Real-Time Systems

Lecture 7: DC Properties II

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Contents & Goals

Last Lecture:

- RDC in discrete time
- Satisfiability and realisability from 0 is decidable for RDC in discrete time

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Facts: (un)decidability properties of DC in continuous time.
  - What's the idea of the considered (un)decidability proofs?
- **Content:**
  - Undecidable problems of DC in continuous time

(Variants of) RDC in Continuous Time

Recall: Restricted DC (RDC)

$\ell = 1 = \lceil 1 \rceil \wedge \neg(\lceil 1 \rceil ; \lceil 1 \rceil)$

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \, ; F_2$$

where P is a state assertion, but with **boolean** observables **only**.

From now on: "RDC $+ \ell = x, \forall x$"

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \, ; F_2 \mid \ell = 1 \mid \ell = x \mid \forall x \bullet F_1$$

Undecidability of Satisfiability/Realisability from 0

Theorem 3.10.
The realisability from 0 problem for DC with **continuous time** is undecidable, not even semi-decidable.

Theorem 3.11.
The satisfiability problem for DC with continuous time is undecidable.

Sketch: Proof of Theorem 3.10

Reduce divergence of **two-counter machines** to realisability from 0:

- Given a two-counter machine $\mathcal{M}$ with final state q_{fin},
- construct a DC formula $F(\mathcal{M}) := encoding(\mathcal{M})$
- such that

  $\mathcal{M}$ **diverges** **if and only if** the DC formula

  $$F(\mathcal{M}) \wedge \neg\Diamond\lceil q_{fin} \rceil$$

  is **realisable from 0**.

- If realisability from 0 was (semi-)decidable, divergence of two-counter machines would be (which it isn't).

Recall: Two-counter machines

A **two-counter** machine is a structure

$$\mathcal{M} = (\mathcal{Q}, q_0, q_{fin}, Prog)$$

where

start of command

- $\mathcal{Q}$ is a finite set of **states**,
- comprising the **initial state** q_0 and the **final state** q_{fin}
- $Prog$ is the **machine program**, i.e. a finite set of **commands** of the form

just syntax

$$q : inc_i : q' \quad \text{and} \quad q : dec_i : q', q'', \qquad i \in \{1, 2\}.$$

could also be: $A(q,q')$, $B(q,q')$, $C(q,q',q'')$, $D(q,q',q'')$

- We assume **deterministic** 2CM: for each $q \in \mathcal{Q}$, at most one command starts in q, and q_{fin} is the only state where no command starts.

2CM Configurations and Computations

- a **configuration** of $\mathcal{M}$ is a triple $K = (q, n_1, n_2) \in \mathcal{Q} \times \mathbb{N}_0 \times \mathbb{N}_0$.

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- The (!) **computation** of $\mathcal{M}$ is a finite sequence of the form ("$\mathcal{M}$ **halts**")

$$K_0 = (q_0, 0, 0) \vdash K_1 \vdash K_2 \vdash \dots \vdash (q_{fin}, n_1, n_2)$$

  or an infinite sequence of the form ("$\mathcal{M}$ **diverges**")

$$K_0 = (q_0, 0, 0) \vdash K_1 \vdash K_2 \vdash \dots$$

2CM Configurations and Computations

current state
current value of counter 1, 2

- a **configuration** of $\mathcal{M}$ is a triple $K = (q, n_1, n_2) \in \mathcal{Q} \times \mathbb{N}_0 \times \mathbb{N}_0$.
- The (!) **computation** of $\mathcal{M}$ is a finite sequence of the form ("$\mathcal{M}$ **halts**")

$$K_0 = (q_0, 0, 0) \vdash K_1 \vdash K_2 \vdash \dots \vdash (q_{fin}, n_1, n_2)$$

$(K_1, K_2) \in \vdash$

or an infinite sequence of the form ("$\mathcal{M}$ **diverges**")

$$K_0 = (q_0, 0, 0) \vdash K_1 \vdash K_2 \vdash \dots$$

- The **transition relation** "$\vdash$" on configurations is defined as follows:

Command	Semantics: $K \vdash K'$
$q : inc_1 : q'$	$(q, n_1, n_2) \vdash (q', n_1 + 1, n_2)$
$q : dec_1 : q', q''$	$(q, 0, n_2) \vdash (q', 0, n_2)$
	$(q, n_1 + 1, n_2) \vdash (q'', n_1, n_2)$
$q : inc_2 : q'$	$(q, n_1, n_2) \vdash (q', n_1, n_2 + 1)$
$q : dec_2 : q', q''$	$(q, n_1, 0) \vdash (q', n_1, 0)$
	$(q, n_1, n_2 + 1) \vdash (q'', n_1, n_2)$

2CM Example

- $\mathcal{M} = (\mathcal{Q}, q_0, q_{fin}, Prog)$
- commands of the form $q : inc_i : q'$ and $q : dec_i : q', q''$, $i \in \{1, 2\}$
- configuration $K = (q, n_1, n_2) \in \mathcal{Q} \times \mathbb{N}_0 \times \mathbb{N}_0$.
-

Command	Semantics: $K \vdash K'$
$q : inc_1 : q'$	$(q, n_1, n_2) \vdash (q', n_1 + 1, n_2)$
$q : dec_1 : q', q''$	$(q, 0, n_2) \vdash (q', 0, n_2)$
	$(q, n_1 + 1, n_2) \vdash (q'', n_1, n_2)$
$q : inc_2 : q'$	$(q, n_1, n_2) \vdash (q', n_1, n_2 + 1)$
$q : dec_2 : q', q''$	$(q, n_1, 0) \vdash (q', n_1, 0)$
	$(q, n_1, n_2 + 1) \vdash (q'', n_1, n_2)$

$Q = \{q_0, q_1, q_{fin}\}$

$Prog = \{q_0 : inc_1 : q_1,$
$q_1 : inc_2 : q_{fin}\}$

$(q_0, 0, 0)$
$\vdash$
$(q_1, 1, 0)$
$\vdash$
$(q_{fin}, 1, 1)$

↳ machine halts

$\hat{Q} = \{\hat{q}_0, \hat{q}_{fin}\}$, $\hat{Prog} = \{\hat{q}_0 : inc_1 : \hat{q}_0\}$

$(\hat{q}_0, 0, 0)$
$\vdash$
$(\hat{q}_0, 1, 0)$
$\vdash$
$(\hat{q}_0, 2, 0)$
$\vdots$

↳ machine diverges

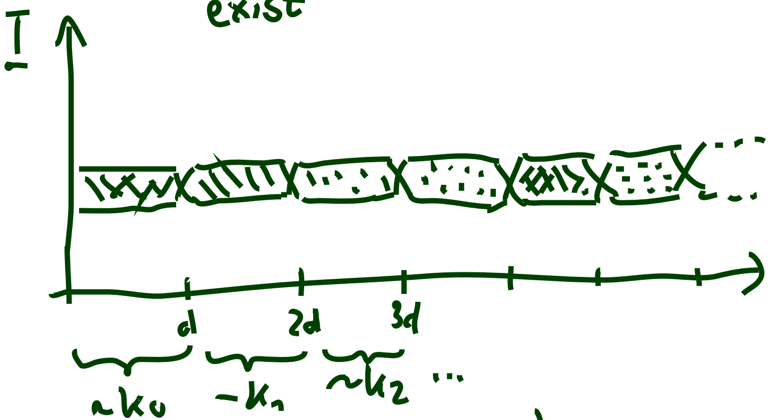
2CM $\mathcal{M}$ diverges

iff

exists $\pi: k_0 \vdash k_1 \vdash k_2 \dots$ (never q_{fin})

iff

exist



("I describes π")

and

$\mathcal{I} \models_0 F(\mathcal{M}) \land \neg\Diamond\lceil q_{fin} \rceil$

$F(\mathcal{M})$ intuitively requires:

- $[n \cdot d, (n+1) \cdot d]$ encodes a configuration
- $[n \cdot d, (n+1) \cdot d]$ and $[(n+1) \cdot d, (n+2) \cdot d]$ are in $\vdash$-relation
- $[0, d]$ encodes $(q_0, 0, 0)$
- if q_{fin} is reached, we stay there

Reducing Divergence to DC realisability: Idea

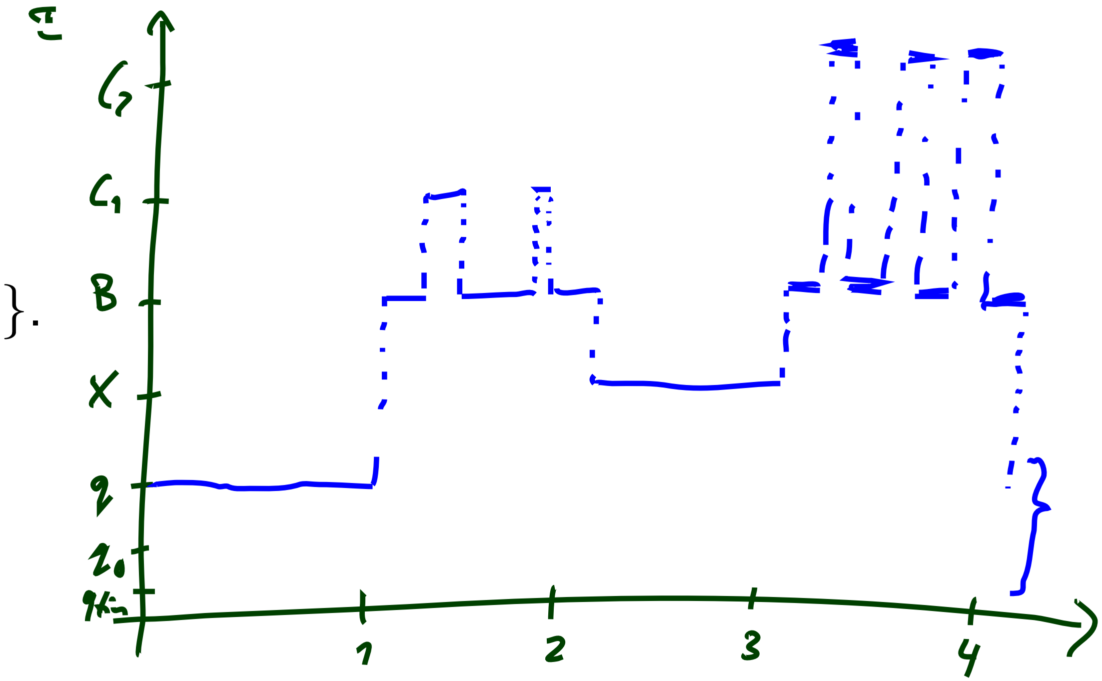
- A single configuration K of $\mathcal{M}$ can be encoded in an interval of length 4; being an encoding interval can be **characterised** by a DC formula.

- An interpretation on 'Time' encodes **the** computation of $\mathcal{M}$ if
  - each interval $[4n, 4(n+1)]$, $n \in \mathbb{N}_0$, **encodes** a configuration K_n,
  - each two subsequent intervals $[4n, 4(n+1)]$ and $[4(n+1), 4(n+2)]$, $n \in \mathbb{N}_0$, encode configurations $K_n \vdash K_{n+1}$ **in transition relation**.

- Being encoding of the run can be **characterised** by DC formula $F(\mathcal{M})$.

- Then $\mathcal{M}$ **diverges** if and only if $F(\mathcal{M}) \land \neg\Diamond\lceil q_{fin} \rceil$ is realisable from 0.

Encoding Configurations



- We use $\mathrm{Obs} = \{\mathrm{obs}\}$ with $\mathcal{D}(\mathrm{obs}) = Q_\mathcal{M} \,\dot{\cup}\, \{C_1, C_2, B, X\}$.

Examples:

- $K = (q, 2, 3)$

$$\begin{pmatrix} \lceil q \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B \rceil ; \lceil C_1 \rceil ; \lceil B \rceil ; \lceil C_1 \rceil ; \lceil B \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil X \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B \rceil ; \lceil C_2 \rceil ; \lceil B \rceil ; \lceil C_2 \rceil ; \lceil B \rceil ; \lceil C_2 \rceil ; \lceil B \rceil \\ \wedge \\ \ell = 1 \end{pmatrix}$$

- $K_0 = (q_0, 0, 0)$

$$\begin{pmatrix} \lceil q_0 \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil X \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B \rceil \\ \wedge \\ \ell = 1 \end{pmatrix}$$

or, using abbreviations, $\lceil q_0 \rceil^1 ; \lceil B \rceil^1 ; \lceil X \rceil^1 ; \lceil B \rceil^1$.

Construction of $F(\mathcal{M})$

In the following, we give DC formulae describing

- the initial configuration,
- the general form of configurations,
- the transitions between configurations,
- the handling of the final state.

$F(\mathcal{M})$ is the conjunction of all these formulae.

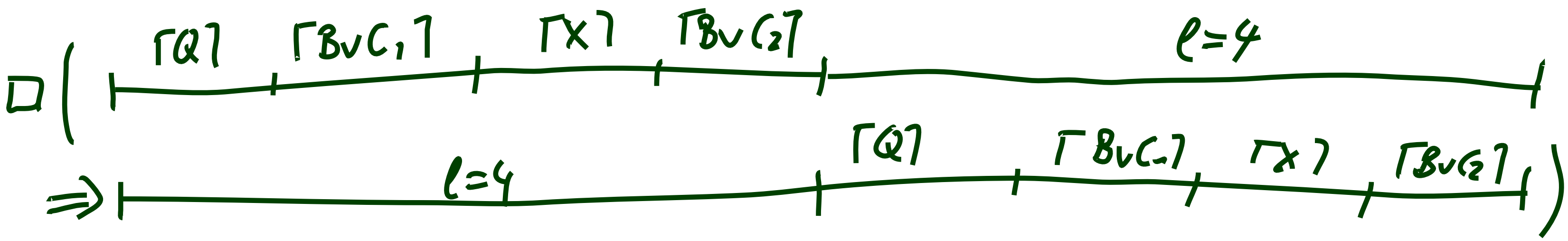
$$F(\mathcal{M}) = init \wedge keep \wedge \dots \wedge \bigwedge_{q: inc; :q' \in Prog_{\mathcal{M}}} F(q: inc; :q') \wedge \bigwedge_{q: dec; :q', q'' \in Prog_{\mathcal{M}}} F(q: dec; :q', q'')$$

Initial and General Configurations

$$init :\iff (\ell \geq 4 \implies \lceil q_0 \rceil^1 \,;\, \lceil B \rceil^1 \,;\, \lceil X \rceil^1 \,;\, \lceil B \rceil^1 \,;\, true)$$

$$keep :\iff \Box(\lceil Q \rceil^1 \,;\, \lceil B \vee C_1 \rceil^1 \,;\, \lceil X \rceil^1 \,;\, \lceil B \vee C_2 \rceil^1 \,;\, \ell = 4 \implies \ell = 4 \,;\, \lceil Q \rceil^1 \,;\, \lceil B \vee C_1 \rceil^1 \,;\, \lceil X \rceil^1 \,;\, \lceil B \vee C_2 \rceil^1)$$

where $Q := \neg(X \vee C_1 \vee C_2 \vee B)$.



Auxiliary Formula Pattern *copy*

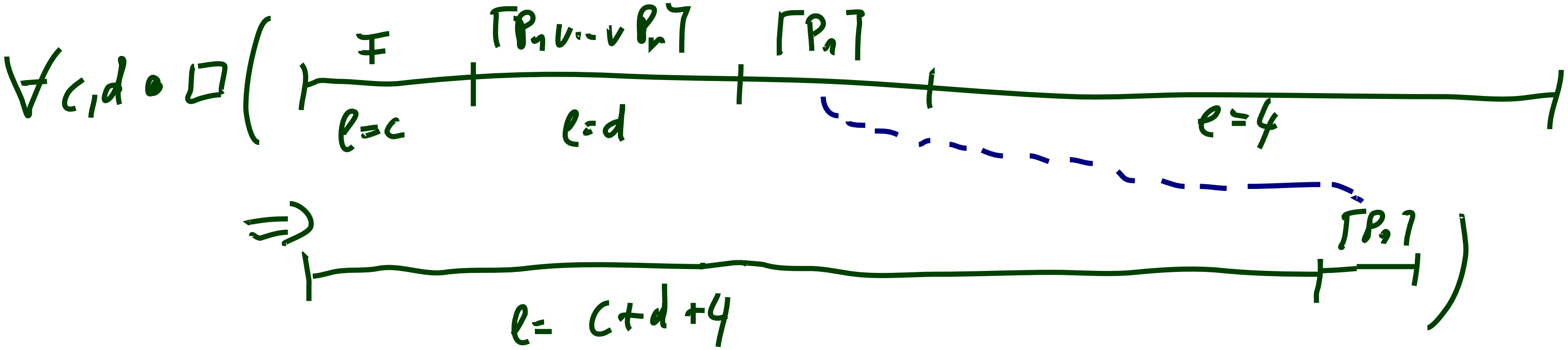
formula / state assertions

$$copy(F, \{P_1, \dots, P_n\}) :\Longleftrightarrow$$

$$\forall\, c, d \bullet \Box((F \wedge \ell = c)\,;\,(\lceil P_1 \vee \dots \vee P_n \rceil \wedge \ell = d)\,;\,\lceil P_1 \rceil\,;\,\ell = 4$$
$$\Longrightarrow\ \ell = c + d + 4\,;\,\lceil P_1 \rceil$$

$$\wedge \ \dots$$

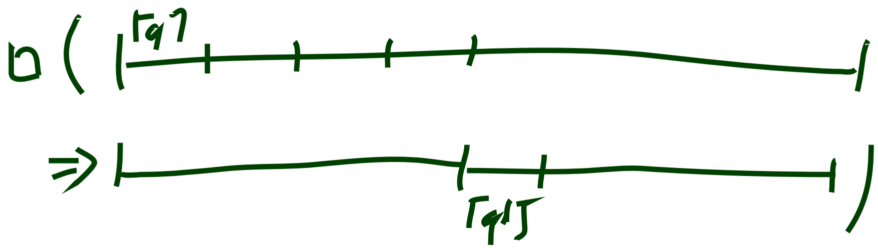
$$\wedge\ \forall\, c, d \bullet \Box((F \wedge \ell = c)\,;\,(\lceil P_1 \vee \dots \vee P_n \rceil \wedge \ell = d)\,;\,\lceil P_n \rceil\,;\,\ell = 4$$
$$\Longrightarrow\ \ell = c + d + 4\,;\,\lceil P_n \rceil$$



$q : inc_1 : q'$ (Increment)

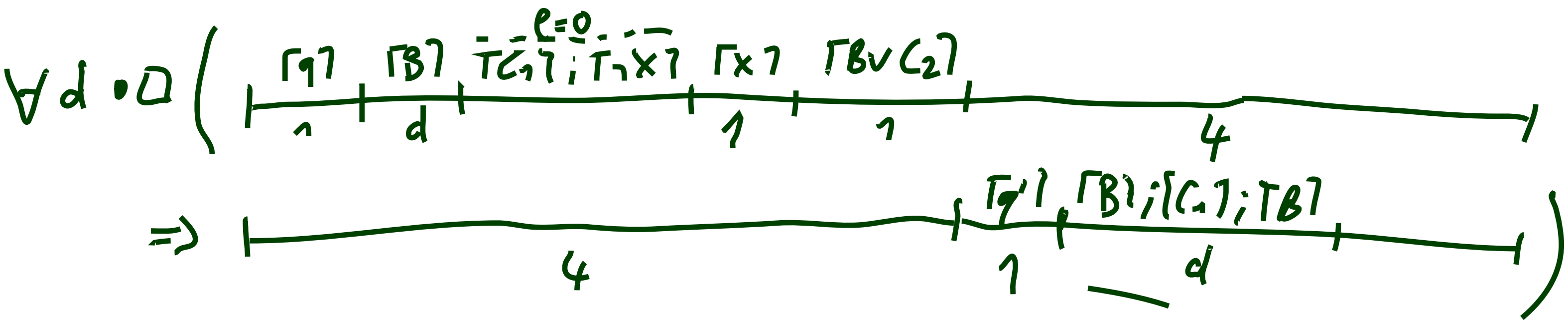
(i) Change state

$$\Box(\lceil q \rceil^1 \,;\, \lceil B \vee C_1 \rceil^1 \,;\, \lceil X \rceil^1 \,;\, \lceil B \vee C_2 \rceil^1 \,;\, \ell = 4 \implies \ell = 4 \,;\, \lceil q' \rceil^1 \,;\, \mathit{true})$$



(ii) Increment counter

$$\forall\, d \bullet \Box(\lceil q \rceil^1 \,;\, \lceil B \rceil^d \,;\, (\ell = 0 \vee \lceil C_1 \rceil \,;\, \lceil \neg X \rceil) \,;\, \lceil X \rceil^1 \,;\, \lceil B \vee C_2 \rceil^1 \,;\, \ell = 4$$
$$\implies \ell = 4 \,;\, \lceil q' \rceil^1 \,;\, (\lceil B \rceil \,;\, \lceil C_1 \rceil \,;\, \lceil B \rceil \wedge \ell = d) \,;\, \mathit{true}$$



$q : inc_1 : q'$ (Increment)

(i) Keep rest of first counter

$$copy(\lceil q \rceil^1 \,;\, \lceil B \vee C_1 \rceil \,;\, \lceil C_1 \rceil, \{B, C_1\})$$

$\mathcal{F}$ $\{P_1, P_2\}$

(ii) Leave second counter unchanged

$$copy(\lceil q \rceil^1 \,;\, \lceil B \vee C_1 \rceil \,;\, \lceil X \rceil^1, \{B, C_2\})$$

$q : dec_1 : q', q''$ (Decrement)

(i) If zero

$$\Box(\lceil q\rceil^1 \,;\, \lceil B\rceil^1 \,;\, \lceil X\rceil^1 \,;\, \lceil B \vee C_2\rceil^1 \,;\, \ell = 4 \implies \ell = 4 \,;\, \lceil q'\rceil^1 \,;\, \lceil B\rceil^1 \,;\, \mathit{true})$$

(ii) Decrement counter

$$\forall\, d \bullet \Box(\lceil q\rceil^1 \,;\, (\lceil B\rceil \,;\, \lceil C_1\rceil \wedge \ell = d) \,;\, \lceil B\rceil \,;\, \lceil B \vee C_1\rceil \,;\, \lceil X\rceil^1 \,;\, \lceil B \vee C_2\rceil^1 \,;\, \ell = 4 \implies \ell = 4 \,;\, \lceil q''\rceil^1 \,;\, \lceil B\rceil^d \,;\, \mathit{true})$$

(iii) Keep rest of first counter

$$\mathit{copy}(\lceil q\rceil^1 \,;\, \lceil B\rceil \,;\, \lceil C_1\rceil \,;\, \lceil B_1\rceil, \{B, C_1\})$$

(iv) Leave second counter unchanged

$$\mathit{copy}(\lceil q\rceil^1 \,;\, \lceil B \vee C_1\rceil \,;\, \lceil X\rceil^1, \{B, C_2\})$$

Final State

$$copy(\lceil q_{fin} \rceil^1 \,;\, \lceil B \vee C_1 \rceil^1 \,;\, \lceil X \rceil \,;\, \lceil B \vee C_2 \rceil^1, \{q_{fin}, B, X, C_1, C_2\})$$

Satisfiability

- Following [Chaochen and Hansen, 2004] we can observe that

  $\mathcal{M}$ **halts if and only if** the DC formula $F(\mathcal{M}) \land \Diamond\lceil q_{fin} \rceil$ is **satisfiable**.

  This yields

  Theorem 3.11. The satisfiability problem for DC with continuous time is undecidable.

  (It is semi-decidable.)

- Furthermore, by taking the contraposition, we see

  $\mathcal{M}$ **diverges** **if and only if** $\mathcal{M}$ does not **halt**

  if and only if $F(\mathcal{M}) \land \neg\Diamond\lceil q_{fin} \rceil$ is **not** satisfiable.

- Thus whether a DC formula is **not satisfiable** is not decidable, not even semi-decidable.

Validity

- By Remark 2.13, F is valid iff $\neg F$ is not satisfiable, so

> **Corollary 3.12.** The validity problem for DC with continuous time is undecidable, not even semi-decidable.

- This provides us with an alternative proof of Theorem 2.23 ("there is no sound and complete proof system for DC"):
  - **Suppose** there were such a calculus $\mathcal{C}$.
  - By Lemma 2.22 it is semi-decidable whether a given DC formula F is a theorem in $\mathcal{C}$.
  - By the soundness and completeness of $\mathcal{C}$, F is a theorem in $\mathcal{C}$ **if and only if** F is valid.
  - Thus it is semi-decidable whether F is valid. **Contradiction.**

Discussion

- Note: the DC fragment defined by the following grammar is **sufficient** for the reduction

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \,;\, F_2 \mid \ell = 1 \mid \ell = x \mid \forall\, x \bullet F_1,$$

  P a state assertion, x a global variable.

- Formulae used in the reduction are abbreviations:

$$\begin{aligned} \ell = 4 &\iff \ell = 1 \,;\, \ell = 1 \,;\, \ell = 1 \,;\, \ell = 1 \\ \ell \geq 4 &\iff \ell = 4 \,;\, \mathit{true} \\ \ell = x + y + 4 &\iff \ell = x \,;\, \ell = y \,;\, \ell = 4 \end{aligned}$$

- Length 1 is not necessary — we can use $\ell = z$ instead, with fresh z.

- This is RDC augmented by "$\ell = x$" and "$\forall\, x$", which we denote by **RDC** $+\ \ell = x, \forall\, x$.

References

References

[Chaochen and Hansen, 2004] Chaochen, Z. and Hansen, M. R. (2004). *Duration Calculus: A Formal Approach to Real-Time Systems*. Monographs in Theoretical Computer Science. Springer-Verlag. An EATCS Series.

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.