot$ $\qquad$ $\Gamma_E \cup \{x = w_3\} \models \bot$

$s_2 : \bot$ $\qquad$ $s_5 : \bot$

No more equations on middle leaf $\Rightarrow \Gamma$ is $(T_{\mathbb{Z}} \cup T_E)$-satisfiable.