Decision Procedures

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Theory of Arrays

Arrays: Quantifier-free Fragment of T_A

$$\Sigma_\mathsf{A} : \{\cdot[\cdot],\ \cdot\langle\cdot \triangleleft \cdot\rangle,\ =\}\ ,$$

where

- $a[i]$ is a binary function representing read of array a at index i;
- $a\langle i \triangleleft v\rangle$ is a ternary function representing write of value v to index i of array a;
- $=$ is a binary predicate. It is not used on arrays.

Axioms of T_A:

1. axioms of (reflexivity), (symmetry), and (transitivity) of T_E
2. $\forall a, i, j.\ i = j \rightarrow a[i] = a[j]$ (array congruence)
3. $\forall a, v, i, j.\ i = j \rightarrow a\langle i \triangleleft v\rangle[j] = v$ (read-over-write 1)
4. $\forall a, v, i, j.\ i \neq j \rightarrow a\langle i \triangleleft v\rangle[j] = a[j]$ (read-over-write 2)

Decision Procedure for T_A

Given quantifier-free conjunctive Σ_A-formula F.
To decide the T_A-satisfiability of F:

Step 1
For every read-over-write term $a\langle i \triangleleft v\rangle[j]$ in F, replace F with the formula

$$(i = j \wedge F\{a\langle i \triangleleft v\rangle[j] \mapsto v\}) \vee$$
$$(i \neq j \wedge F\{a\langle i \triangleleft v\rangle[j] \mapsto a[j]\})$$

Repeat until there are no more read-over-write terms.

Decision Procedure for T_A (cont)

Step 2

Associate array variables a with fresh function symbol f_a.
Replace read terms $a[i]$ with $f_a(i)$.

Step 3

Now F is a T_E-Formula. Decide T_E-satisfiability using the congruence-closure algorithm for each of the disjuncts produced in Step 1.

Example: Consider Σ_A-formula

$$F : \; i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a\langle i_1 \triangleleft v_1\rangle\langle i_2 \triangleleft v_2\rangle[j] \neq a[j] \,.$$

F contains a read-over-write term,

$$a\langle i_1 \triangleleft v_1\rangle\langle i_2 \triangleleft v_2\rangle[j] \neq a[j] \,.$$

Rewrite it to $F_1 \vee F_2$ with:

$$\begin{aligned} F_1 &: i_2 = j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge v_2 \neq a[j] \,, \\ F_2 &: i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a\langle i_1 \triangleleft v_1\rangle[j] \neq a[j] \,. \end{aligned}$$

F_1 does not contain any write terms, so rewrite it to

$$F_1' : \; i_2 = j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge f_a(j) = v_1 \wedge v_2 \neq f_a(j) \,.$$

The first two literals imply that $i_1 = i_2$, contradicting the third literal, so F_1' is T_E-unsatisfiable.

Now, we try the second case (F_2):
F_2 contains the read-over-write term $a\langle i_1 \triangleleft v_1\rangle[j]$. Rewrite it to $F_3 \vee F_4$ with

$$F_3 : i_1 = j \wedge i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge v_1 \neq a[j] ,$$
$$F_4 : i_1 \neq j \wedge i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a[j] \neq a[j] .$$

Rewrite the array reads to

$$F_3' : i_1 = j \wedge i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge f_a(j) = v_1 \wedge v_1 \neq f_a(j) ,$$
$$F_4' : i_1 \neq j \wedge i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge f_a(j) = v_1 \wedge f_a(j) \neq f_a(j) .$$

In F_3' there is a contradiction because of the final two terms. In F_4', there are two contradictions: the first and third literals contradict each other, and the final literal is contradictory. Since F is equisatisfiable to $F_1' \vee F_3' \vee F_4'$, F is T_A-unsatisfiable.

Suppose instead that F does not contain the literal $i_1 \neq i_2$. Is this new formula T_A-satisfiable?

Complexity of Decision Procedure for T_A

Our algorithm has a big disadvantage. Step 1 doubles the size of the formula:

$$(i = j \land F\{a\langle i \lhd v\rangle[j] \mapsto v\}) \lor$$
$$(i \neq j \land F\{a\langle i \lhd v\rangle[j] \mapsto a[j]\})$$

This can be avoided by introducing fresh variables x_{aijv}:

$$F\{a\langle i \lhd v\rangle[j] \mapsto x_{aijv}\} \land$$
$$((i = j \land x_{aijv} = v) \lor (i \neq j \land x_{aijv} = a[j]))$$

However, this is not in the conjunctive fragment of T_E.

There is no way around:

The conjunctive fragment of T_A is NP-complete.

In programming languages, one often needs to express the following concepts:

- Containment $contains(a, \ell, u, e)$: the array a contains element e at some index between ℓ and u.

$$\exists i. \ell \leq i \leq u \land a[i] = e$$

- Sortedness $sorted(a, \ell, u)$: the array a is sorted between index ℓ and index u.

$$\forall i, j. \ell \leq i \leq j \leq u \implies a[i] \leq a[j]$$

- Partitioning $partition(a, \ell_1, u_1, \ell_2, u_2)$: The array elements between ℓ_1 and u_1 are smaller than all elements between ℓ_2 and u_2.

$$\forall i, j. \ell_1 \leq i \leq u_1 \land \ell_2 \leq j \leq u_2 \implies a[i] \leq a[j]$$

These concepts can only be expressed as first-order formulae with quantifiers.

However: the general theory of arrays T_{A} with quantifier is not decidable.

Is there a decidable fragment of T_{A} that contains the above formulae?

Example

We want to prove validity for a formula, such as:

$$\neg contains(a, \ell, u, e) \land e \neq f \to \neg contains(a\langle j \triangleleft f\rangle, \ell, u, e)$$

$$\neg(\exists i. \ell \leq i \leq u \land a[i] = e) \land e \neq f \to \neg(\exists i. \ell \leq i \leq u \land a\langle j \triangleleft f\rangle[i] \neq e).$$

Check satisfiability of negated formula:

$$\neg(\exists i. \ell \leq i \leq u \land a[i] = e) \land e \neq f \land (\exists i. \ell \leq i \leq u \land a\langle j \triangleleft f\rangle[i] \neq e).$$

Negation Normal Form:

$$(\forall i. \ell > i \lor i > u \lor a[i] \neq e) \land e \neq f \land (\exists i. \ell \leq i \land i \leq u \land a\langle j \triangleleft f\rangle[i] = e).$$

or the equisatisfiable formula

$$\forall i. \ell > i \lor i > u \lor a[i] \neq e \land e \neq f \land \ell \leq i_2 \land i_2 \leq u \land a\langle j \triangleleft f\rangle[i_2] = e.$$

We need to handle satisfiability for universal quantifiers.

Array Property Fragment of T_A

Decidable fragment of T_A that includes $\forall$ quantifiers

Array property

Σ_A-formula of form

$$\forall \bar{i}.\ F[\bar{i}] \rightarrow G[\bar{i}]\ ,$$

where $\bar{i}$ is a list of variables.

- index guard $F[\bar{i}]$:

$$\begin{array}{rcl} \text{iguard} & \rightarrow & \text{iguard} \wedge \text{iguard} \mid \text{iguard} \vee \text{iguard} \mid \text{lit} \\ \text{lit} & \rightarrow & \text{var} = \text{var} \mid \mathit{evar} \neq \text{var} \mid \text{var} \neq \mathit{evar} \mid \top \\ \text{var} & \rightarrow & \mathit{evar} \mid \mathit{uvar} \end{array}$$

  where *uvar* is any universally quantified index variable,
  and *evar* is any constant or unquantified variable.
- value constraint $G[\bar{i}]$: a universally quantified index can occur in a value constraint $G[\bar{i}]$ only in a read $a[i]$, where a is an array term. The read cannot be nested; for example, $a[b[i]]$ is not allowed.

Array property Fragment: Boolean combinations of quantifier-free T_A-formulae and array properties

Example: Array Property Fragment

Is this formula in the array property fragment?

$$F \;:\; \forall i.\; i \neq a[k] \rightarrow a[i] = a[k]$$

The antecedent is not a legal index guard since $a[k]$ is not a variable (neither a *uvar* nor an *evar*); however, by simple manipulation

$$F' \;:\; v = a[k] \wedge \forall i.\; i \neq v \rightarrow a[i] = a[k]$$

Here, $i \neq v$ is a legal index guard, and $a[i] = a[k]$ is a legal value constraint. F and F' are equisatisfiable.
This trick works for every term that does not contain a uvar.
However, no manipulation works for:

$$G \;:\; \forall i.\; i \neq a[i] \rightarrow a[i] = a[k]\;.$$

Thus, G is not in the array property fragment.

Example: Array Property Fragment (cont)

Is this formula in the array property fragment?

$$F' : \forall ij.\ i \neq j \rightarrow a[i] \neq a[j]$$

No, the term uvar $\neq$ uvar is not allowed in the index guard. There is no workaround.

Array property fragment and extensionality

Remark: Array property fragment allows expressing equality between arrays (extensionality): two arrays are equal precisely when their corresponding elements are equal.

For given formula

$$F \ : \ \cdots \land a = b \land \cdots$$

with array terms a and b, rewrite F as

$$F' \ : \ \cdots \land (\forall i.\ \top \rightarrow a[i] = b[i]) \land \cdots \ .$$

F and F' are equisatisfiable.
F' is in array property fragment of T_{A}.

Decision Procedure for Array Property Fragment

Basic Idea: Similar to quantifier elimination.

Replace universal quantification

$$\forall i. F[i]$$

by finite conjunction

$$F[t_1] \wedge \ldots \wedge F[t_n].$$

We call $t_1, \ldots, t_n$ the index terms and they depend on the formula.

Example

Consider

$$F \ : \ a\langle i \triangleleft v\rangle = a \wedge a[i] \neq v \ ,$$

which expands to

$$F' \ : \ \forall j.\ a\langle i \triangleleft v\rangle[j] = a[j] \wedge a[i] \neq v \ .$$

Intuitively, only the index i is important:

$$F'' \ : \ \left(\bigwedge_{j \in \{i\}} a\langle i \triangleleft v\rangle[j] = a[j]\right) \wedge a[i] \neq v \ ,$$

or simply

$$a\langle i \triangleleft v\rangle[i] = a[i] \wedge a[i] \neq v \ .$$

Simplifying,

$$v = a[i] \wedge a[i] \neq v \ ,$$

Given array property formula F, decide its T_A-satisfiability by the following steps:

Step 1

Put F in NNF, but do not rewrite inside a quantifier.

Step 2

Apply the following rule exhaustively to remove writes:

$$\frac{F[a\langle i \triangleleft v\rangle]}{F[a'] \wedge a'[i] = v \wedge (\forall j.\, j \neq i \rightarrow a[j] = a'[j])} \text{ for fresh } a' \quad \text{(write)}$$

After an application of the rule, the resulting formula contains at least one fewer write terms than the given formula.

Step 3

Apply the following rule exhaustively to remove existential quantification:

$$\frac{F[\exists \bar{i}.\ G[\bar{i}]]}{F[G[\bar{j}]]} \text{ for fresh } \bar{j} \quad \text{(exists)}$$

Existential quantification can arise during Step 1 if the given formula has a negated array property.

Steps 4-6 accomplish the reduction of universal quantification to finite conjunction. Main idea: select a set of symbolic index terms on which to instantiate all universal quantifiers. The set is sufficient for correctness.

Step 4

From the output F_3 of Step 3, construct the **index set** $\mathcal{I}$:

$$\begin{array}{rcll} & & \{\lambda\} & \\ \mathcal{I} & = & \cup\,\{t \;:\; & \cdot[t] \in F_3 \text{ such that } t \text{ is not a universally quantified variable}\} \\ & & \cup\,\{t \;:\; & t \text{ occurs as an } \textit{evar} \text{ in the parsing of index guards}\} \end{array}$$

This index set is the finite set of indices that need to be examined. It includes

- all terms t that occur in some read $a[t]$ anywhere in F (unless it is a universally quantified variable)
- all terms t (constant or unquantified variable) that are compared to a universally quantified variable in some index guard.
- λ is a fresh constant that represents all other index positions that are not explicitly in $\mathcal{I}$.

Step 5 (Key step)
Apply the following rule exhaustively to remove universal quantification:

$$\frac{H[\forall \bar{i}.\ F[\bar{i}] \rightarrow G[\bar{i}]]}{H\left[\bigwedge_{\bar{i} \in \mathcal{I}^n} \left(F[\bar{i}] \rightarrow G[\bar{i}]\right)\right]} \quad \text{(forall)}$$

where n is the number of quantified variables $\bar{i}$.

Step 6
From the output F_5 of Step 5, construct

$$F_6 \ : \ F_5 \wedge \bigwedge_{i \ \in \ \mathcal{I} \setminus \{\lambda\}} \lambda \neq i \ .$$

The new conjuncts assert that the variable λ introduced in Step 4 is indeed unique.

Step 7
Decide the T_A-satisfiability of F_6 using the decision procedure for the quantifier-free fragment.

Example

Is this $T_A^=$-formula valid?

$$F \;:\; (\forall i.\ i \neq k \rightarrow a[i] = b[i]) \wedge b[k] = v \rightarrow a\langle k \triangleleft v\rangle = b$$

Check satisfiability of:

$$\neg((\forall i.\ i \neq k \rightarrow a[i] = b[i]) \wedge b[k] = v \rightarrow (\forall i.\ a\langle k \triangleleft v\rangle[i] = b[i]))$$

Step 1: NNF

$$F_1 \;:\; (\forall i.\ i \neq k \rightarrow a[i] = b[i]) \wedge b[k] = v \wedge (\exists i.\ a\langle k \triangleleft v\rangle[i] \neq b[i])$$

Step 2: Remove array writes

$$F_2 \;:\; (\forall i.\ i \neq k \rightarrow a[i] = b[i]) \wedge b[k] = v \wedge (\exists i.\ a'[i] \neq b[i]) \\ \wedge\ a'[k] = v \wedge (\forall i.\ i \neq k \rightarrow a'[i] = a[i])$$

Step 3: Remove existential quantifier

$$F_3 \;:\; (\forall i.\ i \neq k \rightarrow a[i] = b[i]) \wedge b[k] = v \wedge a'[j] \neq b[j]$$

Example (cont)

Step 4: Compute index set $\mathcal{I} = \{\lambda, k, j\}$
Step 5+6: Replace universal quantifier:

$$\begin{aligned} F_6 : & (\lambda \neq k \rightarrow a[\lambda] = b[\lambda]) \\ & \wedge (k \neq k \rightarrow a[k] = b[k]) \\ & \wedge (j \neq k \rightarrow a[j] = b[j]) \\ & \wedge b[k] = v \wedge a'[j] \neq b[j] \wedge a'[k] = v \\ & \wedge (\lambda \neq k \rightarrow a'[\lambda] = a[\lambda]) \\ & \wedge (k \neq k \rightarrow a'[k] = a[k]) \\ & \wedge (j \neq k \rightarrow a'[j] = a[j]) \\ & \wedge \lambda \neq k \wedge \lambda \neq j \end{aligned}$$

Case distinction on $j = k$ proves unsatisfiability of F_6.
Therefore F is valid

The importance of λ

Is this formula satisfiable?

$$F : (\forall i. i \neq j \rightarrow a[i] = b[i]) \wedge (\forall i. i \neq k \rightarrow a[i] \neq b[i])$$

The algorithm produces:

$$\begin{aligned} F_6 : \; & \lambda \neq j \rightarrow a[\lambda] = b[\lambda] \\ & \wedge j \neq j \rightarrow a[j] = b[j] \\ & \wedge k \neq j \rightarrow a[k] = b[k] \\ & \wedge \lambda \neq k \rightarrow a[\lambda] \neq b[\lambda] \\ & \wedge j \neq k \rightarrow a[j] \neq b[j] \\ & \wedge k \neq k \rightarrow a[k] \neq b[k] \\ & \wedge \lambda \neq j \wedge \lambda \neq k \end{aligned}$$

The first, fourth and last line give a contradiction!

The importance of λ (cont)

Without λ we had the formula:

$$
\begin{aligned}
F_6' : j \neq j \rightarrow a[j] &= b[j] \\
\wedge\ k \neq j \rightarrow a[k] &= b[k] \\
\wedge\ j \neq k \rightarrow a[j] &\neq b[j] \\
\wedge\ k \neq k \rightarrow a[k] &\neq b[k]
\end{aligned}
$$

which simplifies to:

$$j \neq k \rightarrow a[k] = b[k] \wedge a[j] \neq b[j] .$$

This formula is satisfiable!

Theorem

Consider a Σ_A-formula F from the array property fragment of T_A. The output F_6 of Step 6 of the algorithm is T_A-equisatisfiable to F.

This also works when extending the Logic with an arbitrary theory T with signature Σ for the elements:

Theorem

Consider a $\Sigma_A \cup \Sigma$-formula F from the array property fragment of $T_A \cup T$. The output F_6 of Step 6 of the algorithm is $T_A \cup T$-equisatisfiable to F.

Proof: It is easy to see that steps 1–3 do not change the satisfiability of formula. For step 4–6 we need to show:

$$(1) \quad H[\forall \bar{i}.\ (F[\bar{i}] \rightarrow G[\bar{i}])] \text{ is satisfiable}$$

iff.

$$(2) \quad H[\bigwedge_{\bar{i} \in \mathcal{I}^n} (F[\bar{i}] \rightarrow G[\bar{i}])] \wedge \bigwedge_{i \in \mathcal{I} \setminus \{\lambda\}} \lambda \neq i \text{ is satisfiable.}$$

If the formula (1) is satisfied some Interpretation, then (2) holds in the same interpretation.

If the formula (2) holds in some interpretation I, we construct an interpretation J as follows:

$$\begin{aligned} proj_{\mathcal{I}}(j) &= \begin{cases} i & \text{if } i \in \mathcal{I} \wedge \alpha_I[j] = \alpha_I[i] \\ \lambda & \text{otherwise} \end{cases} \\ \alpha_J[a[j]] &= \alpha_I[a[proj_{\mathcal{I}}(j)]] \\ \alpha_J[x] &= \alpha_I[x] \text{ for every non-array variable and constant} \end{aligned}$$

J interprets the symbols occuring in formula (2) in the same way as I.
Therefore, (2) holds in J.
To prove that formula (1) holds in J, it suffices to show:

$$J \models \bigwedge_{\bar{i} \in \mathcal{I}^n} (F[\bar{i}] \rightarrow G[\bar{i}]) \text{ implies } J \models \forall \bar{i}.\ (F[\bar{i}] \rightarrow G[\bar{i}])$$

Proof of Theorem (cont)

Assume $J \models \bigwedge_{\bar{i} \in \mathcal{I}^n}(F[\bar{i}] \rightarrow G[\bar{i}])$. Show:

$$F[\bar{i}] \rightarrow F[proj_{\mathcal{I}}(\bar{i})] \rightarrow G[proj_{\mathcal{I}}(\bar{i})] \rightarrow G[\bar{i}]$$

The first implication $F[\bar{i}] \rightarrow F[proj_{\mathcal{I}}(\bar{i})]$ can be shown by structural induction over F. Base cases:

- $var_1 = var_2 \rightarrow proj_{\mathcal{I}}(var_1) = proj_{\mathcal{I}}(var_2)$: trivial.
- $evar_1 \neq var_2 \rightarrow proj_{\mathcal{I}}(evar_1) \neq proj_{\mathcal{I}}(var_2)$:
  By definition of $\mathcal{I}$: $evar_1 \in \mathcal{I} \setminus \{\lambda\}$.
  If $evar_1 = proj_{\mathcal{I}}(evar_1) = proj_{\mathcal{I}}(var_2)$, then $var_2 \in \mathcal{I} \setminus \{\lambda\}$, hence
  $evar_1 = proj_{\mathcal{I}}(var_2) = var_2$
- $var_1 \neq evar_2$ analogously.

The induction step is trivial.

The second implication $F[proj_{\mathcal{I}}(\bar{i})] \rightarrow G[proj_{\mathcal{I}}(\bar{i})]$ holds by assumption.

The third implication $G[proj_{\mathcal{I}}(\bar{i})] \implies G[\bar{i}]$ holds because G contains variables i only in array reads $a[i]$. By definition of J: $\alpha_J[a[i]] = \alpha_J[a[proj_{\mathcal{I}}(i)]]$.

Theory of Integer-Indexed Arrays

$\leq$ enables reasoning about subarrays and properties such as subarray is sorted or partitioned.

signature of $T_{\mathsf{A}}^{\mathbb{Z}}$: $\Sigma_{\mathsf{A}}^{\mathbb{Z}} = \Sigma_{\mathsf{A}} \cup \Sigma_{\mathbb{Z}}$

axioms of $T_{\mathsf{A}}^{\mathbb{Z}}$: both axioms of T_{A} and $T_{\mathbb{Z}}$

Array property: $\Sigma_A^{\mathbb{Z}}$-formula of the form

$$\forall \bar{i}.\ F[\bar{i}] \rightarrow G[\bar{i}]\ ,$$

where $\bar{i}$ is a list of integer variables.

- $F[\bar{i}]$ index guard:

$$\begin{array}{rcl} \text{iguard} & \rightarrow & \text{iguard} \wedge \text{iguard} \mid \text{iguard} \vee \text{iguard} \mid \text{lit} \\ \text{lit} & \rightarrow & \text{expr} \leq \text{expr} \mid \text{expr} = \text{expr} \\ \text{expr} & \rightarrow & \mathit{uvar} \mid \text{pexpr} \\ \text{pexpr} & \rightarrow & \text{pexpr}' \\ \text{pexpr}' & \rightarrow & \mathbb{Z} \mid \mathbb{Z} \cdot \mathit{evar} \mid \text{pexpr}' + \text{pexpr}' \end{array}$$

  where *uvar* is any universally quantified integer variable,
  and *evar* is any existentially quantified or free integer variable.
- $G[\bar{i}]$ value constraint:
  Any occurrence of a quantified index variable i must be as a read into an array, $a[i]$, for array term a. Array reads may not be nested; *e.g.*, $a[b[i]]$ is not allowed.

Array property fragment of $T_A^{\mathbb{Z}}$ consists of formulae that are Boolean combinations of quantifier-free $\Sigma_A^{\mathbb{Z}}$-formulae and array properties.

Application: array property fragments

- Array equality $a = b$ in T_{A}: $\quad \forall i.\ a[i] = b[i]$
- Bounded array equality $\mathrm{beq}(a, b, \ell, u)$ in $T_{\mathsf{A}}^{\mathbb{Z}}$:

$$\forall i.\ \ell \leq i \leq u \rightarrow a[i] = b[i]$$

- Universal properties $F[x]$ in T_{A}:

$$\forall i.\ F[a[i]]$$

- Bounded universal properties $F[x]$ in $T_{\mathsf{A}}^{\mathbb{Z}}$:

$$\forall i.\ \ell \leq i \leq u \rightarrow F[a[i]]$$

- Bounded and unbounded sorted arrays $\mathrm{sorted}(a, \ell, u)$ in $T_{\mathsf{A}}^{\mathbb{Z}} \cup T_{\mathbb{Z}}$ or $T_{\mathsf{A}}^{\mathbb{Z}} \cup T_{\mathbb{Q}}$:

$$\forall i, j.\ \ell \leq i \leq j \leq u \rightarrow a[i] \leq a[j]$$

- Partitioned arrays $\mathrm{partitioned}(a, \ell_1, u_1, \ell_2, u_2)$ in $T_{\mathsf{A}}^{\mathbb{Z}} \cup T_{\mathbb{Z}}$ or $T_{\mathsf{A}}^{\mathbb{Z}} \cup T_{\mathbb{Q}}$:

$$\forall i, j,\ \ell_1 \leq i \leq u_1 < \ell_2 \leq j \leq u_2 \rightarrow a[i] \leq a[j]$$

The idea again is to reduce universal quantification to finite conjunction.
Given F from the array property fragment of $T_{\mathsf{A}}^{\mathbb{Z}}$, decide its $T_{\mathsf{A}}^{\mathbb{Z}}$-satisfiability as follows:

Step 1
Put F in NNF.

Step 2
Apply the following rule exhaustively to remove writes:

$$\frac{F[a\langle i \triangleleft e\rangle]}{F[a'] \wedge a'[i] = e \wedge (\forall j.\, j \neq i \rightarrow a[j] = a'[j])} \text{ for fresh } a' \qquad \text{(write)}$$

To meet the syntactic requirements on an index guard, rewrite the third conjunct as

$$\forall j.\, j \leq i - 1 \vee i + 1 \leq j \rightarrow a[j] = a'[j] \,.$$

The Decision Procedure (Step 3–4)

Step 3
Apply the following rule exhaustively to remove existential quantification:

$$\frac{F[\exists \bar{i}.\ G[\bar{i}]]}{F[G[\bar{j}]]} \text{ for fresh } \bar{j} \qquad \text{(exists)}$$

Existential quantification can arise during Step 1 if the given formula has a negated array property.

Step 4
From the output of Step 3, F_3, construct the index set $\mathcal{I}$:

$$\mathcal{I} = \begin{array}{l} \{t \ : \ \cdot[t] \in F_3 \text{ such that } t \text{ is not a universally quantified variable}\} \\ \cup \ \{t \ : \ t \text{ occurs as a pexpr in the parsing of index guards}\} \end{array}$$

If $\mathcal{I} = \emptyset$, then let $\mathcal{I} = \{0\}$. The index set contains all relevant symbolic indices that occur in F_3.

Step 5

Apply the following rule exhaustively to remove universal quantification:

$$\frac{H[\forall \bar{i}.\ F[\bar{i}] \to G[\bar{i}]]}{H\left[\bigwedge_{\bar{i} \in \mathcal{I}^n} \left(F[\bar{i}] \to G[\bar{i}]\right)\right]} \quad \text{(forall)}$$

n is the size of the block of universal quantifiers over $\bar{i}$.

Step 6

F_5 is quantifier-free in the combination theory $T_{\mathsf{A}} \cup T_{\mathbb{Z}}$. Decide the $(T_{\mathsf{A}} \cup T_{\mathbb{Z}})$-satisfiability of the resulting formula.

Example

$\Sigma_A^{\mathbb{Z}}$-formula:

$$F : \begin{array}{l} (\forall i.\ \ell \leq i \leq u \rightarrow a[i] = b[i]) \\ \wedge \neg(\forall i.\ \ell \leq i \leq u + 1 \rightarrow a\langle u + 1 \triangleleft b[u + 1]\rangle[i] = b[i]) \end{array}$$

In NNF, we have

$$F_1 : \begin{array}{l} (\forall i.\ \ell \leq i \leq u \rightarrow a[i] = b[i]) \\ \wedge (\exists i.\ \ell \leq i \leq u + 1 \wedge a\langle u + 1 \triangleleft b[u + 1]\rangle[i] \neq b[i]) \end{array}$$

Step 2 produces

$$F_2 : \begin{array}{l} (\forall i.\ \ell \leq i \leq u \rightarrow a[i] = b[i]) \\ \wedge (\exists i.\ \ell \leq i \leq u + 1 \wedge a'[i] \neq b[i]) \\ \wedge\ a'[u + 1] = b[u + 1] \\ \wedge (\forall j.\ j \leq u + 1 - 1 \vee u + 1 + 1 \leq j \rightarrow a[j] = a'[j]) \end{array}$$

Step 3 removes the existential quantifier by introducing a fresh constant k:

$$F_3 : \quad \begin{array}{l} (\forall i.\ \ell \leq i \leq u \rightarrow a[i] = b[i]) \\ \wedge\ \ell \leq k \leq u + 1 \wedge a'[k] \neq b[k] \\ \wedge\ a'[u+1] = b[u+1] \\ \wedge\ (\forall j.\ j \leq u + 1 - 1 \vee u + 1 + 1 \leq j \rightarrow a[j] = a'[j]) \end{array}$$

Simplifying,

$$F_3' : \quad \begin{array}{l} (\forall i.\ \ell \leq i \leq u \rightarrow a[i] = b[i]) \\ \wedge\ \ell \leq k \leq u + 1 \wedge a'[k] \neq b[k] \\ \wedge\ a'[u+1] = b[u+1] \\ \wedge\ (\forall j.\ j \leq u \vee u + 2 \leq j \rightarrow a[j] = a'[j]) \end{array}$$

The index set is

$$\mathcal{I} = \{k, u+1\} \cup \{\ell, u, u+2\},$$

which includes the read terms k and $u+1$ and the terms ℓ, u, and $u+2$ that occur as pexprs in the index guards.

Step 5 rewrites universal quantification to finite conjunction over this set:

$$F_5 : \quad \begin{array}{l} \bigwedge\limits_{i \in \mathcal{I}} (\ell \leq i \leq u \rightarrow a[i] = b[i]) \\ \wedge\ \ell \leq k \leq u + 1 \wedge a'[k] \neq b[k] \\ \wedge\ a'[u+1] = b[u+1] \\ \wedge \bigwedge\limits_{j \in \mathcal{I}} (j \leq u \vee u + 2 \leq j \rightarrow a[j] = a'[j]) \end{array}$$

Expanding the conjunctions according to the index set $\mathcal{I}$ and simplifying according to trivially true or false antecedents (*e.g.*, $\ell \leq u + 1 \leq u$ simplifies to $\bot$, while $u \leq u \vee u + 2 \leq u$ simplifies to $\top$) produces:

$$F_5' : \begin{array}{lr} (\ell \leq k \leq u \rightarrow a[k] = b[k]) & (1) \\ \wedge\ (\ell \leq u \rightarrow a[\ell] = b[\ell] \wedge a[u] = b[u]) & (2) \\ \wedge\ \ell \leq k \leq u + 1 & (3) \\ \wedge\ a'[k] \neq b[k] & (4) \\ \wedge\ a'[u+1] = b[u+1] & (5) \\ \wedge\ (k \leq u \vee u + 2 \leq k \rightarrow a[k] = a'[k]) & (6) \\ \wedge\ (\ell \leq u \vee u + 2 \leq \ell \rightarrow a[\ell] = a'[\ell]) & (7) \\ \wedge\ a[u] = a'[u] \wedge a[u+2] = a'[u+2] & (8) \end{array}$$

$(T_\mathsf{A} \cup T_\mathbb{Z})$-unsatisfiability of this quantifier-free $(\Sigma_\mathsf{A} \cup \Sigma_\mathbb{Z})$-formula can be decided using the techniques of Combination of Theories.

Informally, $\ell \leq k \leq u + 1$ (3)

- If $k \in [\ell, u]$ then $a[k] = b[k]$ (1). Since $k \leq u$ then $a[k] = a'[k]$ (6), contradicting $a'[k] \neq b[k]$ (4).
- if $k = u + 1$, $a'[k] \neq b[k] = b[u+1] = a'[u+1] = a'[k]$ by (4) and (5), a contradiction.

Hence, F is $T_\mathsf{A}^\mathbb{Z}$-unsatisfiable.

Correctness of Decision Procedure

Theorem

Consider a $\Sigma_A^{\mathbb{Z}} \cup \Sigma$-formula F from the array property fragment of $T_A^{\mathbb{Z}} \cup T$. The output F_5 of Step 5 of the algorithm is $T_A^{\mathbb{Z}} \cup T$-equisatisfiable to F.

Proof: The proof proceeds using the same strategy as for T_{A}.
It is easy to see that steps 1–3 do not change the satisfiability of formula.
For step 4–5 we need to show:

$$(1) \quad H[\forall \bar{i}.\ (F[\bar{i}] \rightarrow G[\bar{i}])] \text{ is satisfiable}$$

iff.

$$(2) \quad H[\textstyle\bigwedge_{\bar{i} \in \mathcal{I}^n} (F[\bar{i}] \rightarrow G[\bar{i}])] \text{ is satisfiable.}$$

$\Rightarrow$: Obviously formula (1) implies formula (2).

Proof of Theorem (cont)

If the formula (2) holds in some interpretation $I = (D_I, \alpha_I)$, we construct an interpretation $J = (D_J, \alpha_J)$ with $D_J := D_I$ and

$$proj_{\mathcal{I}}(j) = \begin{cases} max\{\alpha_I[i] | i \in \mathcal{I} \land \alpha_I[i] \leq \alpha_I[j]\} & \text{if for some } i \in \mathcal{I}\text{:} \\ & \alpha_I[i] \leq \alpha_I[j] \\ min\{\alpha_I[i] | i \in \mathcal{I} \land \alpha_I[i] \geq \alpha_I[j]\} & \text{otherwise} \end{cases}$$

$$\alpha_J[a[j]] = \alpha_I[a[proj_{\mathcal{I}}(j)]]$$

$$\alpha_J[x] = \alpha_I[x] \text{ for every non-array variable and constant}$$

J interprets the symbols occuring in formula (2) in the same way as I.
Therefore, (2) holds in J.
To prove that formula (1) holds in J, it suffices to show:

$$J \models \bigwedge_{\bar{i} \in \mathcal{I}^n} (F[\bar{i}] \rightarrow G[\bar{i}]) \text{ implies } J \models \forall \bar{i}.\ (F[\bar{i}] \rightarrow G[\bar{i}])$$

Proof of Theorem (cont)

Assume $J \models \bigwedge_{\bar{i} \in \mathcal{I}^n} (F[\bar{i}] \rightarrow G[\bar{i}])$. Show:

$$F[\bar{i}] \rightarrow F[proj_{\mathcal{I}}(\bar{i})] \rightarrow G[proj_{\mathcal{I}}(\bar{i})] \rightarrow G[\bar{i}]$$

The first implication $F[\bar{i}] \rightarrow F[proj_{\mathcal{I}}(\bar{i})]$ can be shown by structural induction over F. Base cases:

- $expr_1 \leq expr_2$: see exercise.
- $expr_1 = expr_2$: follows from first case since it is equivalent to

$$expr_1 \leq expr_2 \wedge expr_2 \leq expr_1 .$$

The induction step is trivial.
The second implication $F[proj_{\mathcal{I}}(\bar{i})] \rightarrow G[proj_{\mathcal{I}}(\bar{i})]$ holds by assumption.
The third implication $G[proj_{\mathcal{I}}(\bar{i})] \implies G[\bar{i}]$ holds because G contains variables i only in array reads $a[i]$. By definition of J: $\alpha_J[a[i]] = \alpha_J[a[proj_{\mathcal{I}}(i)]]$.