Introduction to Alternating Finite Automata

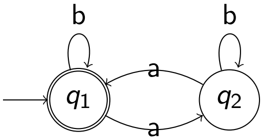
Pascal Raiola

July 24th, 2013

Outline

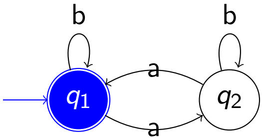
Accepting with DFAs

Example: ababa



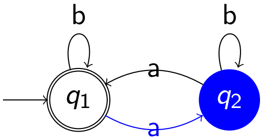
Accepting with DFAs

Example: $q_1$ababa



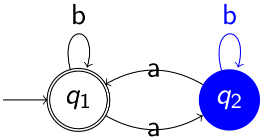
Accepting with DFAs

Example: a$q_2$baba



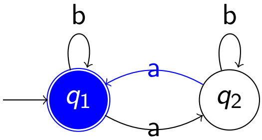
Accepting with DFAs

Example: $\text{ab}q_2\text{aba}$



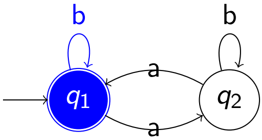
Accepting with DFAs

Example: $\color{blue}{\mathsf{aba}q_1}\mathsf{ba}$



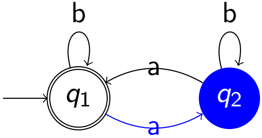
Accepting with DFAs

Example: ababq$_1$a



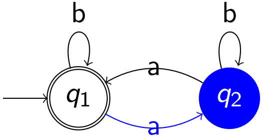
Accepting with DFAs

Example: $\color{blue}{\text{ababa}q_2}$



Accepting with DFAs

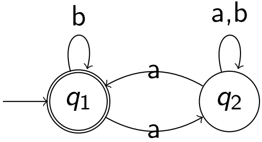
Example: ababaq_2



$\Rightarrow$ Not accepted.

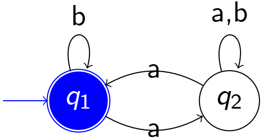
Accepting with NFAs

Example: ababa



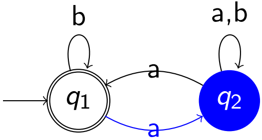
Accepting with NFAs

Example: $\{q_1\}$ababa



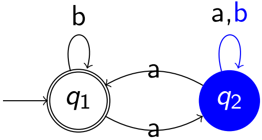
Accepting with NFAs

Example: a$\{q_2\}$baba



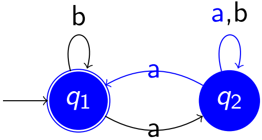
Accepting with NFAs

Example: ab$\{q_2\}$aba



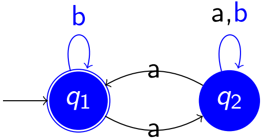
Accepting with NFAs

Example: aba$\{q_1, q_2\}$ba



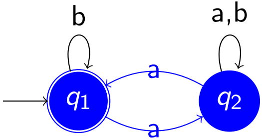
Accepting with NFAs

Example: $\text{abab}\{q_1, q_2\}\text{a}$



Accepting with NFAs

Example: $\text{ababa}\{q_1, q_2\}$



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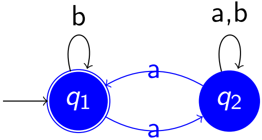

At least one accepting state $\Rightarrow$ Accepted.

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- but accept the same class of languages.

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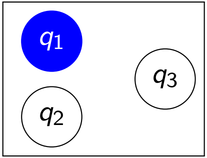
- but accept the same class of languages.

Can it be even more general?

Restrictions (NFA)

If we know: 

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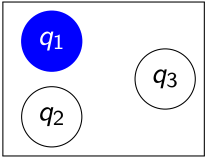
If we know: 

$\Rightarrow$ Reading a $\Rightarrow$

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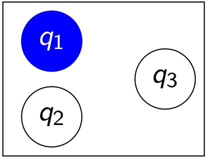

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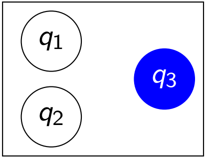

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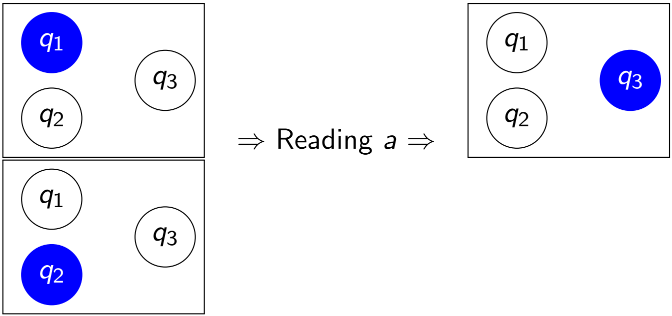

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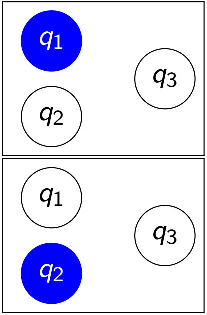

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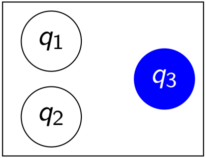

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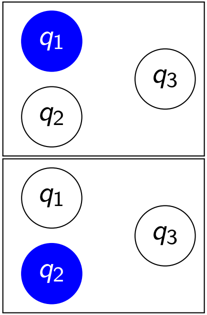

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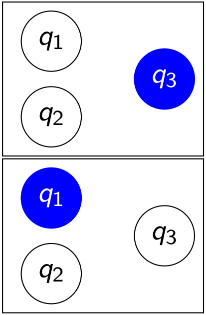

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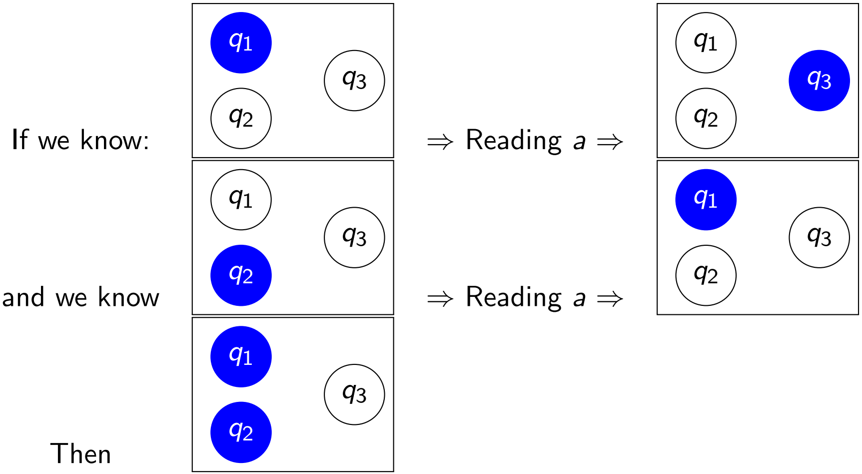

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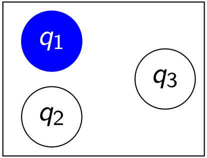
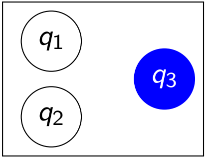
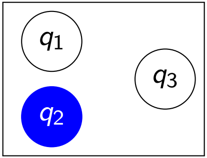
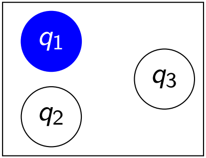
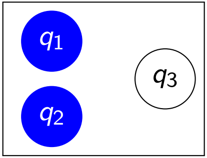
If we know:  $\Rightarrow$ Reading a $\Rightarrow$ 

and we know  $\Rightarrow$ Reading a $\Rightarrow$ 

Then  $\Rightarrow$ Reading a $\Rightarrow$

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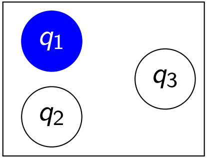

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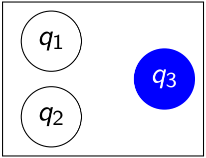

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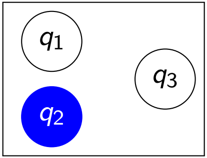

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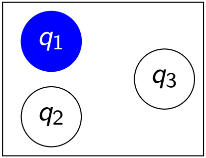

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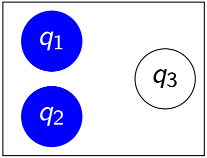

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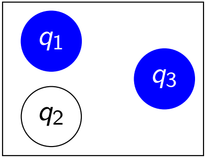

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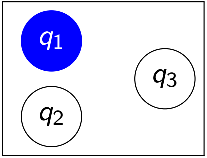

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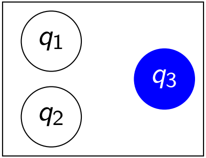

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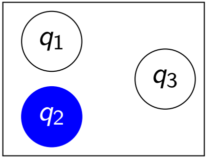

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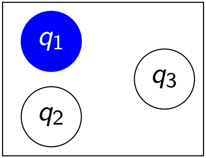

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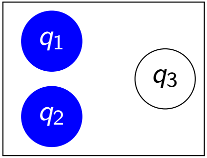

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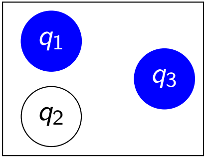

The transition can be more general!

Acceptance condition

- DFAs accept iff the run ends in a final state.

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- NFAs accept iff the run ends in a set containing at least one final state.

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- NFAs accept iff the run ends in a set containing at least one final state.
- More general: A function h deciding acceptance for each subset of Q:

$$h : 2^Q \to \{0, 1\}$$

Formal definition: h-AFA & r-AFA

An h-AFA/r-AFA is a 5-tuple (Q, Σ, g, h, F), where

- Q is the finite set of states,
- Σ is the input alphabet,

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$$Q = \{q_1, q_2, q_3, q_4, q_5\}, F = \{q_2, q_3\}$$
$$\Rightarrow f = (0 \quad , 1 \; , 1 \quad , 0 \quad , 0)$$

Formal definition: *h*-AFA & *r*-AFA

- The transition function $g : Q \times \Sigma \times 2^Q \to \{0, 1\}$ is universalized from getting just one letter as an input to a whole word:

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- Notation: $g(w, v) := (g(q, w, v))_{q \in Q}$.

Acceptance

An input w is accepted by an h-AFA iff

$$h(g(w, f)) = 1$$

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and by an r-AFA iff

$$h(g(w^R, f)) = 1.$$

Example: r-AFA

Let $A = (Q, \Sigma, g, h, F)$ be an r-AFA with

- $Q = \{q_1, q_2\}$,
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- and g is given by

$$g(a, (q_1, q_2)) = (q_1 \vee \overline{q_2}, \overline{q_1} \wedge \overline{q_2})$$
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Example: r-AFA

Let $w = ab$, then w is accepted by A as follows:

$$h(g(w^R, f))$$

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&= h(g(b, g(a, (0, 1)))) \\
&= h(g(b, (0 \vee \overline{1}, \overline{0} \wedge \overline{1}))) \\
&= h(g(b, (0, 0)))
\end{aligned}
$$

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Example: r-AFA

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Equivalence of DFAs and r-AFAs: "DFA $\Rightarrow$ r-AFA"

Let $A_D = (Q_D, \Sigma, \delta, s, F_D)$ be a DFA. Let $A_A = (Q_A, \Sigma, g, h, F_A)$ be an r-AFA with:

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Highly inefficient (see next talk)

Equivalence of DFAs and r-AFAs: "r-AFA $\Rightarrow$ DFA":

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- g and h as in the next slide.

Example:

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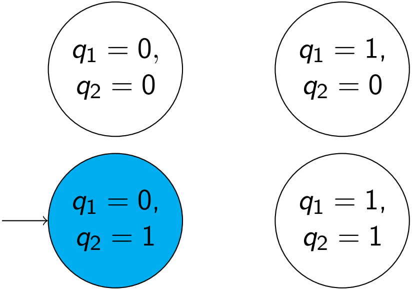

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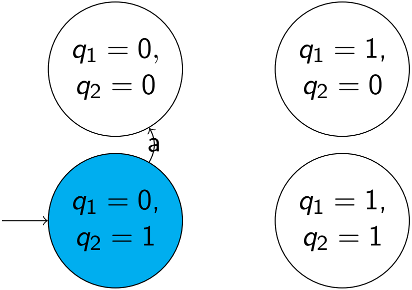

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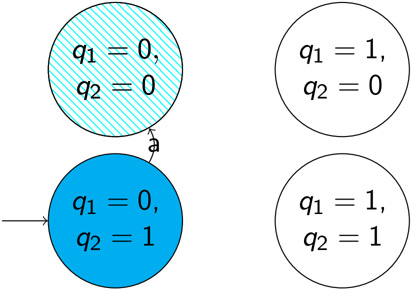

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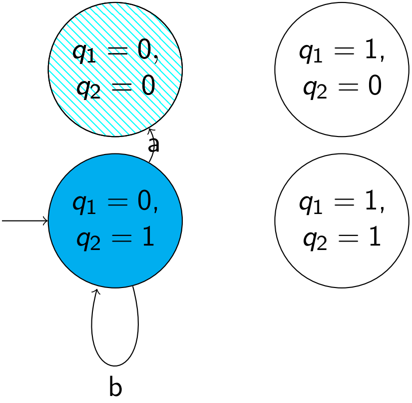

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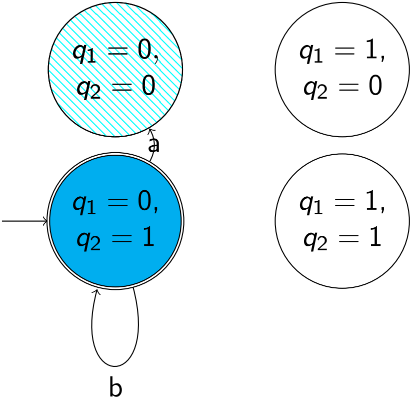

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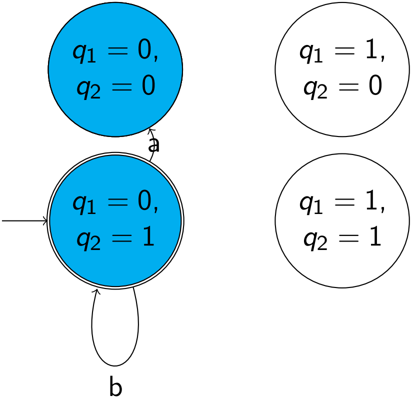

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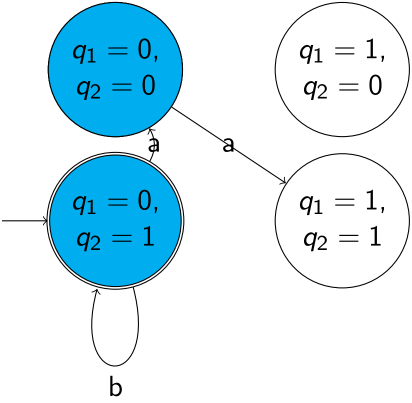

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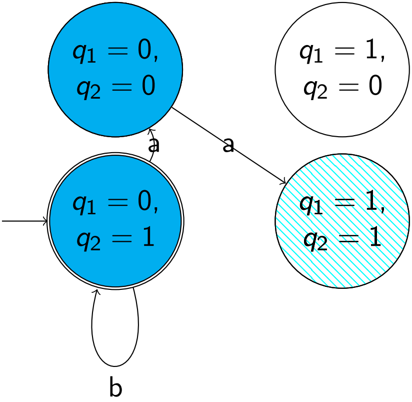

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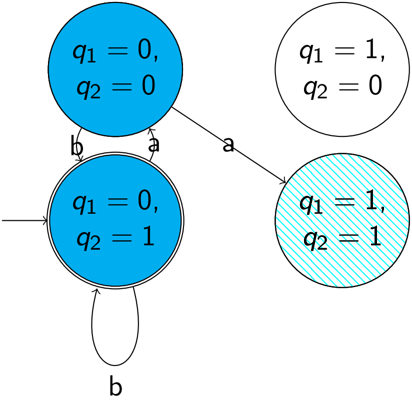

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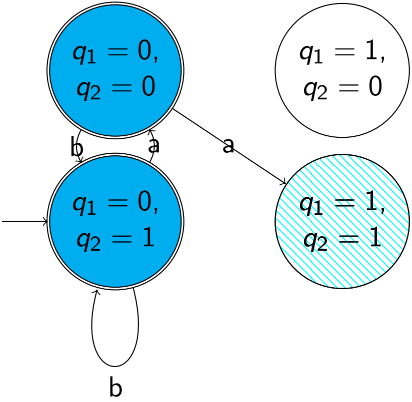

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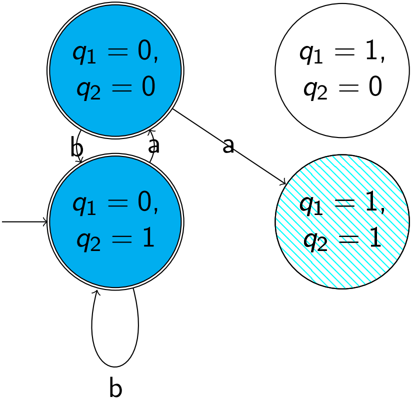

And so on...

Equivalence of DFAs and r-AFAs

r-AFAs $\sim$ regular languages

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Regular languages are closed under reversion

Equivalence of DFAs and r-AFAs

r-AFAs $\sim$ regular languages

Regular languages are closed under reversion

$\Rightarrow$ h-AFAs $\sim$ regular languages

Concatenation of two r-AFAs

Two r-AFAs:

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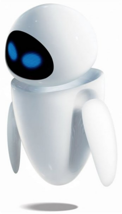

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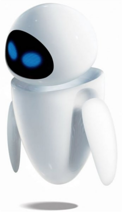


Target: r-AFA $A = (Q, \Sigma, g, h, F)$ with $L(A) = L(A_1) \cdot L(A_2)$.

Concatenation of two r-AFAs: Idea

•Zacatecas

Concatenation of two r-AFAs: Idea

Z•acatecas

Concatenation of two r-AFAs: Idea

Za•catecas

Concatenation of two r-AFAs: Idea

Za•catecas

•catecas

A_1: Za

Concatenation of two r-AFAs: Idea

Zac•atecas

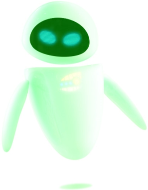
c•atecas

A_1: Za

Concatenation of two r-AFAs: Idea

Zaca•tecas

ca•tecas



A_1: Za

Concatenation of two r-AFAs: Idea

Zacat•ecas

cat•ecas

A_1: Za

Concatenation of two r-AFAs: Idea

Zacat•ecas

cat•ecas

A_1: Za

•ecas

A_1: Zacat

Concatenation of two r-AFAs: Idea

Zacate•cas

cate•cas

A_1: Za

e•cas

A_1: Zacat

Concatenation of two r-AFAs: Idea

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A_1: Za

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A_1: Zacate

Concatenation of two r-AFAs: Idea

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A_1: Zacate

Concatenation of two r-AFAs: Idea

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A_1: Zacate

Concatenation of two r-AFAs: Idea

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cas•

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Each subset $x \in 2^{Q_2}$ is associated to a state p_x.

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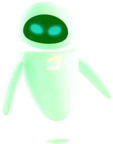
$$F = F_1 \cup \{p_{f_2}\}$$

Concatenation of two r-AFAs: Accepting

h has only to care for A_2, formally:

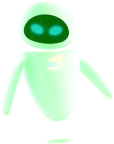


A_1: Zacat

Concatenation of two r-AFAs: Accepting

h has only to care for A_2, formally:

$$h(v) = 1 \quad \Leftrightarrow \quad \exists x \in [0, 2^m - 1].\ \underbrace{v_{n+x}}_{\rightsquigarrow p_x} = 1 \wedge h_2(x) = 1$$



A_1: Zacat

Concatenation of two r-AFAs: g on the first n states

A has to run A_1 on the whole input word without any possibility of interruption:

$$g(a, v)|_{Q_1} = g_1(a, v|_{Q_1})$$

Concatenation of two r-AFAs: g on the last 2^m states

Copies of A_2 should work parallel on the states p_k ($k \geq 0$)

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$\Rightarrow$ p_k should be reached from p_j
iff
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formally:

For all $k \geq 0, k \neq f_2$

$$g(p_k, a, v) = 1 \quad \Leftrightarrow \quad \exists j \in [0, 2^m - 1].\ v_{n+j} = 1 \wedge g_2(a, j) = k$$

Concatenation of two r-AFAs: Special treatment for p_{f_2}

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Formally:

$$g(p_{f_2}, a, v) = 1 \quad \Leftrightarrow \quad (\exists j \in [0, 2^m - 1].\ v_{n+j} = 1 \wedge g_2(a, j) = f_2) \\ \vee\ h_1(g(a, v)|_{Q_1}) = 1$$

Concatenation of two r-AFAs

Then $L(A) = L(A_1) \cdot L(A_2)$.

Sources

Literature:

- Efficient implementation of regular languages using reversed alternating finite automata, K. Salomaa, X. Wu, S. Yu, Theoretical Computer Science, Elsevier, 17 January 2000
- Implementing Reversed Alternating Finite Automaton (r-AFA) Operations, S. Huerter, K. Salomaa, Xiuming Wu, S. Yu

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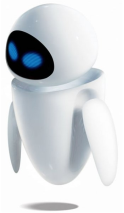
Pictures:

- A_1: Larry D. Moore CC BY-SA 3.0.
- A_2: Disney/Pixar

Thank you!



Additional operations: Union and intersection

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- $Q = Q_1 \cup Q_2$
- $g(q, a, u) = \begin{cases} g_1(q, a, u|_{Q_1}) & q \in Q_1 \\ g_2(q, a, u|_{Q_2}) & q \in Q_2 \end{cases}$

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- $h = h_1 \vee h_2$ (union) resp. $h = h_1 \wedge h_2$ (intersection).
- $F = F_1 \cup F_2$

Additional operations: Complemet

For the complement B of an AFA A, define $h_B = \overline{h_A}$.