Real-Time Systems

Lecture 04: Duration Calculus II

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Contents & Goals

Last Lecture:

- Started DC Syntax and Semantics: Symbols, State Assertions

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Read (and at best also write) Duration Calculus terms.
- **Content:**
  - Duration Calculus continued

Duration Calculus Cont'd

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

$I : Obs \to (Time \to \mathcal{D})$

$X_I : Time \to \mathcal{D}(X)$

(i) **Symbols:**

$0, 1, 3.14, \ldots f, g,$ $\quad true, false, =, <, >, \leq, \geq,$ $\quad x, y, z,$ $\quad X, Y, Z,$ $\quad d$

(ii) **State Assertions:**

$I[\![P]\!] : Time \to \{0,1\}$

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$I[\![\theta]\!] : Intv \times Val \to \mathbb{R}$

$$\theta ::= x \mid \ell \mid \int P \mid f(\theta_1, \ldots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Terms: Syntax

- **Duration terms** (DC terms or just terms) are defined by the following grammar:

$$\theta ::= x \mid \ell \mid \int P \mid f(\theta_1, \ldots, \theta_n)$$

where x is a global variable, ℓ and $\int$ are special symbols, P is a state assertion, and f a function symbol (of arity n).

- ℓ is called **length operator**, $\int$ is called **integral operator**
- Notation: we may write function symbols in **infix notation** as usual, i.e. write $\theta_1 + \theta_2$ instead of $+(\theta_1, \theta_2)$.

> Definition 1. [*Rigid*]
> A term **without** length and integral symbols is called rigid.

Example: $x + (y \cdot z) - 27$ is rigid

Terms: Semantics

- Closed **intervals** in the time domain

$$\text{Intv} := \{[b, e] \mid b, e \in \text{Time and } b \leq e\}$$

Point intervals: $[b, b]$

- Let GVar be the set of global variables.
  A valuation of GVar is a function
  $\mathcal{V} : GVar \to \mathbb{R}$
  We use Val to denote the set of all valuations of GVar, i.e. Val = $(GVar \to \mathbb{R})$.

Terms: Semantics

- The **semantics** of a **term** is a function

$$\mathcal{I}[\![\theta]\!] : \mathsf{Val} \times \mathsf{Intv} \to \mathbb{R}$$

i.e. $\mathcal{I}[\![\theta]\!](\mathcal{V}, [b,e])$ is the real number that θ denotes under interpretation $\mathcal{I}$ and valuation $\mathcal{V}$ in the interval $[b,e]$.

- The value is defined **inductively** on the structure of θ:

$$\mathcal{I}[\![x]\!](\mathcal{V}, [b,e]) = \mathcal{V}(x),$$

$$\mathcal{I}[\![\ell]\!](\mathcal{V}, [b,e]) = e - b$$

$$\mathcal{I}[\![\textstyle\int P]\!](\mathcal{V}, [b,e]) = \int_b^e P_{\mathcal{I}}(t)\, dt$$

$$\mathcal{I}[\![f(\theta_1, \dots, \theta_n)]\!](\mathcal{V}, [b,e]) = \hat{f}\left(\mathcal{I}[\![\theta_1]\!](\mathcal{V},[b,e]), \dots, \mathcal{I}[\![\theta_n]\!](\mathcal{V},[b,e])\right)$$

Terms: Semantics Well-defined?

- So, $\mathcal{I}[\![\int P]\!](\mathcal{V}, [b,e])$ is $\int_b^e P_{\mathcal{I}}(t)\, dt$ — but does the integral always exist?
- IOW: is there a $P_{\mathcal{I}}$ which is not (Riemann-)integrable? Yes. For instance

$$P_{\mathcal{I}}(t) = \begin{cases} 1 & , \text{if } t \in \mathbb{Q} = \{ \frac{p}{q} \mid p,q \in \mathbb{Z} \} \\ 0 & , \text{if } t \notin \mathbb{Q} \end{cases}$$

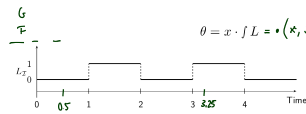
- To exclude such functions, DC considers only interpretations $\mathcal{I}$ satisfying the following condition of **finite variability**:

  For each state variable X and each interval $[b,e]$ there is a **finite partition** of $[b,e]$ such that the interpretation $X_{\mathcal{I}}$ is **constant on each part**.

  Thus on each interval $[b,e]$ the function $X_{\mathcal{I}}$ has only **finitely many points of discontinuity**.

Terms: Example

$L = G \wedge \neg F$



$$\theta = x \cdot \textstyle\int L = \cdot(x, \int L)$$

$$\mathcal{V}(x) = 20.$$

- $\theta_1 = \int 0 + x = +(\int 0, x)$

  $\mathcal{I}[\![\theta_1]\!](\mathcal{V}, [1,2]) = \hat{+}\left(\mathcal{I}[\![\int 0]\!](\mathcal{V},[1,2]), \mathcal{I}[\![x]\!](\mathcal{V},[1,2])\right) = \hat{+}(0, 20) = 20 \in \mathbb{R}$ ($\hat{+} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $0 \in \mathbb{R}$, $20 \in \mathbb{R}$)

  $\mathcal{I}[\![\int 0]\!](\mathcal{V}, [1,2]) = \int_1^2 \underbrace{0_{\mathcal{I}}(t)}_{\in \mathbb{R}}\, dt = \int_1^2 \underbrace{\mathcal{I}[\![0]\!](t)}_{\in \mathbb{R} \,:\, \mathsf{Time} \to \{0,1\}}\, dt = 0 \in \mathbb{R}$

  $\mathcal{I}[\![x]\!](\mathcal{V}, [1,2]) = \mathcal{V}(x) = 20 \in \mathbb{R}$

- $\mathcal{I}[\![\int L]\!](\mathcal{V}, [0.5, 3.25]) = \int_{0.5}^{3.25} L_{\mathcal{I}}(t)\, dt = 1.25$

  $\mathcal{I}[\![\theta]\!](\mathcal{V}, [0.5, 3.25]) = \hat{\cdot}(20, 1.25) = 25$

Terms: Remarks

"finitely many points don't matter"

Remark 2.5. The semantics $\mathcal{I}[\![\theta]\!]$ of a term is insensitive against changes of the interpretation $\mathcal{I}$ at individual time points.

Let $\mathcal{I}_1, \mathcal{I}_2$ be interpretations such that $\mathcal{I}_1(X)(t) = \mathcal{I}_2(X)(t)$ for all X for all $t \in \mathsf{Time}$ except for $t_0 \in \mathsf{Time}$.
Then $\mathcal{I}_1[\![\theta]\!](\mathcal{V}, [b,e]) = \mathcal{I}_2[\![\theta]\!](\mathcal{V}, [b,e])$.

Remark 2.6. The semantics $\mathcal{I}[\![\theta]\!](\mathcal{V}, [b,e])$ of a **rigid** term does not depend on the interval $[b,e]$.

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$0, 1, 3.14, \; a \in \mathbb{R}, f, g, \quad true, false, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \dots, \theta_n)$$

(iv) **Formulae:** $\mathcal{I}[\![F]\!] : \mathsf{Val} \times \mathsf{Intv} \to \{tt, ff\}$

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Formulae: Syntax

- The set of **DC formulae** is defined by the following grammar:

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

where p is a predicate symbol, θ_i a term, x a global variable.

- **chop operator**: ';'
- **atomic formula**: $p(\theta_1, \dots, \theta_n)$
- **rigid formula**: all terms are rigid
- **chop free**: ';' doesn't occur
- usual notion of **free** and **bound** (global) variables

- Note: quantification only over (**first-order**) global variables, not over (**second-order**) state variables.

Formulae: Priority Groups

- To avoid parentheses, we define the following five priority groups from highest to lowest priority:
  - $\neg$ **(negation)**
  - **;** **(chop)**
  - $\wedge, \vee$ **(and/or)**
  - $\implies, \iff$ **(implication/equivalence)**
  - $\exists, \forall$ **(quantifiers)**

Examples:

- $\neg F \,;\, F \vee H$ — $(\neg F; F) \vee H$ ✓ / $((\neg F); F) \vee H$ / $(\neg F); (F \vee H)$
- $\forall x \bullet (F \wedge G)$

Syntactic Substitution...

...of a term θ for a variable x in a formula F.

- We use

$$F[x := \theta]$$

to denote the formula that results from performing the following steps:

(i) transform F into $\tilde{F}$ by (consistently) renaming bound variables such that no free occurrence of x in $\tilde{F}$ appears within a quantified subformula $\exists\, z \bullet G$ or $\forall\, z \bullet G$ for some z occurring in θ,

(ii) textually replace all free occurrences of x in $\tilde{F}$ by θ.

Examples: $F := (x \geq y \implies \exists\, z \bullet z \geq 0 \wedge x = y + z)$, $\theta_1 := \ell$, $\theta_2 := \ell + z$,

- $F[x := \theta_1] = (\ell \geq y \implies \exists z \bullet z \geq 0 \wedge \ell = y + z)$
- $F[x := \theta_2] = (\ell + z \geq y \implies \exists \tilde{z} \bullet \tilde{z} \geq 0 \wedge \ell + z = y + \tilde{z})$

Formulae: Semantics

- The **semantics** of a **formula** is a function

$$\mathcal{I}[\![F]\!] : \mathsf{Val} \times \mathsf{Intv} \to \{\mathrm{tt}, \mathrm{ff}\}$$

i.e. $\mathcal{I}[\![F]\!](\mathcal{V}, [b, e])$ is the truth value of F under interpretation $\mathcal{I}$ and valuation $\mathcal{V}$ in the interval $[b, e]$.

- This value is defined **inductively** on the structure of F:

$\mathcal{I}[\![p(\theta_1, \dots, \theta_n)]\!](\mathcal{V}, [b, e]) = \hat{p}(\mathcal{I}[\![\theta_1]\!](\mathcal{V}, [b,e]), \dots, \mathcal{I}[\![\theta_n]\!](\mathcal{V}, [b,e]))$

$\mathcal{I}[\![\neg F_1]\!](\mathcal{V}, [b, e]) = \mathrm{tt}$ iff $\mathcal{I}[\![F_1]\!](\mathcal{V}, [b,e]) = \mathrm{ff}$

$\mathcal{I}[\![F_1 \wedge F_2]\!](\mathcal{V}, [b, e]) = \mathrm{tt}$ iff $\mathcal{I}[\![F_i]\!](\mathcal{V}, [b,e]) = \mathrm{tt}$, $i = 1, 2$

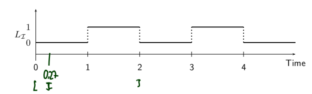
$\mathcal{I}[\![\forall\, x \bullet F_1]\!](\mathcal{V}, [b, e]) = \mathrm{tt}$ iff for all $a \in \mathbb{R}$ — the symbol!
$\mathcal{I}[\![F_1[x := a]]\!](\mathcal{V}, [b,e]) = \mathrm{tt}$

$\mathcal{I}[\![F_1 \,;\, F_2]\!](\mathcal{V}, [b, e]) =$ iff there is an $m \in [b, e]$ such that
$\mathcal{I}[\![F_1]\!](\mathcal{V}, [b, m]) = \mathrm{tt}$ and $\mathcal{I}[\![F_2]\!](\mathcal{V}, [m, e]) = \mathrm{tt}$

Formulae: Example

$$F := \underbrace{\textstyle\int L = 0}_{\ell = 1} \,;\, \underbrace{\textstyle\int L = 1}_{\ell = 1}$$

$\int(X = \text{green}) > 1$
$\int(L = 1) > 2$



- $\mathcal{I}[\![F]\!](\mathcal{V}, [0, 2]) = \mathrm{tt}$ (*)

choose $m = 1.0$: $\mathcal{I}[\![\int L = 0]\!](\mathcal{V}, [0, 1]) = \mathrm{tt}$ because $\mathcal{I}[\![\int L]\!]([0, 1]) = \int_0^1 L_\mathcal{I}(t)\,dt = 0$
and $\mathcal{I}[\![=(\int L, 0)]\!]([0, 1]) = \underbrace{\hat{=}(0, \hat{0})}_{\text{all math., not DC symbols}} = \mathrm{tt}$

$\mathcal{I}[\![\int L = 1]\!](\mathcal{V}, [1, 2]) = \hat{=}(1, \hat{1}) = \mathrm{tt}$

what about
$m = 0.27$: witness for (*) → chop point not necessarily unique
$m = 1.75$: no witness for (*)

References

References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.