Timed automata

- Measure time: finite set $C$ of clocks $x, y, z, \ldots$
- Clocks increase their value implicitly as time progresses
- All clocks proceed at rate 1
Timed automata

- Measure time: finite set $C$ of clocks $x, y, z, \ldots$
- Clocks increase their value implicitly as time progresses
- All clocks proceed at rate 1
- Limited clock access:

Reading: Clock constraints

\[
g ::= x < c \mid x \leq c \mid x > c \mid x \geq c \mid g \land g
\]

with $c \in \mathbb{N}$ ($c \in \mathbb{Q}$) and $x \in C$.

Syntactic sugar: $true$, $x \in [c_1, c_2)$, $c_1 \leq x < c_2$, $x = c$, ...

\[ACC(C): \text{set of atomic clock constraints over } C\]

\[CC(C): \text{set of clock constraints over } C\]

Writing: Clock reset sets value to 0
Semantics of clock constraints

Given a set $C$ of clocks, a clock valuation $\nu : C \rightarrow \mathbb{R}_{\geq 0}$ assigns a non-negative value to each clock. We use $V_C$ to denote the set of clock valuations for the clock set $C$.

**Definition**

For a set $C$ of clocks, $x \in C$, $\nu \in V_C$, $c \in \mathbb{N}$, and $g, g' \in CC(C)$, let $\models \subseteq V_C \times CC(C)$ be defined by

\[
\begin{align*}
\nu \models x < c & \text{ iff } \nu(x) < c \\
\nu \models x \leq c & \text{ iff } \nu(x) \leq c \\
\nu \models x > c & \text{ iff } \nu(x) > c \\
\nu \models x \geq c & \text{ iff } \nu(x) \geq c \\
\nu \models g \land g' & \text{ iff } \nu \models g \text{ and } \nu \models g'
\end{align*}
\]
Semantics of clock access

Definition

- For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in C$.

- For a valuation $\nu \in V_C$ and a clock $x \in C$ we define $\text{reset } x \text{ in } \nu$ to be the valuation which equals $\nu$ except on $x$ whose value is 0:

$$ (\text{reset } x \text{ in } \nu)(y) = \begin{cases} 
\nu(y) & \text{if } y \neq x \\ 
0 & \text{else}
\end{cases} $$
Semantics of clock access

Definition

For a set $C$ of clocks, $\nu \in V_C$, and $c \in \mathbb{N}$ we denote by $\nu + c$ the valuation with $(\nu + c)(x) = \nu(x) + c$ for all $x \in C$.

For a valuation $\nu \in V_C$ and a clock $x \in C$ we define reset $x$ in $\nu$ to be the valuation which equals $\nu$ except on $x$ whose value is 0:

$$(\text{reset } x \text{ in } \nu)(y) = \begin{cases} 
\nu(y) & \text{if } y \neq x \\
0 & \text{else}
\end{cases}$$

What does it mean?

- $\nu + 9$
- reset $x$ in $(\nu + 9)$
- $(\text{reset } x \text{ in } \nu) + 9$
- reset $x$ in (reset $y$ in $\nu$)
Timed automata

A **timed automaton** is a special hybrid system:

- All variables are **clocks**.
- **Edges** are defined by
  - source and target locations,
  - a label,
  - a **guard**: clock constraint specifying enabling,
  - a set of clocks to be **reset**.
- **Invariants** are clock constraints.
Timed automaton

Definition (Syntax of timed automata)

A timed automaton \( T = (\text{Loc}, \mathcal{C}, \text{Lab}, \text{Edge}, \text{Inv}, \text{Init}) \) is a tuple with

- \( \text{Loc} \) is a finite set of locations,
- \( \mathcal{C} \) is a finite set of clocks,
- \( \text{Lab} \) is a finite set of synchronization labels,
- \( \text{Edge} \subseteq \text{Loc} \times \text{Lab} \times (\mathcal{C}(\mathcal{C}) \times 2^\mathcal{C}) \times \text{Loc} \) is a finite set of edges,
- \( \text{Inv} : \text{Loc} \rightarrow \mathcal{C}(\mathcal{C}) \) is a function assigning an invariant to each location, and
- \( \text{Init} \subseteq \Sigma \) with \( \nu(x) = 0 \) for all \( x \in \mathcal{C} \) and all \( (l, \nu) \in \text{Init} \).

We call the variables in \( \mathcal{C} \) clocks. We also use the notation \( l \xrightarrow{a:g,C} l' \) to state that there exists an edge \((l, a, (g, C'), l') \in \text{Edge}\).

Note: (1) no explicite activities given (2) restricted logic for constraints
Analogously to Kripke structures, we can additionally define
- a set of atomic propositions $AP$ and
- a labeling function $L : Loc \rightarrow 2^{AP}$

to model further system properties.
Operational semantics

\[(l, a, (g, R), l') \in Edge\]
\[\nu \models g \quad \nu' = \text{reset } R \text{ in } \nu \quad \nu' \models Inv(l')\]

\[\frac{}{(l, \nu) \xrightarrow{a} (l', \nu')}\]

Rule \text{Discrete}

\[t > 0 \quad \nu' = \nu + t \quad \nu' \models Inv(l)\]

\[\frac{}{(l, \nu) \xrightarrow{t} (l, \nu')}\]

Rule \text{Time}
Operational semantics

\[(l, a, (g, \mathcal{R}), l') \in \text{Edge} \]
\[
\nu \models g \quad \nu' = \text{reset } \mathcal{R} \text{ in } \nu \quad \nu' \models \text{Inv}(l') \]

\[\frac{(l, \nu) \xrightarrow{a} (l', \nu')}{} \]

Rule \text{Discrete}

\[t > 0 \quad \nu' = \nu + t \quad \nu' \models \text{Inv}(l) \]

\[\frac{(l, \nu) \xrightarrow{t} (l, \nu')}{} \]

Rule \text{Time}

- **Execution step:** \(\rightarrow = \xrightarrow{a} \cup \xrightarrow{t}\)
- **Path:** \(\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots\)
- **Run:** path \(\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \ldots\) with \(\sigma_0 = (l_0, \nu_0), \ l_0 \in \text{Init}, \ \nu_0(x) = 0\) f.a. \(x \in \mathcal{C}\) and \(\nu_0 \in \text{Inv}(l_0)\)
- **Reachability** of a state: exists a run leading to the state
Examples:

- Light switch
- Controller from the railroad crossing example
- Simplified railroad crossing
- Parallel composition for the simplified railroad crossing
Example: Timed Automaton

\[ x \geq 2, \ \text{reset}(x) \]
Example: Timed Automaton

$x \geq 2, \text{reset}(x)$

$q_1$
Example: Timed Automaton

$x \geq 2$, \(\text{reset}(x)\)
Example: Timed Automaton

\[ x \geq 2, \ \text{reset}(x) \]
Example: Timed Automaton

$2 \leq x \leq 3$, reset($x$)
In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point where the pursued started, so that the slower must always hold a lead.”

—Aristotle, Physics VI:9, 239b15

- Not all paths of a timed automata represent realistic behaviour.
- Three essential phenomena: time convergence, timelock, zenoness.
Definition

For a timed automaton \( \mathcal{T} = (\text{Loc}, \mathcal{C}, \text{Lab}, \text{Edge}, \text{Inv}, \text{Init}) \). we define 
\[
\text{ExecTime} : (\text{Lab} \cup \mathbb{R}^\geq 0) \rightarrow \mathbb{R}^\geq 0
\]
with

- \( \text{ExecTime}(a) = 0 \) for \( a \in \text{Lab} \) and
- \( \text{ExecTime}(d) = d \) for \( d \in \mathbb{R}^\geq 0 \).

Furthermore, for \( \rho = s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} s_2 \xrightarrow{\alpha_2} \ldots \) we define

\[
\text{ExecTime}(\rho) = \sum_{i=0}^{\infty} \text{ExecTime}(\alpha_i).
\]

A path is time-divergent iff \( \text{ExecTime}(\rho) = \infty \), and time-convergent otherwise.

- Time-convergent paths are not realistic, and are not considered in the semantics.
- Note: their existence cannot be avoided (in general).
Timelock

Definition

For a state $\sigma \in \Sigma$ let $Paths_{div}(\sigma)$ be the set of time-divergent paths starting in $\sigma$.

A state $\sigma \in \Sigma$ contains a timelock iff $Paths_{div}(\sigma) = \emptyset$.

A timed automaton is timelock-free iff none of its reachable states contains a timelock.

Timelocks are modeling flows and should be avoided.
Zenoness

Definition

An infinite path fragment $\pi$ is zeno iff it is time-convergent and infinitely many discrete actions are executed within $\pi$. A timed automaton is non-zeno iff no zeno path starts in an initial state.

- Zeno paths represent nonrealizable behaviour, since their execution would require infinitely fast processors.
- Thus zeno paths are modeling flows and should be avoided.
- To check whether a timed automaton is non-zeno is algorithmically difficult.
- Instead, sufficient conditions are considered that are simple to check, e.g., by static analysis.
Theorem (Sufficient condition for non-zenoness)

Let $\mathcal{T}$ be a timed automaton with clocks $C$ such that for every control cycle

$$
l_0 \xrightarrow{\alpha_1:g_1} C_1 l_1 \xrightarrow{\alpha_2:g_2} C_2 l_2 \ldots \xrightarrow{\alpha_n:g_n} C_n l_n = l_0
$$

in $\mathcal{T}$ there exists a clock $x \in C$ such that

- $x \in C_i$ for some $0 < i \leq n$, and
- for all evaluations $\nu \in V$ there exist some $0 < j \leq n$ and $d \in \mathbb{N}^{>0}$ with

$$
\nu(x) < d \quad \text{implies} \quad (\nu \not\models \text{Inv}(l_j) \text{ or } \nu \not\models g_j).
$$

Then $\mathcal{T}$ is non-zeno.
How to describe the behaviour of timed automata?

Logic: **TCTL**, a real-time variant of CTL

**Syntax:**

State formulae

\[ \psi ::= \text{true} \mid a \mid g \mid \psi \land \psi \mid \neg \psi \mid E\phi \mid A\phi \]

Path formulae:

\[ \phi ::= \psi \ U^J \psi \]

with \( J \subseteq \mathbb{R}^{\geq 0} \) is an interval with integer bounds (open right bound may be \( \infty \)).
TCTL syntax

- Syntactic sugar:

\[
\begin{align*}
\mathcal{F}^J \psi & := \text{true} \cup^J \psi \\
\mathcal{E} \mathcal{G}^J \psi & := \neg \mathcal{A} \mathcal{F}^J \neg \psi \\
\mathcal{A} \mathcal{G}^J \psi & := \neg \mathcal{E} \mathcal{F}^J \neg \psi \\
\psi_1 \cup \psi_1 & := \psi_1 \cup^{[0,\infty)} \psi_2 \\
\mathcal{F} \psi & := \mathcal{F}^{[0,\infty)} \psi \\
\mathcal{G} \psi & := \mathcal{G}^{[0,\infty)} \psi
\end{align*}
\]

- Note: no next-time operator
**TCTL semantics**

**Definition (TCTL semantics)**

Let $\mathcal{T} = (\text{Loc}, \mathcal{C}, \text{Lab}, \text{Edge}, \text{Inv}, \text{Init})$ be a timed automaton, $AP$ a set of atomic propositions, and $L : \text{Loc} \rightarrow 2^{AP}$ a state labeling function. The function $\models$ assigns a truth value to each TCTL state and path formulae as follows:

- $\sigma \models true$
- $\sigma \models a$ iff $a \in L(\sigma)$
- $\sigma \models g$ iff $\sigma \models g$
- $\sigma \models \neg \psi$ iff $\sigma \not\models \psi$
- $\sigma \models \psi_1 \land \psi_2$ iff $\sigma \models \psi_1$ and $\sigma \models \psi_2$
- $\sigma \models E\varphi$ iff $\pi \models \varphi$ for some $\pi \in \text{Paths}_{\text{div}}(\sigma)$
- $\sigma \models A\varphi$ iff $\pi \models \varphi$ for all $\pi \in \text{Paths}_{\text{div}}(\sigma)$.

where $\sigma \in \Sigma$, $a \in AP$, $g \in ACC(C)$, $\psi$, $\psi_1$ and $\psi_2$ are TCTL state formulae, and $\varphi$ is a TCTL path formula.
Meaning of $\mathcal{U}$: a time-divergent path satisfies $\psi_1 \mathcal{U}^J \psi_2$ whenever at some time point in $J$ property $\psi_2$ holds and at all previous time instants $\psi_1 \lor \psi_2$ is satisfied.
Definition (TCTL semantics)

For a time-divergent path $\pi = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \ldots$ we define $\pi \models \psi_1 \mathcal{U}^J \psi_2$ iff

- $\exists i \geq 0. \sigma_i + d \models \psi_1$ for some $d \in [0, d_i]$ with
  \[ (\sum_{k=0}^{i-1} d_k) + d \in J, \text{ and} \]

- $\forall j \leq i. \sigma_j + d' \models \psi_1 \lor \psi_2$ for any $d' \in [0, d_j]$ with
  \[ (\sum_{k=0}^{j-1} d_k) + d' \leq (\sum_{k=0}^{i-1} d_k) + d \]

where $d_i = \text{ExecTime}(\alpha_i)$. 


Satisfaction set

Definition

For a timed automaton $\mathcal{T}$ with clocks $C$ and locations $Loc$, and a TCTL state formula $\psi$ the satisfaction set $Sat(\psi)$ is defined by

$$Sat(\psi) = \{ s \in \Sigma | s \models \psi \}.$$ 

$\mathcal{T}$ satisfies $\psi$ iff $\psi$ holds in all initial states:

$$\mathcal{T} \models \psi \iff \forall l_0 \in Init. (l_0, \nu_0) \models \psi$$

where $\nu_0(x) = 0$ for all $x \in C$. 

TCTL vs. CTL

- TCTL formulae with intervals \([0, \infty)\) may be considered as CTL formulae
- However, there is a difference due to time convergent paths
- TCTL ranges over time-divergent paths, whereas CTL over all paths!
TCTL model checking

Input: timed automaton $T$, TCTL formula $\psi$
Output: the answer to the question if $T \models \psi$

1. Eliminate the timing parameters from $\psi$, resulting in $\hat{\psi}$;
2. Make a finite abstraction of the state space
3. Construct abstract transition system $RTS$ with $T \models \psi$ iff $RTS \models \hat{\psi}$.
4. Apply CTL model checking to check whether $RTS \models \hat{\psi}$;
5. Return the model checking result.
TCTL model checking

Input: timed automaton $\mathcal{T}$, TCTL formula $\psi$
Output: the answer to the question if $\mathcal{T} \models \psi$

1. Eliminate the timing parameters from $\psi$, resulting in $\hat{\psi}$;
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3. Construct abstract transition system $RTS$ with $\mathcal{T} \models \psi$ iff $RTS \models \hat{\psi}$.
4. Apply CTL model checking to check whether $RTS \models \hat{\psi}$;
5. Return the model checking result.
1. Eliminating timing parameters

Let $\mathcal{T}$ be a timed automaton with clock set $C$ and atomic propositions $AP$. Let $\mathcal{T}' = \mathcal{T} \oplus z$ result from $\mathcal{T}$ by adding a fresh clock which never gets reset.

For any state $\sigma$ of $\mathcal{T}$ it holds that:

$$\sigma \models \text{TCTL}_E (\psi_1 U J \psi_2) \iff \text{reset}(z) \text{ in } \sigma \models \text{TCTL}_E ((\psi_1 \lor \psi_2) U ((z \in J \land \psi_2)))$$

$$\sigma \models \text{TCTL}_A (\psi_1 U J \psi_2) \iff \text{reset}(z) \text{ in } \sigma \models \text{TCTL}_A ((\psi_1 \lor \psi_2) U ((z \in J \land \psi_2)))$$

$$\sigma \models \text{TCTL}_E F \leq 2 \psi_1 \iff \text{reset}(z) \text{ in } \sigma \models \text{TCTL}_E F ((z \leq 2) \land \psi_1)$$

$$\sigma \models \text{TCTL}_E G \leq 2 \psi_1 \iff \text{reset}(z) \text{ in } \sigma \models \text{TCTL}_E G ((z \leq 2) \rightarrow \psi_1)$$
1. Eliminating timing parameters

Let $\mathcal{T}$ be a timed automaton with clock set $C$ and atomic propositions $AP$. Let $\mathcal{T}' = \mathcal{T} \oplus z$ result from $\mathcal{T}$ by adding a fresh clock which never gets reset.

For any state $\sigma$ of $\mathcal{T}$ it holds that

$$\sigma \models TCTL_E (\psi_1 U J \psi_2) \iff \text{reset}(z) \in \sigma \models TCTL_E ((\psi_1 \vee \psi_2) U ((z \in J) \land \psi_2))$$

For any state $\sigma$ of $\mathcal{T}$ it holds that

$$\sigma \models TCTL_A (\psi_1 U J \psi_2) \iff \text{reset}(z) \in \sigma \models TCTL_A ((\psi_1 \vee \psi_2) U ((z \in J) \land \psi_2))$$

For any state $\sigma$ of $\mathcal{T}$ it holds that

$$\sigma \models TCTL_E (\psi_1) \iff \text{reset}(z) \in \sigma \models TCTL_E (z \leq 2 \land \psi_1)$$

For any state $\sigma$ of $\mathcal{T}$ it holds that

$$\sigma \models TCTL_E (\psi_1) \iff \text{reset}(z) \in \sigma \models TCTL_E (z \leq 2 \rightarrow \psi_1)$$
Let $\mathcal{T}$ be a timed automaton with clock set $\mathcal{C}$ and atomic propositions $AP$. Let $\mathcal{T}' = \mathcal{T} \oplus z$ result from $\mathcal{T}$ by adding a fresh clock which never gets reset.

For any state $\sigma$ of $\mathcal{T}$ it holds that

1. $\sigma \models_{TCTL} E(\psi_1 \mathcal{U}^J \psi_2)$ iff $\text{reset}(z)$ in $\sigma \models_{TCTL} E((\psi_1 \lor \psi_2) \mathcal{U} ((z \in J) \land \psi_2))$. 


1. Eliminating timing parameters

Let $\mathcal{T}$ be a timed automaton with clock set $C$ and atomic propositions $AP$. Let $\mathcal{T}' = \mathcal{T} \oplus z$ result from $\mathcal{T}$ by adding a fresh clock which never gets reset.

For any state $\sigma$ of $\mathcal{T}$ it holds that

1. $\sigma \models_{TCTL} E(\psi_1) U^J \psi_2$ iff $\text{reset}(z)$ in $\sigma \models_{TCTL} E((\psi_1 \lor \psi_2) U ((z \in J) \land \psi_2))$.

2. $\sigma \models_{TCTL} A(\psi_1) U^J \psi_2$ iff $\text{reset}(z)$ in $\sigma \models_{TCTL} A((\psi_1 \lor \psi_2) U ((z \in J) \land \psi_2))$. 
1. Eliminating timing parameters

Let $\mathcal{T}$ be a timed automaton with clock set $\mathcal{C}$ and atomic propositions $\mathcal{AP}$. Let $\mathcal{T}' = \mathcal{T} \oplus z$ result from $\mathcal{T}$ by adding a fresh clock which never gets reset.

For any state $\sigma$ of $\mathcal{T}$ it holds that

1. $\sigma \models_{\mathcal{T}CTL} E(\psi_1) U^J \psi_2$ iff $\text{reset}(z) \in \sigma$ and $\models_{\mathcal{T}CTL} E((\psi_1 \lor \psi_2) U ((z \in J) \land \psi_2))$.

2. $\sigma \models_{\mathcal{T}CTL} A(\psi_1) U^J \psi_2$ iff $\text{reset}(z) \in \sigma$ and $\models_{\mathcal{T}CTL} A((\psi_1 \lor \psi_2) U ((z \in J) \land \psi_2))$.

3. $\sigma \models_{\mathcal{T}CTL} E F^{\leq 2} \psi_1$ iff $\text{reset}(z) \in \sigma$ and $\models_{\mathcal{T}CTL} E F((z \leq 2) \land \psi_1)$.
1. Eliminating timing parameters

Let $T$ be a timed automaton with clock set $C$ and atomic propositions $AP$. Let $T' = T \oplus z$ result from $T$ by adding a fresh clock which never gets reset.

For any state $\sigma$ of $T$ it holds that

1. $\sigma \models TCTL E(\psi_1) U^J \psi_2$ iff $\text{reset}(z)$ in $\sigma \models TCTL E((\psi_1 \lor \psi_2) U ((z \in J) \land \psi_2))$.

2. $\sigma \models TCTL A(\psi_1) U^J \psi_2$ iff $\text{reset}(z)$ in $\sigma \models TCTL A((\psi_1 \lor \psi_2) U ((z \in J) \land \psi_2))$.

3. $\sigma \models TCTL EF^{\leq 2} \psi_1$ iff $\text{reset}(z)$ in $\sigma \models TCTL EF((z \leq 2) \land \psi_1)$.

4. $\sigma \models TCTL EG^{\leq 2} \psi_1$ iff $\text{reset}(z)$ in $\sigma \models TCTL EG((z \leq 2) \rightarrow \psi_1)$.
TCTL model checking

Input: timed automaton $T$, TCTL formula $\psi$
Output: the answer to the question if $T \models \psi$

1. Eliminate the timing parameters from $\psi$, resulting in $\hat{\psi}$;
2. Make a finite abstraction of the state space
3. Construct abstract transition system $RTS$ with $T \models \psi$ iff $RTS \models \hat{\psi}$.
4. Apply CTL model checking to check whether $RTS \models \hat{\psi}$;
5. Return the model checking result.
Keywords:
Finite abstraction
Equivalence relation, equivalence classes
Bisimulation

And what does it mean in our context?
We search for an equivalence relation $\cong$ on states, such that equivalent states satisfy the same (sub)formulae $\psi'$ occurring in the timed automaton $T$ or in the specification $\psi$:

$$\sigma \cong \sigma' \implies (\sigma \models \psi' \iff \sigma' \models \psi').$$

Since the set of such (sub)formulae is finite, we strive for a finite number of equivalence classes.
Bisimulation for two LSTs

Definition

Let $LSTS_1 = (\Sigma_1, Lab_1, Edge_1, Init_1), LSTS_2 = (\Sigma_2, Lab_2, Edge_2, Init_2)$ be two state transition systems, $AP$ a set of atomic propositions, and $L_1 : \Sigma_1 \rightarrow 2^{AP}$ and $L_2 : \Sigma_2 \rightarrow 2^{AP}$ labeling functions over $AP$. A **bisimulation** for $(LSTS_1, LSTS_2)$ is an equivalence relation $\approx \subseteq \Sigma_1 \times \Sigma_2$ such that for all $\sigma_1 \approx \sigma_2$

1. $L(\sigma_1) = L(\sigma_2)$
2. for all $\sigma'_1 \in \Sigma_1$ with $\sigma_1 \xrightarrow{a} \sigma'_1$ there exists $\sigma'_2 \in \Sigma_2$ such that $\sigma_2 \xrightarrow{a} \sigma'_2$ and $\sigma'_1 \approx \sigma'_2$. 
Bisimulation for a single LSTS

Definition

Let \( LSTS = (\Sigma, Lab, Edge, Init) \) be a state transition system, \( AP \) a set of atomic propositions, and \( L : \Sigma \rightarrow 2^{AP} \) a labeling function over \( AP \). A bisimulation for \( LSTS \) is an equivalence relation \( \approx \subseteq \Sigma \times \Sigma \) such that for all \( \sigma_1 \approx \sigma_2 \)

1. \( L(\sigma_1) = L(\sigma_2) \)
2. for all \( \sigma'_1 \in \Sigma \) with \( \sigma_1 \xrightarrow{a} \sigma'_1 \) there exists \( \sigma'_2 \in \Sigma \) such that \( \sigma_2 \xrightarrow{a} \sigma'_2 \) and \( \sigma'_1 \approx \sigma'_2 \).
Definition

Let $T = (\text{Loc}, C, \text{Lab}, \text{Edge}, \text{Inv}, \text{Init})$ be a timed automaton, $AP$ a set of atomic propositions, and $L : \Sigma \rightarrow 2^{AP}$.

A \textit{time abstract bisimulation} on $T$ is an equivalence relation $\approx \subseteq \Sigma \times \Sigma$ such that for all $\sigma_1, \sigma_2 \in \Sigma$ satisfying $\sigma_1 \approx \sigma_2$

- $L(\sigma_1) = L(\sigma_2)$
- for all $\sigma_1' \in \Sigma$ with $\sigma_1 \xrightarrow{a} \sigma_1'$ there is a $\sigma_2' \in \Sigma$ such that $\sigma_2 \xrightarrow{a} \sigma_2'$ and $\sigma_1' \approx \sigma_2'$
- for all $\sigma_1' \in \Sigma$ with $\sigma_1 \xrightarrow{t} \sigma_1'$ there is a $\sigma_2' \in \Sigma$ such that $\sigma_2 \xrightarrow{t} \sigma_2'$ and $\sigma_1' \approx \sigma_2'$. 
Bisimulation

Lemma

Assume a timed automaton $T$ with state space $\Sigma$, and a bisimulation $\approx \subseteq \Sigma \times \Sigma$ on $T$.

Then for all $\sigma, \sigma' \in \Sigma$ with $\sigma \approx \sigma'$ we have that for each path

$$\pi : \sigma \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \sigma_2 \xrightarrow{\alpha_3} \ldots$$

of $T$ there exists a path

$$\pi' : \sigma' \xrightarrow{\alpha'_1} \sigma'_1 \xrightarrow{\alpha'_2} \sigma'_2 \xrightarrow{\alpha'_3} \ldots$$

of $T$ such that for all $i$

- $\sigma_i \approx \sigma'_i$,
- $\alpha_i = \alpha'_i$ if $\alpha_i \in \text{Lab}$ and
- $\alpha_i, \alpha'_i \in \mathbb{R}_{\geq 0}$ otherwise.
Now, back to timed automata. How could such a bisimulation look like?

Since, in general,
- the atomic propositions assigned to and
- the paths starting at
different locations in $T$ are different, only states $(l, \nu)$ and $(l', \nu')$ satisfying $l = l'$ should be equivalent.
2. Finite state space abstraction

Equivalent states should satisfy the same atomic clock constraints.

Notation:

- Integral part of $r \in \mathbb{R}$: $\lfloor r \rfloor = \max \{ c \in \mathbb{N} \mid c \leq r \}$
- Fractional part of $r \in \mathbb{R}$: $\text{frac}(r) = r - \lfloor r \rfloor$

For clock constraints $x < c$ with $c \in \mathbb{N}$ we have:

$$\nu \models x < c \iff \nu(x) < c \iff \lfloor \nu(x) \rfloor < c.$$  

For clock constraints $x \leq c$ with $c \in \mathbb{N}$ we have:

$$\nu \models x \leq c \iff \nu(x) \leq c \iff \lfloor \nu(x) \rfloor < c \lor (\lfloor \nu(x) \rfloor = c \land \text{frac}(\nu(x)) = 0).$$

i.e., only states $(l, \nu)$ and $(l, \nu')$ satisfying

$$\lfloor \nu(x) \rfloor = \lfloor \nu'(x) \rfloor \text{ and } \text{frac}(\nu(x)) = 0 \text{ iff } \text{frac}(\nu'(x)) = 0$$

for all $x \in C$ should be equivalent.
Problem: It would generate infinitely many equivalence classes!

Let $c_x$ be the largest constant which a clock $x$ is compared to in $T$ or in $\psi$. Then there is no observation which could distinguish between the $x$-values in $(l, \nu)$ and $(l, \nu')$ if $\nu(x) > c_x$ and $\nu'(x) > c_x$. I.e., only states $(l, \nu)$ and $(l, \nu')$ satisfying

\[
(\nu(x) > c_x \land \nu'(x) > c_x) \lor \\
([\nu(x)] = [\nu'(x)] \land \text{frac}(\nu(x)) = 0 \text{ iff } \text{frac}(\nu'(x)) = 0)
\]

for all $x \in C$ should be equivalent.
2. Finite state space abstraction

- $x = 3$
- $y = 2$

Constraints:
- $2 < x < 3$
- $1 < y < 2$
- $0 < y < 1$
As the following example illustrates, we must make a further refinement of the abstraction, since it does not distinguish between states satisfying different formulae.
2. Finite state space abstraction

\[
\begin{align*}
x & \geq 2 \\
y & < 1
\end{align*}
\]
What we need is a refinement taking the order of the fractional parts of the clock values into account. However, again only for values below the largest constants to which the clocks get compared. I.e., only states \( (l, \nu) \) and \( (l, \nu') \) satisfying

\[
(\nu(x), \nu'(x) > c_x \land \nu(y), \nu'(y) > c_x) \quad \lor \\
(\frac{\nu(x)}{} < \frac{\nu(y)}{} \text{ iff } \frac{\nu'(x)}{} < \frac{\nu'(y)}{}) \quad \land \\
\frac{\nu(x)}{} = \frac{\nu(y)}{} \text{ iff } \frac{\nu'(x)}{} = \frac{\nu'(y)}{}) \quad \land \\
\frac{\nu(x)}{} > \frac{\nu(y)}{} \text{ iff } \frac{\nu'(x)}{} > \frac{\nu'(y)}{}))
\]

for all \( x, y \in C \) should be equivalent.

Because of symmetry the following is also sufficient:

\[
(\nu(x), \nu'(x) > c_x \land \nu(y), \nu'(y) > c_y) \quad \lor \\
(\frac{\nu(x)}{} \leq \frac{\nu(y)}{} \text{ iff } \frac{\nu'(x)}{} \leq \frac{\nu'(y)}{})
\]

for all \( x, y \in C \) should be equivalent.
2. Finite state space abstraction

![Diagram showing finite state space abstraction with indices and constraints.](image)

\[ c_y = 2 \]
\[ c_x = 4 \]

*finite index*
2. Finite state space abstraction

**Definition**

For a timed automaton $T$ and a TCTL formula $\psi$, both over a clock set $\mathcal{C}$, we define the **clock equivalence relation** $\cong \subseteq \Sigma \times \Sigma$ by $(l, \nu) \cong (l', \nu')$ iff $l = l'$ and

- for all $x \in \mathcal{C}$, either $\nu(x) > c_x \land \nu'(x) > c_x$ or
  
  $$\lfloor \nu(x) \rfloor = \lfloor \nu'(x) \rfloor \land \left( \frac{\nu(x)}{c_x} = 0 \right) \iff \frac{\nu'(x)}{c_x} = 0$$

- for all $x, y \in \mathcal{C}$ if $\nu(x), \nu'(x) \leq c_x$ and $\nu(y), \nu'(y) \leq c_x$ then

  $$\frac{\nu(x)}{c_x} \leq \frac{\nu(y)}{c_x} \iff \frac{\nu'(x)}{c_x} \leq \frac{\nu'(y)}{c_x}.$$  

The **clock region** of an evaluation $\nu \in V$ is the set $[\nu] = \{ \nu' \in V \mid \nu \cong \nu' \}$. The **clock region** of a state $\sigma = (l, \nu) \in \Sigma$ is the set $[\sigma] = \{ (l, \nu') \in \Sigma \mid \nu \cong \nu' \}$. 
2. Finite state space abstraction

Lemma

Clock equivalence is a bisimulation over $AP' = AP \cup ACC(T) \cup ACC(\psi)$. 
TCTL model checking

Input: timed automaton $\mathcal{T}$, TCTL formula $\psi$
Output: the answer to the question if $\mathcal{T} \models \psi$

1. Eliminate the timing parameters from $\psi$, resulting in $\hat{\psi}$;
2. Make a finite abstraction of the state space
3. Construct abstract transition system $RTS$ with $\mathcal{T} \models \psi$ iff $RTS \models \hat{\psi}$.
4. Apply CTL model checking to check whether $RTS \models \hat{\psi}$;
5. Return the model checking result.
3. The abstract transition system

We have defined regions as abstract states, now we connect them by abstract transitions.

Two kinds of transitions: time and discrete
Definition

The clock region \( r_\infty = \{ \nu \in V \mid \forall x \in C. \ \nu(x) > c_x \} \) is called unbounded.

Let \( r, r' \) be two clock regions. The region \( r' \) is the successor clock region of \( r \), denoted by \( r' = \text{succ}(r) \), if either

- \( r = r' = r_\infty \), or
- \( r \neq r_\infty, r \neq r' \), and for all \( \nu \in r \):

\[
\exists d \in \mathbb{R}_{>0}. \ (\nu + d \in r' \ \land \ \forall 0 \leq d' \leq d. \ \nu + d' \in r \cup r').
\]

The successor state region is defined as \( \text{succ}((l, r)) = (l, \text{succ}(r)). \)
\( \alpha : x \geq 2, \text{reset}(x) \)

\[ \mathbb{E}\mathcal{F}^{(0,2]} x = 0 \]
\[ l \rightarrow \alpha : x \geq 2, \text{reset}(x) \]

\[ \mathbb{E}_{\mathcal{F}^{(0,2]}} x = 0 \]
\[ x \geq 2 : \alpha, \text{reset}(x) \]
Definition

Let $\mathcal{T} = (\text{Loc}, C, \text{Lab}, \text{Edge}, \text{Inv}, \text{Init})$ be a non-zeno timelock-free timed automaton with an atomic proposition set $AP$ and a labeling function $L$, and let $\hat{\psi}$ be an unbounded TCTL formula over $C$ and $AP$. The region transition system of $\mathcal{T}$ for $\hat{\psi}$ is a labelled state transition system $\mathcal{RTS}(\mathcal{T}, \psi) = (\Sigma', \text{Lab}', \text{Edge}', \text{Init}')$ with atomic propositions $AP'$ and a labeling function $L'$ such that

- $\Sigma'$ the finite set of all state regions
- $\text{Init}' = \{[\sigma] \mid \sigma \in \text{Init}\}$
- $AP' = AP \cup ACC(\mathcal{T}) \cup ACC(\hat{\psi})$
- $L'(l, r) = L(l) \cup \{g \in AP' \setminus AP \mid r \models g\}$

and
3. The abstract transition system

Definition

\[ (l, a, (g, C), l') \in \text{Edge} \]
\[ r \models g \quad r' = \text{reset}(C) \text{ in } r \quad r' \models \text{Inv}(l') \]
\[ \frac{(l, r) \xrightarrow{a} (l', r')} {\text{Rule Discrete}} \]

\[ r \models \text{Inv}(l) \quad \text{succ}(r) \models \text{Inv}(l) \]
\[ \frac{(l, r) \xrightarrow{t} (l, \text{succ}(r))} {\text{Rule Time}} \]
3. The abstract transition system

Lemma

For non-zeno $T$ and $\pi = s_0 \rightarrow s_1 \rightarrow \ldots$ an initial, infinite path of $T$:

- if $\pi$ is time-convergent, then there is an index $j$ and a state region $(l, r)$ such that $s_i \in (l, r)$ for all $i \geq j$.
- if there is a state region $(l, r)$ with $r \neq r_\infty$ and an index $j$ such that $s_i \in (l, r)$ for all $i \geq j$ then $\pi$ is time-convergent.

Lemma

For a non-zeno timed automaton $T$ and a TCLT formula $\psi$:

$$T \models_{TCTL} \psi \iff RTS(T, \hat{\psi}) \models_{CTL} \hat{\psi}$$
TCTL model checking

Input: timed automaton $\mathcal{T}$, TCTL formula $\psi$
Output: the answer to the question if $\mathcal{T} \models \psi$

1. Eliminate the timing parameters from $\psi$, resulting in $\hat{\psi}$;
2. Make a finite abstraction of the state space
3. Construct abstract transition system $RTS$ with $\mathcal{T} \models \psi$ iff $RTS \models \hat{\psi}$.
4. Apply CTL model checking to check whether $RTS \models \hat{\psi}$;
5. Return the model checking result.
The procedure is quite similar to CTL model checking for finite automata.

One difference:
- Handling nested time bounds in TCTL formulae
TCTL model checking

Input: timed automaton $\mathcal{T}$, TCTL formula $\psi$
Output: the answer to the question if $\mathcal{T} \models \psi$

1. Eliminate the timing parameters from $\psi$, resulting in $\hat{\psi}$;
2. Make a finite abstraction of the state space
3. Construct abstract transition system $RTS$
   $\mathcal{T} \models \psi$ iff $RTS \models \hat{\psi}$
4. Apply CTL model checking to check whether $RTS \models \hat{\psi}$;
5. Return the model checking result.
Given a state transition system and a CTL formula $\psi$, CTL model checking labels the states recursively with the sub-formulae of $\psi$ inside-out.

- The labeling with atomic propositions $a \in AP$ is given by a labeling function.
- Given the labelings for $\psi_1$ and $\psi_2$, we label a state with $\psi_1 \land \psi_2$ iff the state is labeled with both $\psi_1$ and $\psi_2$.
- Given the labeling for $\psi$, we label a state with $\neg\psi$ iff the state is not labeled with $\psi$. 
Given the labeling for $\psi$, we label a state with $\text{EX}\psi$ iff there is a successor state labeled with $\psi$.

Given the labeling for $\psi_1$ and $\psi_2$, we

- label all with $\psi_2$ labeled states additionally with $\text{E}\psi_1 \cup \psi_2$, and
- label all states that have the label $\psi_1$ and have a successor state with the label $\text{E}\psi_1 \cup \psi_2$ also with $\text{E}\psi_1 \cup \psi_2$ iteratively until a fixed point is reached.

Given the labeling for $\psi$, we label a state with $\text{AX}\psi$ iff all successor states are labeled with $\psi$.

Given the labeling for $\psi_1$ and $\psi_2$, we

- label all with $\psi_2$ labeled states additionally with $\text{A}\psi_1 \cup \psi_2$, and
- label all states that have the label $\psi_1$ and all of their successor states have the label $\text{A}\psi_1 \cup \psi_2$ also with $\text{A}\psi_1 \cup \psi_2$ iteratively until a fixed point is reached.