# 10 Randomized algorithms

Albert-Ludwigs-Universität Freiburg

# Randomized algorithms

- Classes of randomized algorithms
- Randomized Quicksort
- Randomized primality test
- Cryptography

# 1. Classes of ramdomized algorithms

- **Las Vegas** algorithms
  always correct; expected running time ("probably fast")

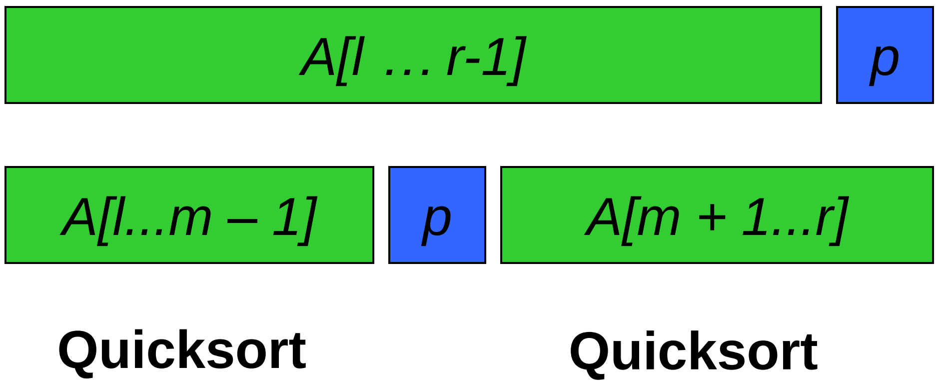
  Example: randomized Quicksort

- **Monte Carlo** algorithms (mostly correct):
  probably correct; guaranteed running time

  Example: randomized primality test

# 2. Quicksort

Unsorted range *A*[*l*, *r*] in array *A*



# Quicksort

**Algorithm:** *Quicksort*

**Input:** unsorted range $[l, r]$ in array $A$

**Output:** sorted range $[l, r]$ in array $A$

**if** $r > l$

  **then** choose pivot element $p = A[r]$

    $m = divide(A, l, r)$

      /* Divide $A$ according to $p$:

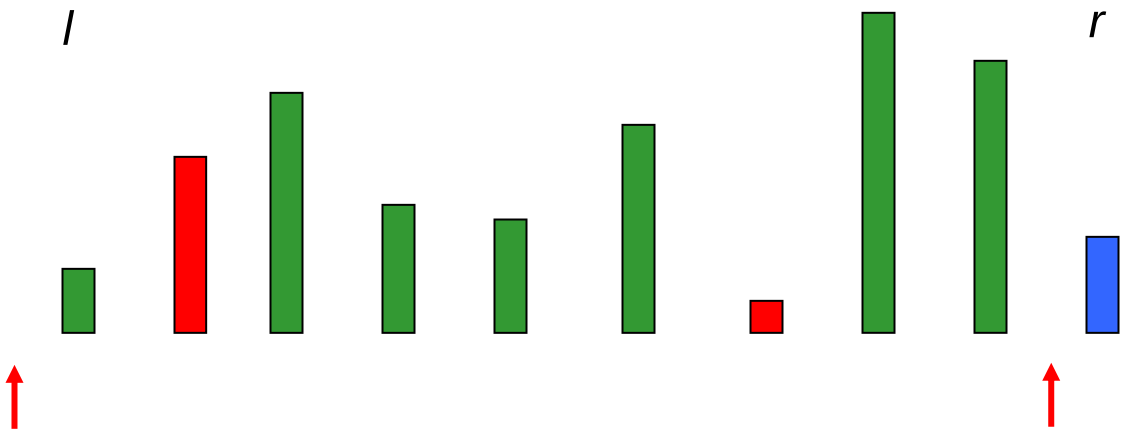
        $A[l],....,A[m-1] \leq p \leq A[m+1],...,A[r]$

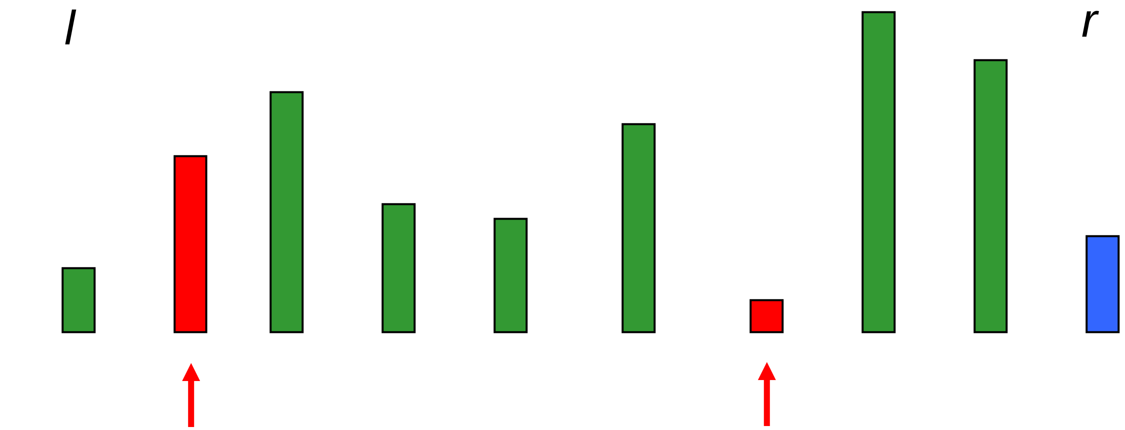
      */

    Quicksort($A, l, m - 1$)

    Quicksort ($A, m + 1, r$)

# The *divide* step



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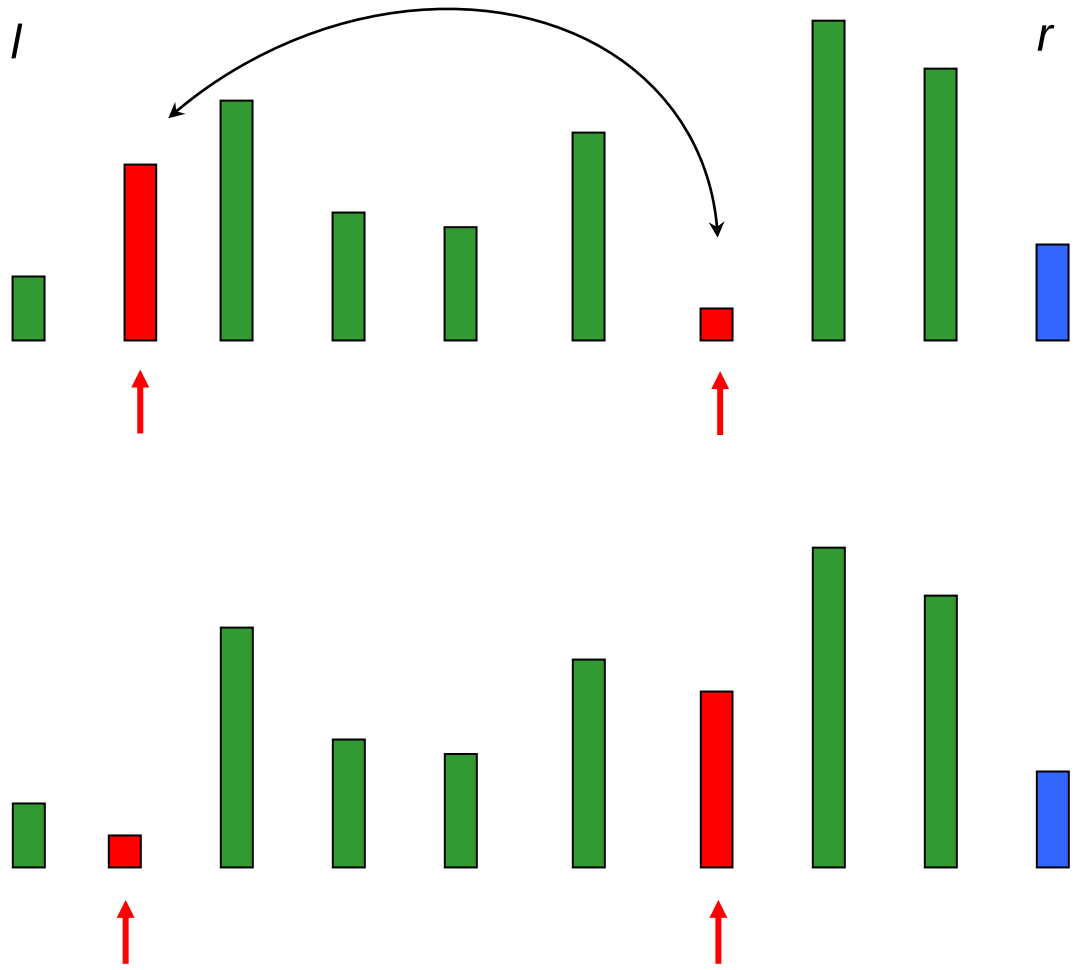
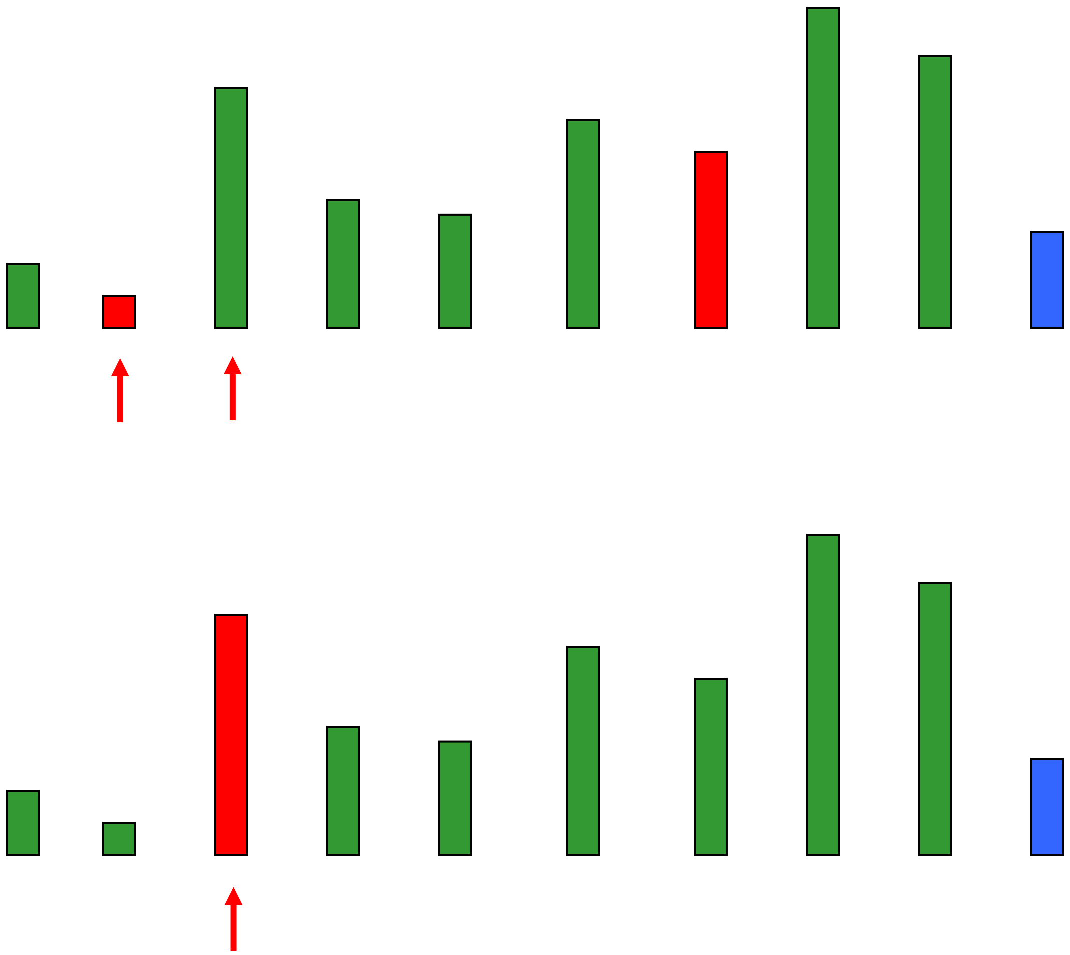
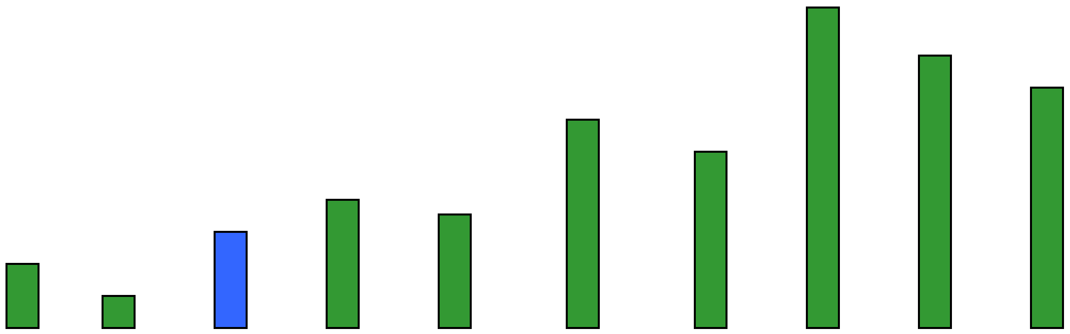


# The *divide* step



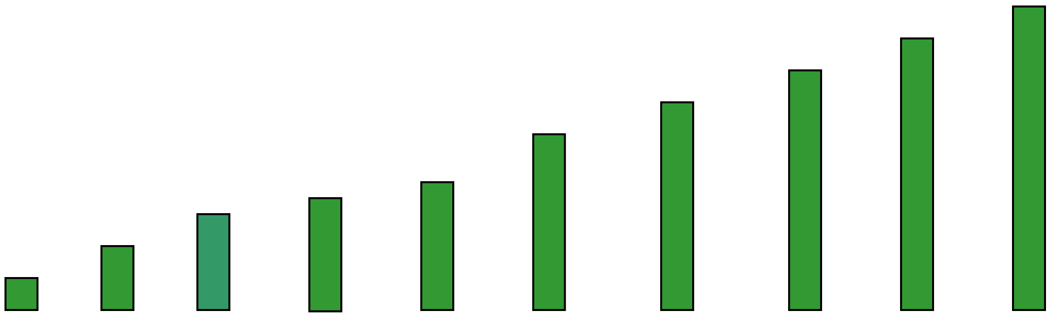
# The *divide* step



*divide*(*A*, *l* , *r*):

- returns the index of the pivot element in *A*
- can be done in time $O(r - l)$

# Worst case input



*n* elements*:*

Running time: $(n\text{-}1) + (n\text{-}2) + \ldots + 2 + 1 = n \cdot (n\text{-}1)/2$

# 3. Randomized Quicksort

**Algorithm:** Quicksort

**Input:** unsorted range [$l$, $r$] in array $A$

**Output:** sorted range [$l$, $r$] in array $A$

```
if  r > l
    then randomly choose a pivot element p = A[i] in range [l, r]
         swap A[i] and A[r]
         m = divide(A, l, r)
         /* Divide A according to p:
               A[l],....,A[m – 1] ≤ p ≤ A[m + 1],...,A[r]
          */
          Quicksort(A, l, m - 1)
          Quicksort(A, m + 1, r)
```

# Analysis

$n$ elements; let $S_i$ be the $i$-th smallest element

- $S_1$ is chosen as pivot with probability $1/n$:
- Sub-problems of sizes $0$ and $n$-1

•
•
•

- $S_k$ is chosen as pivot with probability $1/n$:
- Sub-problems of sizes $k$-1 and $n$-$k$

•
•
•

- $S_n$ is chosen as pivot with probability $1/n$:
- Sub-problems of sizes $n$-1 and $0$

# Analysis

**Expected running time**:

$$E(T(n)) = \frac{1}{n}\sum_{k=0}^{n-1}(E(T(k)) + E(T(n-k-1))) + \Theta(n)$$

$$= \frac{2}{n}\sum_{k=0}^{n-1} E(T(k)) + \Theta(n)$$

$$\leq \frac{2}{n}\sum_{k=2}^{n-1} E(T(k)) + \Theta(n)$$

absorb k=0,k=1 in $\Theta(n)$

# Analysis

$$E(T(n)) = \frac{2}{n}\sum_{k=2}^{n-1} E(T(k)) + \Theta(n)$$

Prove : $E[T(n)] \leq cn \lg n$ for $n \geq 2$ and some $c > 0$

base case :

choose c large enough s.t. E[$T(n)$]$\leq cn \lg n$ for n = 2

Inductive step :

substitute inductive hypotesis

$$E(T(n)) \leq \frac{2}{n}\sum_{k=2}^{n-1} ck \lg k + \Theta(n)$$

# Analysis

$$E(T(n)) \le \frac{2c}{n}\sum_{k=2}^{n-1} k \lg k + \Theta(n)$$

$$(\sum_{k=2}^{n-1} k \lg k \le \frac{n^2}{2}\lg n - \frac{n^2}{8})$$

(Proof as exercise!)

$$E(T(n)) \le \frac{2c}{n}(\frac{n^2}{2}\lg n - \frac{n^2}{8}) + \Theta(n)$$

$$\le cn \lg n - (\frac{cn}{4} - \Theta(n))$$

# Analysis

$$E(T(n)) \leq cn\lg n - (\frac{cn}{4} - \Theta(n))$$

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desired

Should be >=0

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desired

Should be >=0

$E(T(n)) \le cn \lg n$ where we choose $c$ large enough s.t. $cn/4$ dominate $\Theta(n)$

# 4. Primality test

**Definition:**

An integer $p \geq 2$ is prime **iff** ($a \mid p \rightarrow a = 1$ **or** $a = p$).

**Algorithm:** deterministic primality test (naive)

**Input:** integer $n \geq 2$

**Output:** answer to the question: Is $n$ prime?

**if** $n = 2$ **then return** true
**if** $n$ even **then return** false
**for** $i = 1$ **to** $\sqrt{n}/2$ **do**
  **if** $2i + 1$ divides $n$
  **then return** false
**return** true

Complexity: $\Theta(\sqrt{n})$

# Primality test

**Goal:**

Randomized method

- Polynomial time complexity (in the length of the input)
- If answer is "not prime", then $n$ is not prime
- If answer is "prime", then the probability that $n$ is not prime is at most $p>0$

$k$ iterations: probability that $n$ is not prime is at most $p^k$

# Primality test

**Observation:**

Each odd prime number $p$ divides $2^{p-1} - 1$.

**Examples:** $p = 17$, $2^{16} - 1 = 65535 = 17 * 3855$

$p = 23$, $2^{22} - 1 = 4194303 = 23 * 182361$

**Simple primality test:**

1 Calculate $z = 2^{n-1} \bmod n$

2 if $z = 1$

3 then $n$ is possibly prime

4 else $n$ is definitely not prime

Advantage: This only takes polynomial time

# Simple primality test

**Definition:**

*n* is called pseudoprime to base 2, if *n* is not prime and

$$2^{n-1} \bmod n = 1.$$

**Example:** $n = 11 * 31 = 341$

$$2^{340} \bmod 341 = 1$$

# Randomized primality test

**Theorem:** (Fermat‘s little theorem)

If $p$ prime and $0 < a < p$, then

$$a^{p-1} \bmod p = 1.$$

**Definition:**

$n$ is pseudoprime to base $a$, if $n$ not prime and

$$a^{n-1} \bmod n = 1.$$

**Example:** $n = 341$, $a = 3$

$$3^{340} \bmod 341 = 56 \neq 1$$

# Randomized primality test

**Algorithm:** Randomized primality test 1

Randomly choose $a \in [2, n\text{-}1]$
Calculate $a^{n-1}$ mod $n$
**if** $a^{n-1}$ mod $n = 1$
    **then** $n$ is possibly prime
    **else** $n$ is definitely not prime

# Carmichael numbers

**Problem:** Carmichael numbers

**Definition:** An integer *n* is called Carmichael number if

$$a^{n-1} \bmod n = 1$$

for all *a* with GCD(*a*, *n*) = 1. **(GCD = greatest common divisor)**

**Example:**

Smallest Carmichael number: 561 = 3 * 11 * 17

# Randomized primality test 2

**Theorem:**

If $p$ prime and $0 < a < p$, then the only solutions to the equation

$$a^2 \bmod p = 1$$

are $a = 1$ and $a = p - 1$.

**Definition:**

$a$ is called non-trivial square root of 1 mod $n$, if

$$a^2 \bmod n = 1 \quad \text{and} \quad a \neq 1, n - 1.$$

**Example:** $n = 35$

$$6^2 \bmod 35 = 1$$

# Fast exponentiation

**Idea:**

During the computation of $a^{n-1}$ ($0 < a < n$ randomly chosen), test whether there is a non-trivial square root 1 mod $n$.

**Method for the computation of $a^n$:**

**Case 1**: [$n$ is even]

$a^n = a^{n/2} * a^{n/2}$

**Case 2**: [$n$ is odd]

$a^n = a^{(n-1)/2} * a^{(n-1)/2} * a$

# Fast exponentiation

**Example:**

$a^{62} = (a^{31})^2$

$a^{31} = (a^{15})^2 * a$

$a^{15} = (a^{7})^2 * a$

$a^{7} \quad = (a^{3})^2 * a$

$a^{3} \quad = (a)^2 * a$

Complexity: $O(\log^2 a^n \log n)$

# Fast exponentiation

```
boolean isProbablyPrime;

power(int a, int p, int n) {
    /* computes a^p mod n and checks during the
       computation whether there is an x with
       x^2 mod n = 1 and x ≠ 1, n-1 */

    if (p == 0) return 1;
    x = power(a, p/2, n)
    result = (x * x) % n;
```

# Fast exponentiation

```
/* check whether x^2 mod n = 1 and x ≠ 1, n-1 */
    if (result == 1 && x != 1 && x != n -1 )
        isProbablyPrime = false;

    if (p % 2 == 1)
        result = (a * result) % n;

    return result;
}
```

Complexity: $O(\log^2 n \log p)$

# Randomized primality test 2

```
primalityTest(int n) {
    /* carries out the randomized primality test for
       a randomly selected a */

    a = random(2, n-1);

    isProbablyPrime = true;

    result = power(a, n-1, n);

    if (result != 1 || !isProbablyPrime)
        return false;
    else
        return true;
}
```

# Randomized primality test 2

**Theorem:**

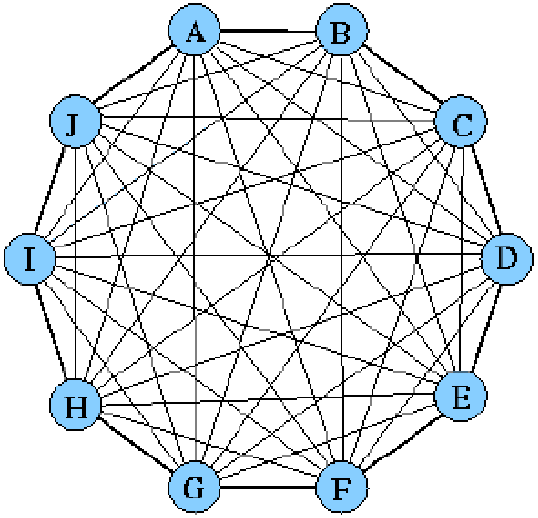
If $n$ is not prime, there are at most ${}^{(n-1)}/_{4}$ integers $0 < a < n$, for which the algorithm primalityTest fails.

# Application: cryptosystems

## Traditional encryption of messages with secret keys

**Disadvantages:**

1. The key *k* has to be exchanged between A and B before the transmission of the message.
2. For messages between *n* parties *n*(*n*-1)/2 keys are required.




**Advantage:**

Encryption and decryption can be computed very efficiently.

# Desired properties of cryptographic systems

- confidential transmission
- integrity of data
- authenticity of the sender
- reliable transmission

# Public-key cryptosystems

Diffie and Hellman (1976)

**Idea:** Each participant A has two keys:

1. a public key $P_A$ accessible to every other participant
2. a private (or: secret) key $S_A$ only known to $A$.

# Public-key cryptosystems

$D$ = set of all legal messages,
e.g. the set of all bit strings of finite length

$$P_A, S_A : D \to D$$

**Three conditions**:

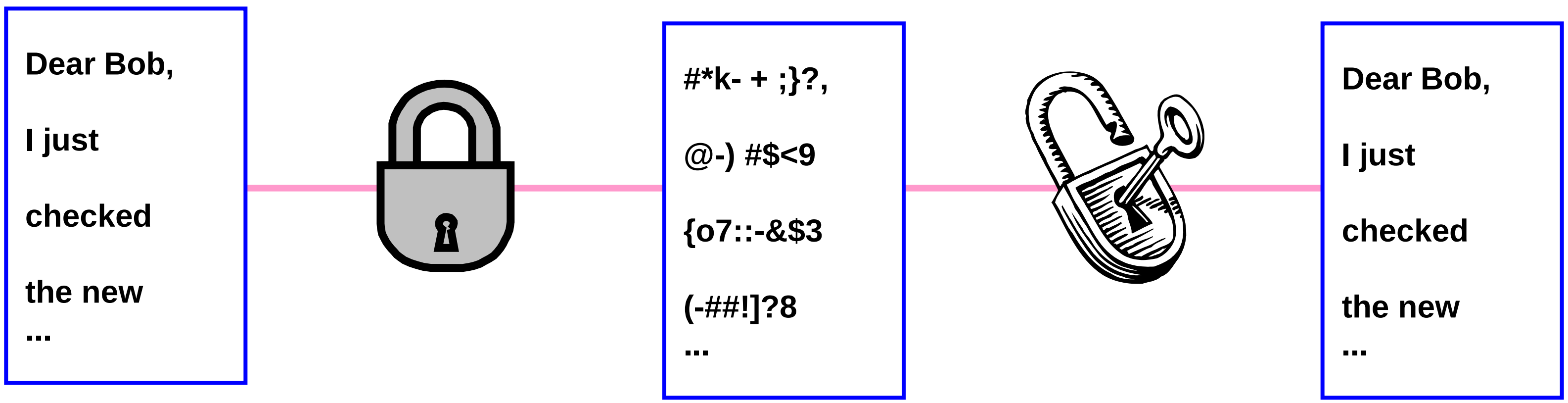
1. $P_A$ and $S_A$ can be computed efficiently

2. $S_A(P_A(M)) = M$ and $P_A(S_A(M)) = M$
($P_A$ is the inverse function of $S_A$ and vice-versa)

3. $S_A$ cannot be computed from $P_A$ (without unreasonable effort)

# Encryption in a public-key cryptosystem

*A* sends a message *M* to *B*.



# Encryption in a public-key cryptosystem

1. ***A*** accesses ***B***'s public key $P_B$ (from a public directory or directly from *B*).
2. ***A*** computes the encrypted message $C = P_B(M)$ and sends *C to* ***B***.
3. After ***B*** has received message *C*, ***B*** decrypts the message with his own private key $S_B$: $M = S_B(C)$

# Generating a digital signature

***A*** sends a digitally signed message $M'$ to ***B***:

1. ***A*** computes the digital signature $\sigma$ for $M'$ with her own private key:

$$\sigma = S_A(M')$$

2. ***A*** sends the pair $(M', \sigma)$ to ***B***.

3. After receiving $(M', \sigma)$, ***B*** verifies the digital signature:

$$P_A(\sigma) = M'$$

$\sigma$ can by verified by anybody via the public $P_A$.

# RSA cryptosystems

R. Rivest, A. Shamir, L. Adleman

Generating the public and private keys:

1. Randomly select two primes $p$ and $q$ of similar size, each with $l+1$ bits ($l \geq 500$).

2. Let $n = p \cdot q$

3. Let $e$ be an integer that does not divide $(p - 1) \cdot (q - 1)$.

4. Calculate $d = e^{-1} \bmod (p - 1)(q - 1)$

   i.e.:

$$d \cdot e \equiv 1 \bmod (p - 1)(q - 1)$$

# RSA cryptosystems

5. Publish $P = (e, n)$ as public key

6. Keep $S = (d, p, q)$ as private key

Divide message (described in a binary string) in blocks of size $2 \cdot l$.

Interpret each block $M$ as a binary number: $0 \leq M < 2^{2 \cdot l}$

$$P(M) = M^e \bmod n \qquad\qquad S(C) = C^d \bmod n$$