Real-Time Systems

Lecture 05: Duration Calculus III

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Contents & Goals

Last Lecture:

- DC Syntax and Semantics: Terms, Formulae

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Read (and at best also write) Duration Calculus formulae – including abbreviations.
  - What is Validity/Satisfiability/Realisability for DC formulae?
  - How can we prove a design correct?
- **Content:**
  - Duration Calculus Abbreviations
  - Basic Properties
  - Validity, Satisfiability, Realisability

Duration Calculus Cont'd

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$$f, g, \quad true, false, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:** $\mathcal{I}[\![P]\!] : \mathrm{Time} \to \{0,1\}$

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:** $\mathcal{I}[\![\theta]\!] : \mathrm{Vals} \times \mathrm{Intv} \to \mathbb{R}$

$$\theta ::= x \mid \ell \mid \int P \mid f(\theta_1, \ldots, \theta_n)$$

(iv) **Formulae:** $\mathcal{I}[\![F]\!] : \mathrm{Val} \times \mathrm{Intv} \to \{\mathrm{ff}, \mathrm{tt}\}$

$$F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

true ; F ; true

(v) **Abbreviations:**

$$\lceil\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Formulae: Remarks

Remark 2.10. [*Rigid and chop-free*] Let F be a duration formula, $\mathcal{I}$ an interpretation, $\mathcal{V}$ a valuation, and $[b, e] \in \mathsf{Intv}$.

- If F is **rigid**, then

$$\forall [b', e'] \in \mathsf{Intv} : \mathcal{I}[\![F]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F]\!](\mathcal{V}, [b', e']).$$

- If F is **chop-free** or θ is **rigid**, then in the calculation of the semantics of F, every occurrence of θ denotes the same value.

rigid: $(x>0) \wedge \ldots \vee (x>0)\wedge \ldots$

$(\int L > 0) \wedge \ldots \vee (\int L > 0) \ldots$

Substitution Lemma

Lemma 2.11. [*Substitution*]
Consider a formula F, a global variable x, and a term θ such that F is **chop-free** or θ is **rigid**.
Then for all interpretations $\mathcal{I}$, valuations $\mathcal{V}$, and intervals $[b, e]$,

$$\mathcal{I}[\![F[x := \theta]]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F]\!](\mathcal{V}[x := d], [b, e])$$

where $d = \mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$.

VALID

$\mathcal{V}(x) = 3$
$[b,e] = [5, 11]$

- $F := ((\ell = x) ; (\ell = x)) \implies (\ell = 2 \cdot x) \quad \theta := \ell$

$d = \mathcal{I}[\![\ell]\!](\mathcal{V}, [b,e]) = 6$

$(6 = x) ; (6 = x) \Rightarrow 6 = 2 \cdot x$ ↯ NOT VALID

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$$f, g, \quad true, false, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \ldots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

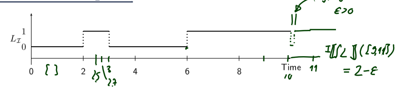
$$\lceil\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Duration Calculus Abbreviations

Abbreviations

- $\lceil\rceil := \ell = 0$ **(point interval)**
- $\lceil P \rceil := (\int P = \ell) \wedge \ell > 0$ **(almost everywhere)**
- $\lceil P \rceil^t := \lceil P \rceil \wedge \ell = t$ **(for time t)**
- $\lceil P \rceil^{\leq t} := \lceil P \rceil \wedge \ell \leq t$ **(up to time t)**
- $\Diamond F := true \,;\, F \,;\, true$ **(for some subinterval)**
- $\Box F := \neg \Diamond \neg F$ **(for all subintervals)**

Abbreviations: Examples

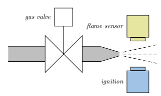


$\mathcal{I}[\![\ \int L = 0\ ]\!](\mathcal{V}, [0,2]) =$ tt		
$\mathcal{I}[\![\ \int L = 1\ ]\!](\mathcal{V}, [2,6]) =$ tt		
$\mathcal{I}[\![\ \int L = 0 \,;\, \int L = 1\ ]\!](\mathcal{V}, [0,6]) =$ tt	for $m \in [0,2]$	
$\mathcal{I}[\![\ \lceil \neg L \rceil\ ]\!](\mathcal{V}, [0,2]) =$ tt		$\int \neg L = \ell \wedge \ell > 0$
$\mathcal{I}[\![\ \lceil L \rceil\ ]\!](\mathcal{V}, [2,3]) =$ tt		
$\mathcal{I}[\![\ \lceil \neg L \rceil \,;\, \lceil L \rceil\ ]\!](\mathcal{V}, [0,3]) =$ tt	for $m = 2$ (unique!)	
$\mathcal{I}[\![\ \lceil \neg L \rceil \,;\, \lceil L \rceil \,;\, \lceil \neg L \rceil\ ]\!](\mathcal{V}, [0,6]) =$ tt	$m_1 = 2,\ m_2 = 3$	
$\mathcal{I}[\![\ \Diamond \lceil L \rceil\ ]\!](\mathcal{V}, [0,6]) =$ tt	$m_1 = 2,\ m_2 = 3$	true ; $\lceil L \rceil$; true
$\mathcal{I}[\![\ \Diamond \lceil \neg L \rceil\ ]\!](\mathcal{V}, [0,6]) =$ tt		
$\mathcal{I}[\![\ \Diamond \lceil \neg L \rceil^2\ ]\!](\mathcal{V}, [0,6]) =$ tt	$m_1 = 0,\ m_2 = 2$	can't get this
$\mathcal{I}[\![\ \lceil \neg L \rceil^2 \,;\, \lceil \neg L \rceil^1 \,;\, \lceil \neg L \rceil^3\ ]\!](\mathcal{V}, [0,6]) =$ ff		
$\mathcal{I}[\![\ \lceil \neg L \rceil^2 \,;\, \lceil L \rceil^1 \,;\, \lceil \neg L \rceil^3\ ]\!](\mathcal{V}, [0,6]) =$ ff		

Duration Calculus: Looking back

- Duration Calculus is an **interval logic**.
- Formulae are evaluated in an (**implicitly given**) interval.

Back to our gas burner:



- G, F, I, H : ~~Time~~ $\mathcal{D} = \{0, 1\}$
- Define L ~~: Time → {0,1}~~ as $G \wedge \neg F$.

Strangest operators:

- **everywhere** — Example: $\lceil G \rceil$
  (Holds in a given interval $[b, e]$ iff the gas valve is open almost everywhere.)
- **chop** — Example: $(\lceil \neg I \rceil \,;\, \lceil I \rceil \,;\, \lceil \neg I \rceil) \implies \ell \geq 1$
  (Ignition phases last at least one time unit.)
- **integral** — Example: $\ell \geq 60 \implies \int L \leq \frac{\ell}{20}$
  (At most 5% leakage time within intervals of at least 60 time units.)

DC Validity, Satisfiability, Realisability

Validity, Satisfiability, Realisability

Let $\mathcal{I}$ be an interpretation, $\mathcal{V}$ a valuation, $[b,e]$ an interval, and F a DC formula.

- $\mathcal{I}, \mathcal{V}, [b,e] \models F$ ("F **holds** in $\mathcal{I}$, $\mathcal{V}$, $[b,e]$") iff $\quad \mathcal{I}[\![F]\!](\mathcal{V},[b,e]) = \mathrm{tt}$.
- F is called **satisfiable** iff it holds in some $\mathcal{I}$, $\mathcal{V}$, $[b,e]$.
- $\mathcal{I}, \mathcal{V} \models F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F") iff $\quad \forall [b,e] \in \mathsf{Intv} : \mathcal{I}, \mathcal{V}, [b,e] \models F$.
- F is called **realisable** iff some $\mathcal{I}$ and $\mathcal{V}$ realise F.
- $\mathcal{I} \models F$ ("$\mathcal{I}$ **realises** F") iff $\quad \forall \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models F$.
- $\models F$ ("F is **valid**") iff $\quad \forall$ interpretation $\mathcal{I} : \mathcal{I} \models F$.

Validity vs. Satisfiability vs. Realisability

Remark 2.13. For all DC formulae F,

- F is satisfiable iff $\neg F$ is not valid,
  F is valid iff $\neg F$ is not satisfiable.
- If F is valid then F is realisable, but not vice versa.
- If F is realisable then F is satisfiable, but not vice versa.

Examples: Valid? Realisable? Satisfiable?

- $\mathcal{I}, \mathcal{V}, [b,e] \models F$ ("F **holds** in $\mathcal{I}$, $\mathcal{V}$, $[b,e]$") iff $\quad \mathcal{I}[\![F]\!](\mathcal{V},[b,e]) = \mathrm{tt}$.
- F is called **satisfiable** iff it holds in some $\mathcal{I}$, $\mathcal{V}$, $[b,e]$.
- $\mathcal{I}, \mathcal{V} \models F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F") iff $\quad \forall [b,e] \in \mathsf{Intv} : \mathcal{I}, \mathcal{V}, [b,e] \models F$.
- F is called **realisable** iff some $\mathcal{I}$ and $\mathcal{V}$ realise F.
- $\mathcal{I} \models F$ ("$\mathcal{I}$ **realises** F") iff $\quad \forall \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models F$.
- $\models F$ ("F is **valid**") iff $\quad \forall$ interpretation $\mathcal{I} : \mathcal{I} \models F$.

	Satisfiable	Realisable	Valid
$\ell \geq 0$	✓	✓	✓
$\ell = \int 1$	✓	✓	✓
$\ell = 30 \iff \ell = 10 \,;\, \ell = 20$	✓	✓	✓
$((F \,;\, G) \,;\, H) \iff (F \,;\, (G \,;\, H))$	✓	✓	✓
$\int L \leq x$	✓	✓	×
$\ell = 2$	✓	×	×

Initial Values

- $\mathcal{I}, \mathcal{V} \models_0 F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F **from** 0") iff

  $$\forall t \in \mathsf{Time} : \mathcal{I}, \mathcal{V}, [0,t] \models F.$$

- F is called **realisable from** 0 iff some $\mathcal{I}$ and $\mathcal{V}$ realise F from 0.
- Intervals of the form $[0,t]$ are called **initial intervals**.
- $\mathcal{I} \models_0 F$ ("$\mathcal{I}$ **realises** F **from** 0") iff $\quad \forall \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models_0 F$.
- $\models_0 F$ ("F is **valid from** 0") iff $\quad \forall$ interpretation $\mathcal{I} : \mathcal{I} \models_0 F$.

Initial or not Initial...

For all interpretations $\mathcal{I}$, valuations $\mathcal{V}$, and DC formulae F,

(i) $\mathcal{I}, \mathcal{V} \models F$ implies $\mathcal{I}, \mathcal{V} \models_0 F$, but not vice versa,

(ii) if F is realisable then F is realisable from 0, but not vice versa,

(iii) F is valid iff F is valid from 0.

Specification and Semantics-based Correctness Proofs of Real-Time Systems with DC

Methodology: Ideal World...

(i) Choose a collection of **observables** 'Obs'.

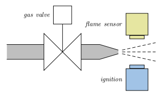
(ii) Provide the **requirement/specification** 'Spec' as a conjunction of DC formulae (over 'Obs').

(iii) Provide a description 'Ctrl' of the **controller** in form of a DC formula (over 'Obs').

(iv) We say 'Ctrl' is **correct** (wrt. 'Spec') iff

$$\models_0 \text{Ctrl} \implies \text{Spec}.$$

Gas Burner Revisited



(i) Choose **observables**:

- two boolean observables G and F (i.e. Obs $= \{G, F\}$, $\mathcal{D}(G) = \mathcal{D}(F) = \{0, 1\}$)
- $G = 1$: gas valve open (output)
- $F = 1$: have flame (input)
- define $L := G \wedge \neg F$ (leakage)

(ii) Provide the **requirement**:

$$\text{Req} :\iff \Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)$$

Gas Burner Revisited

(iii) Provide a description 'Ctrl' of the **controller** in form of a DC formula (over 'Obs'). Here, firstly consider a **design**:

- Des-1 : $\iff \Box(\lceil L \rceil \implies \ell \leq 1)$
- Des-2 : $\iff \Box(\lceil L \rceil \,;\, \lceil \neg L \rceil \,;\, \lceil L \rceil \implies \ell > 30)$

(iv) Prove **correctness**:

- We want (or do we want $\models_0 \ldots$?):

$$\models (\text{Des-1} \wedge \text{Des-2} \implies \text{Req}) \qquad \text{(Thm. 2.16)}$$

- We do show

$$\models \text{Req-1} \implies \text{Req} \qquad \text{(Lem. 2.17)}$$

with the simplified requirement

$$\text{Req-1} := \Box(\ell \leq 30 \implies \textstyle\int L \leq 1),$$

- and we show

$$\models (\text{Des-1} \wedge \text{Des-2}) \implies \text{Req-1}. \qquad \text{(Lem. 2.19)}$$

Gas Burner Revisited: Lemma 2.17

Claim:

$$\models \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}} \implies \underbrace{\Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)}_{\text{Req}}$$

Proof:

- Assume 'Req-1'.
- Let $L_\mathcal{I}$ be any interpretation of L, and $[b, e]$ an interval with $e - b \geq 60$.
- Show "$20 \cdot \int L \leq \ell$", i.e.

$$\mathcal{I}[\![20 \cdot \textstyle\int L \leq \ell]\!](\mathcal{V}, [b,e]) = tt$$

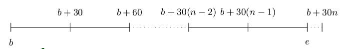
i.e.

$$20 \cdot \int_b^e L_\mathcal{I}(t)\,dt \leq (e-b)$$

Gas Burner Revisited: Lemma 2.17

$$\models \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}} \implies \Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)$$

- Set $n := \lceil \frac{e-b}{30} \rceil$, i.e. $n \in \mathbb{N}$ with $n - 1 < \frac{e-b}{30} \leq n$, and split the interval



$$20 \cdot \int_b^e L_\mathcal{I}(t)\,dt$$

$$= 20 \cdot \left(\sum_{i=0}^{n-2} \int_{b+30i}^{b+30(i+1)} L_\mathcal{I}(t)\,dt + \int_{b+30(n-1)}^{e} L_\mathcal{I}(t)\,dt \right)$$

$$\{\text{Req-1}\} \leq 20 \cdot \sum_{i=0}^{n-2} 1 + 20 \cdot 1$$

$$= 20 \cdot n$$

$$\{n-1 < \tfrac{e-b}{30}\} < 20 \left(\frac{e-b}{30} + 1 \right)$$

$$= \frac{2}{3}(e-b) + 20$$

$$\boxed{\{e-b \geq 60 \iff 20 \leq \tfrac{1}{3}(e-b)\}} \leq e-b$$

$$\boxed{\{e-b \geq 60 \iff 20 \leq \tfrac{1}{3}(e-b)\}}$$

Some Laws of the DC Integral Operator

Theorem 2.18
For all state assertions P and all real numbers $r_1, r_2 \in \mathbb{R}$,

(i) $\models \int P \leq \ell$,

(ii) $\models (\int P = r_1) \,;\, (\int P = r_2) \implies \int P = r_1 + r_2$,

(iii) $\models \lceil \neg P \rceil \implies \int P = 0$,

(iv) $\models \lceil\rceil \implies \int P = 0$.

Gas Burner Revisited: Lemma 2.18

(i) $\models \int P \leq \ell$, (iv) $\models \lceil\rceil \implies \int P = 0$.
(ii) $\models (\int P = r_1) ; (\int P = r_2) \implies \int P = r_1 + r_2$,
(iii) $\models \lceil \neg P \rceil \implies \int P = 0$,

Claim:

$$\models (\underbrace{\Box(\lceil L \rceil \implies \ell \leq 1)}_{\text{Des-1}} \wedge \underbrace{\Box(\lceil L \rceil ; \lceil \neg L \rceil ; \lceil L \rceil \implies \ell > 30)}_{\text{Des-2}}) \implies \underbrace{\Box(\ell \leq 30 \implies \int L \leq 1)}_{\text{Req-1}}$$

Proof: $\ell \leq 30$

{finite variab.} $\implies \lceil\rceil$
$\vee \lceil L \rceil ; (\lceil\rceil \vee \lceil \neg L \rceil)$
$\vee \lceil \neg L \rceil ; (\lceil\rceil \vee \lceil L \rceil)$
$\vee \lceil L \rceil ; \lceil L \rceil \vee \lceil \neg L \rceil$ (*)
$\vee \ell \leq 30 \wedge \Diamond(\lceil L \rceil ; \lceil \neg L \rceil ; \lceil L \rceil)$ } ↯ Des 2

{Des-2} $\implies$ (*)

{Des-1} $\implies \lceil\rceil$
$\vee (\ell \leq 1) ; (\lceil\rceil \vee \lceil \neg L \rceil)$
$\vee \lceil \neg L \rceil ; (\lceil\rceil \vee \ell \leq 1)$
$\vee \lceil \neg L \rceil ; \ell \leq 1 ; \lceil \neg L \rceil$

{(i)} $\implies \lceil\rceil$
$\vee (\int L \leq 1) ; (\lceil\rceil \vee \lceil \neg L \rceil)$
$\vee \lceil \neg L \rceil ; (\lceil\rceil \vee \int L \leq 1)$
$\vee \lceil \neg L \rceil ; \int L \leq 1 ; \lceil \neg L \rceil$

{iii, iv} $\implies \int L = 0$
$\vee (\int L \leq 1) ; (\int L = 0 \vee \int L = 0)$
$\vee (\int L = 0) ; (\int L = 0 \vee \int L \leq 1)$
$\vee \int L = 0 ; \int L \leq 1 ; \int L = 0$

{ii} $\implies \int L \leq 1$

$\square$

References

References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.