Real-Time Systems

Lecture 04: Duration Calculus II

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Contents & Goals

Last Lecture:
- Started DC Syntax and Semantics: Symbols, State Assertions

This Lecture:
- Educational Objectives: Capabilities for following tasks/questions.
  - Read (and at best also write) Duration Calculus terms.
- Content:
  - Duration Calculus continued
Duration Calculus Cont’d

Duration Calculus: Overview

We will introduce three (or five) syntactical “levels”:

(i) Symbols:

\[0, 1, \ldots, \pi, g, \text{true, false, =, <, >, ≤, ≥, } x, y, z, \text{ X, Y, Z, d}\]

(ii) State Assertions:

\[P ::= 0 \mid 1 \mid X = d \mid \neg P \mid P_1 \land P_2\]

(iii) Terms:

\[\theta ::= x \mid \ell \mid f \mid f(\theta_1, \ldots, \theta_n)\]

(iv) Formulae:

\[F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \land F_2 \mid \forall x \bullet F_1 \mid F_1 ; F_2\]

(v) Abbreviations:

\[[], [P], [P]^t, [P]^{≤t}, \diamond F, \Box F\]
Terms: Syntax

- **Duration terms** (DC terms or just terms) are defined by the following grammar:

\[ \theta ::= x | \ell | \int P | f(\theta_1, \ldots, \theta_n) \]

where \( x \) is a global variable, \( \ell \) and \( \int \) are special symbols, \( P \) is a state assertion, and \( f \) a function symbol (of arity \( n \)).

- \( \ell \) is called length operator, \( \int \) is called integral operator

- Notation: we may write function symbols in **infix notation** as usual, i.e. write \( \theta_1 + \theta_2 \) instead of \( +((\theta_1, \theta_2)) \).

Definition 1. [Rigid]
A term **without** length and integral symbols is called rigid.

**Example:** \( x + (y - 2) \cdot \ell \) is rigid

Terms: Semantics

- Closed **intervals** in the time domain

\[ \text{Intv} := \{ [b, e] | b, e \in \text{Time and } b \leq e \} \]

**Point intervals:** \([b, b]\)

- Let \( \text{GVar} \) be the set of global variables.
  A **valuation** of \( \text{GVar} \) is a function
  \[ \mathcal{D}: \text{GVar} \rightarrow \mathbb{R} \]

  We use \( \text{Val} \) to denote the set of all valuations of \( \text{GVar} \), i.e. \( \text{Val} = (\text{GVar} \rightarrow \mathbb{R}) \).
The semantics of a term is a function
\[ I[\theta] : \text{Val} \times \text{Intv} \rightarrow \mathbb{R} \]
i.e. \( I[\theta](V, [b, e]) \) is the real number that \( \theta \) denotes under interpretation \( I \) and valuation \( V \) in the interval \([b, e]\).

The value is defined \textit{inductively} on the structure of \( \theta \):
\[
\begin{align*}
I[x](V, [b, e]) &= V(x), \\
I[\ell](V, [b, e]) &= e - b, \\
I[fP](V, [b, e]) &= \int_{b}^{e} \frac{P_{f}(t)}{t} \, dt, \\
I[f(\theta_{1}, \ldots, \theta_{n})](V, [b, e]) &= \hat{f} \left( I[f\theta_{1}](V, [b, e]), \ldots, I[f\theta_{n}](V, [b, e]) \right)
\end{align*}
\]

So, \( I[fP](V, [b, e]) \) is \( \int_{b}^{e} P_{f}(t) \, dt \) — but does the integral always exist?

IOW: is there a \( P_{f} \) which is not (Riemann-)integrable? Yes. For instance
\[
P_{f}(t) = \begin{cases} 
1, & \text{if } t \in \mathbb{Q} \\
0, & \text{if } t \notin \mathbb{Q}
\end{cases}
\]

To exclude such functions, DC considers only interpretations \( I \) satisfying the following condition of \textit{finite variability}:

For each state variable \( X \) and each interval \([b, e]\) there is a \textit{finite partition} of \([b, e]\) such that the interpretation \( X_{I} \) is constant on each part.

Thus on each interval \([b, e]\) the function \( X_{I} \) has only finitely many points of discontinuity.
Terms: Example

\[ L = G \land \neg F \]

\[ \theta = x \cdot \int L = \langle x, IL \rangle \]

\[ V(x) = 20. \]

Remark 2.5. The semantics \( I[\theta] \) of a term is insensitive against changes of the interpretation \( I \) at individual time points.

Let \( I_1, I_2 \) be interpretations such that \( I_1(x, t) = I_2(x, t) \) for all \( x \) and all \( t \) in \( \bigcup_{t \in T} W(t) \), except for \( t \in T_{12} \).

Then, \( I_1[I[\theta]](V, [b, e]) = I_2[I[\theta]](V, [b, e]) \).

Remark 2.6. The semantics \( I[\theta](V, [b, e]) \) of a rigid term does not depend on the interval \([b, e]\).
Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) Symbols:

\( a \in \mathbb{R}, f, g, \quad true, false, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d \)

(ii) State Assertions:

\[ P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \land P_2 \]

(iii) Terms:

\[ \theta ::= x \mid \ell \mid \int P \mid f(\theta_1, \ldots, \theta_n) \]

(iv) Formulae:

\[ F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \land F_2 \mid \forall x \bullet F_1 \mid F_1 ; F_2 \]

(v) Abbreviations:

\[ [], [P], [P]^t, [P]^{\leq t}, \diamond F, \Box F \]

Formulae: Syntax

- The set of DC formulae is defined by the following grammar:

\[ F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \land F_2 \mid \forall x \bullet F_1 \mid F_1 ; F_2 \]

where \( p \) is a predicate symbol, \( \theta_i \) a term, \( x \) a global variable.

- chop operator: ‘;’
- atomic formula: \( p(\theta_1, \ldots, \theta_n) \)
- rigid formula: all terms are rigid
- chop free: ‘;’ doesn’t occur
- usual notion of free and bound (global) variables

Note: quantification only over (first-order) global variables, not over (second-order) state variables.
**Formulae: Priority Groups**

- To avoid parentheses, we define the following five priority groups from highest to lowest priority:
  - \( \neg \) (negation)
  - \( ; \) (chop)
  - \( \land, \lor \) (and/or)
  - \( \Rightarrow, \iff \) (implication/equivalence)
  - \( \exists, \forall \) (quantifiers)

**Examples:**
- \( \neg F ; F \lor H \)
- \( \forall x \left( F \land G \right) \)

**Syntactic Substitution...**

...of a term \( \theta \) for a variable \( x \) in a formula \( F \).

- We use
  \[ F[x := \theta] \]
  to denote the formula that results from performing the following steps:
  (i) transform \( F \) into \( \tilde{F} \) by (consistently) renaming bound variables such that no free occurrence of \( x \) in \( \tilde{F} \) appears within a quantified subformula \( \exists z \bullet G \) or \( \forall z \bullet G \) for some \( z \) occurring in \( \theta \),
  (ii) textually replace all free occurrences of \( x \) in \( \tilde{F} \) by \( \theta \).

**Examples:**
- \( F := (x \geq y \Rightarrow \exists z \bullet z \geq 0 \land x = y + z) \), \( \theta_1 := \ell, \theta_2 := \ell + z \)
- \( F[x := \theta_1] = (\ell \geq y \Rightarrow \exists z \bullet z \geq 0 \land \ell = y + z) \)
- \( F[x := \theta_2] = (\ell \geq y \Rightarrow \exists z \bullet z \geq 0 \land \ell + z \geq y + \ell) \)
**Formulae: Semantics**

- The **semantics** of a *formula* is a function

\[ I[F] : \text{Val} \times \text{Intv} \rightarrow \{ \text{tt}, \text{ff} \} \]

i.e. \( I[F](V, [b, e]) \) is the truth value of \( F \) under interpretation \( I \) and valuation \( V \) in the interval \([b, e]\).

- This value is defined **inductively** on the structure of \( F \):

\[
\begin{align*}
I[\theta_1, \ldots, \theta_n] & (V, [b, e]) = \left( I \in C2 (V, [b, e]) \right) \left( \theta_1, \ldots, \theta_n \right) \\
I[\neg F_1] & (V, [b, e]) = \text{tt} \iff I \in C2 (V, [b, e]) = \text{ff} \\
I[F_1 \land F_2] & (V, [b, e]) = \text{tt} \iff I \in C2 (V, [b, e]) = \text{ff} \\
I[\forall x \cdot F_1] & (V, [b, e]) = \text{tt} \iff \text{for all } a \in \mathbb{R}, I \in C2 (V, [b, e]) = \text{ff} \\
I[F_1 ; F_2] & (V, [b, e]) = \text{iff } \text{there is an } m \in [b, e] \text{ such that } I \in C2 (V, [b, e]) = \text{ff}.
\end{align*}
\]

**Formulae: Example**

\[
F := \int_{0}^{x=1} (t^2) = \int_{0}^{1} (t^2) = 1
\]

\[
\left\lfloor (x^2 \geq 0) \right\rfloor > 1
\]

\[
\left\lfloor (x^2 = 0) \right\rfloor > 1
\]

\[
\left\lfloor (x = 0) \right\rfloor > 1
\]
References