3 Trees: traversal and analysis of standard search trees

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

Binary search trees

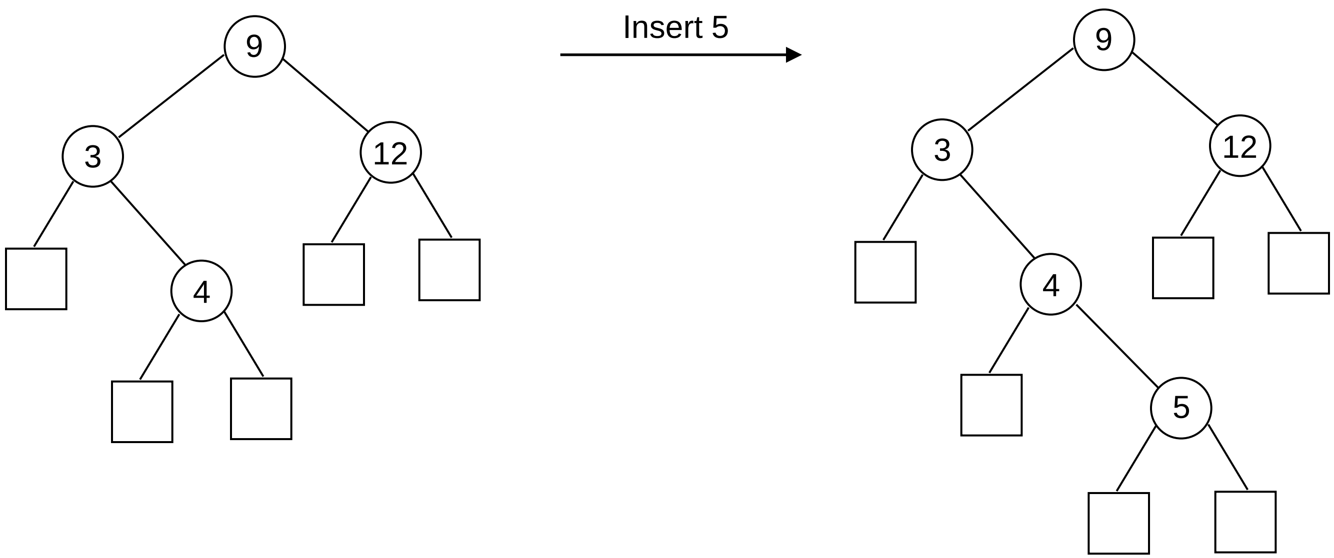
- Binary trees for storing sets of keys, such that the operations are supported:
  - find
  - insert
  - delete

- Search tree property:

  all keys in the left subtree of a node *p* are smaller than the key of *p*, and the key of *p* is smaller than all keys in the right subtree of *p*.

- Implementation:

Standard binary search trees (8)



- Tree structure depends on the order of insertions into the initially empty tree
- Height can increase linearly, but it can also be in O(log *n*), more precisely $\lceil \log 2\,(n+1) \rceil$.

Traversal of trees

Traversal of the nodes of a tree

- for output
- for calculating the sum, average, number of keys ...
- for changing the structure

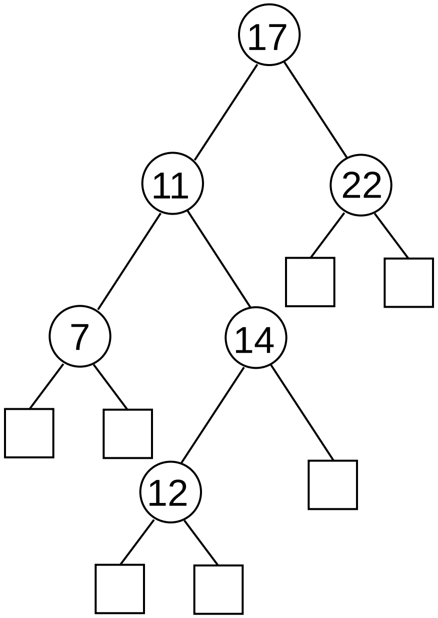
Most important traversal orders:

- preorder = NLR (Node-Left-Right)
  first visit the root, then recursively the left and right subtree (if existent)
- postorder = LRN
- inorder = LNR
- the mirror image versions of 1-3

Preorder

Preorder traversal is recursively defined as follows:

- Traversal of all nodes of a binary tree with root *p* in preorder:
  - visit *p*,
  - traverse the left subtree of *p* in preorder,
  - traverse the right subtree of *p* in preorder.



Preorder implementation

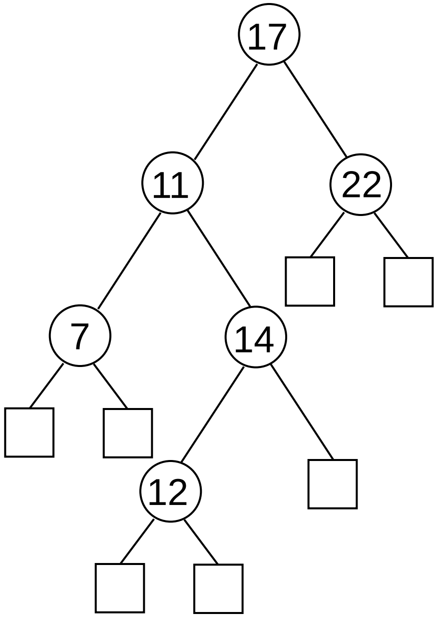
```
// Preorder Node-Left-Right
   void preOrder (){
       preOrder(root);
       System.out.println ();
   }
   void preOrder(SearchNode n){
       if (n == null) return;
       System.out.print (n.content+" ");
       preOrder(n.left);
       preOrder(n.right);
   }

   // Postorder Left-Right-Node
   void postOrder(){
       postOrder(root);
       System.out.println ();
   }
   // ...
```

Inorder

- The traversal order is: first the left subtree, then the root, then the right subtree:

```
 // Inorder Left-Node-Right
void inOrder(){
    inOrder(root);
    System.out.println ();
}
void inOrder(SearchNode n){
    if (n == null) return;
    inOrder(n.left);
    System.out.print (n.content+" ");
    inOrder(n.right);
}
```

Example



Preorder:

17, 11, 7, 14, 12, 22

Postorder:

7, 12, 14, 11, 22, 17

Inorder:

7, 11, 12, 14, 17, 22

Sorting with standard search trees

Idea: Create a search tree for the input sequence and output the keys by an inorder traversal.

Remark: Depending on the input sequence, the search tree may degenerate.

Complexity**:** Depends on internal path length

Worst case: Sorted input: $\Rightarrow \Omega(n^2)$ steps.

Best case: We get a complete search tree of minimal height of about log *n*. Then *n* insertions and outputs are possible in time O(*n* log *n*).

Average case: ?

Analysis of search trees

Two possible approaches to determine the internal path length:

1. Random tree analysis, i.e. average over all possible permutations of keys to be inserted (into the initially empty tree).

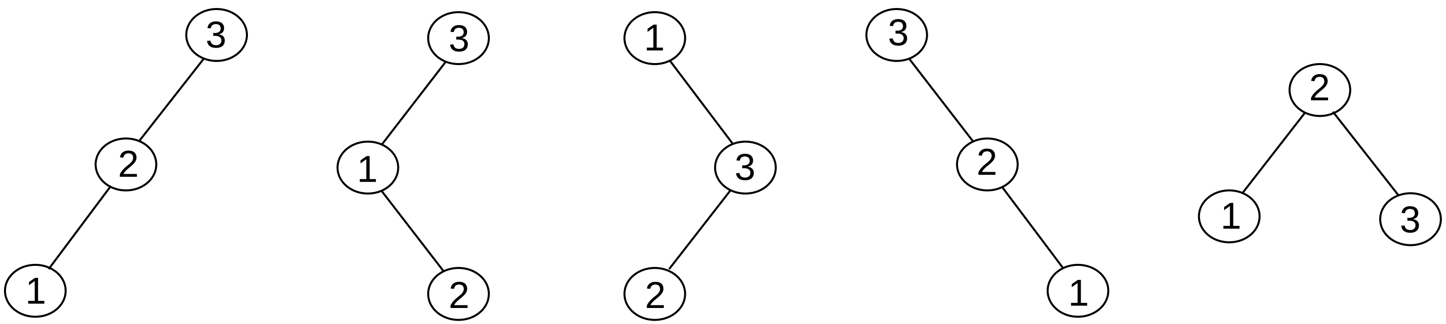
2. Shape analysis, i.e. average over all structurally different trees with n keys .

Difference of the expected values for the internal path:

1. $\approx 1.386\ n \log_2 n - 0.846 \cdot n + O(\log n)$

2. $\approx n \cdot \sqrt{\pi n} + O(n)$

Reason for the difference



3,2,1   3,1,2   1,3,2   3,2,1   2,1,3 und 2,3,1

$\longrightarrow$ Random tree analysis counts more balanced trees more often.

Internal path length

Internal path length *I*: measure for judging the quality of a search tree *t*.

Recursive definition:

1. If *t* is empty, then $I(t) = 0$
2. For a tree *t* with left subtree t_l and right subtree t_r :

$$I(t) = I(t_l) + I(t_r) + \#nodes\ in\ t$$

Apparently:

$$I(t) = \sum_p (depth(p) + 1)$$

p internal node of t

Average search path length

For a tree *t* the average search path length is defined by:

$$D(t) \;:= I(t)/n, n \;=\; \#\text{ nodes in } t.$$

Question: What is the size of *D*(*t*) in the

- best
- worst
- average

case for a tree *t* with *n* internal nodes?

Internal path: best case

We obtain a complete binary tree

Internal path: worst case

Random trees

- Without loss of generality, let $\{1,\ldots,n\}$ be the keys to be inserted.
- Let $s_1,\ldots, s_n$ be a random permutation of these keys.
- Hence, the probability that s_1 has the value k, $P(s_1=k) = 1/n$.
- If k is the first key, k will be stored in the root.
- Then the left subtree contains k-1 elements (the keys 1, …, k-1) and the right subtree contains n-k elements (the keys k+1, …,n).

Expected internal path length

- $EI(n)$: Expectation for the internal path length of a randomly generated binary search tree with n nodes
- Apparently we have:

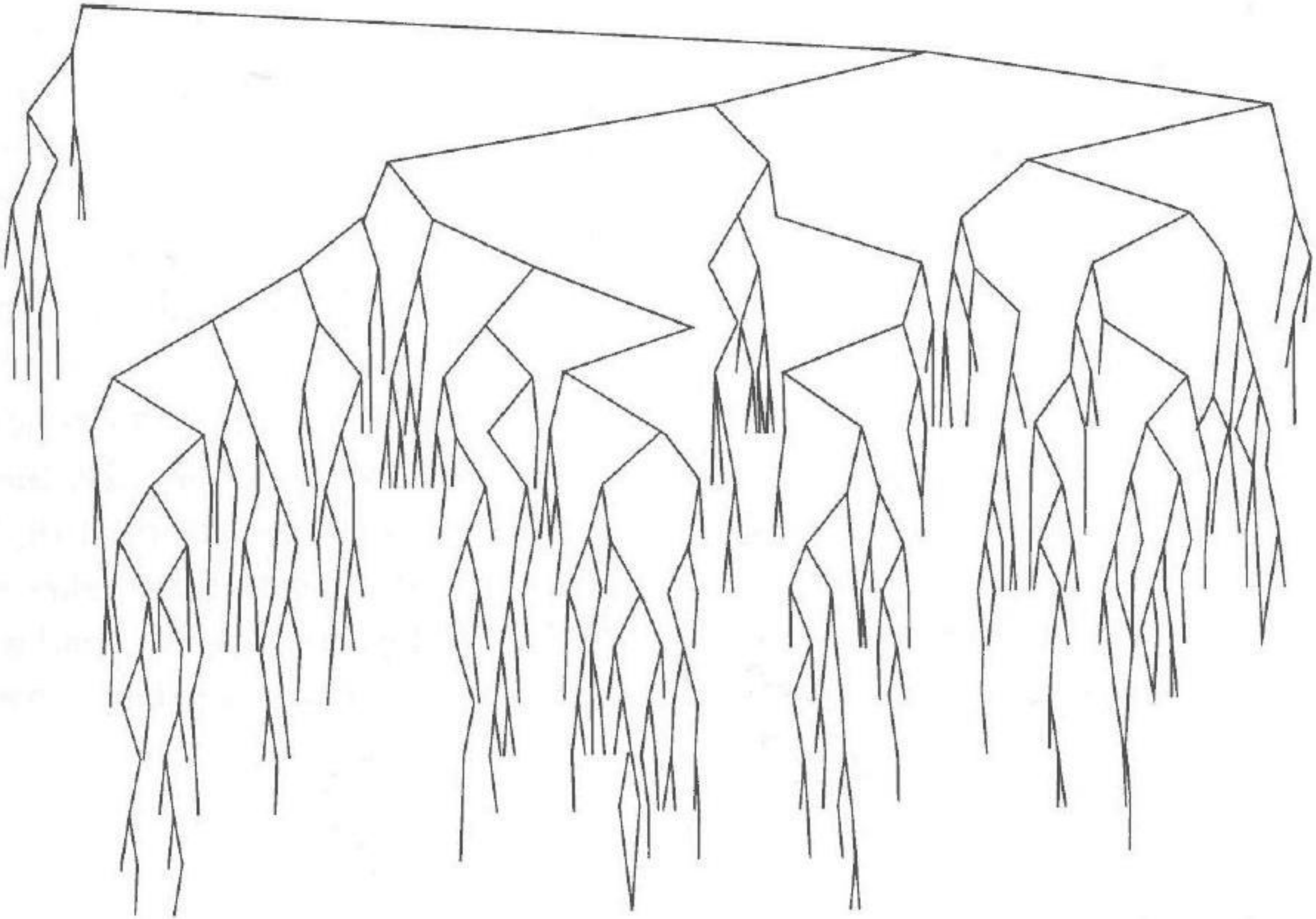
$$
\begin{aligned}
EI(0) &= 0 \\
EI(1) &= 1 \\
EI(n) &= \frac{1}{n}\sum_{k=1}^{n}\left(EI(j-1) + EI(n-k) + n\right) \\
&= n + \frac{1}{n}\left(\sum_{k=1}^{n} EI(k-1) + \sum_{k=1}^{n} EI(n-k)\right)
\end{aligned}
$$

- Claim: $EI(n) \approx 1.386 n \log_2 n - 0.846 n + O(\log n)$.

Observation

- Search, insertion and deletion of a key in a randomly generated binary search tree with n keys can be done, on average, in $O(\log_2 n)$ steps.
- In the worste case, average cost for search, insertion and deletion can be linear in the number of items.
- One can show that the average distance of a node from the root in a randomly generated tree is only about 40% above the optimal value.
- However, by the restriction to the symmetrical successor, the behaviour becomes worse.
- If n^2 update operations are carried out in a randomly generated search tree with n keys, the expected average search path is only $\Theta(\sqrt{n})$.

Typical binary tree for a random sequence of keys



Resulting binary tree after n^2 updates