# Decision Procedures

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# DPLL(T)

# Satisfiability and Conjunctive Theories

Suppose we have a $T_{\mathbb{Q}}$-formulae that is not conjunctive:

$$(x \geq 0 \rightarrow y > z) \land (x + y \geq z \rightarrow y \leq z) \land (y \geq 0 \rightarrow x \geq 0) \land x + y \geq z$$

Our approach so far: Converting to DNF.
Yields in 8 conjuncts that have to be checked separately.

Is there a more efficient way to prove unsatisfiability?

# CNF and Propositional Core

Suppose we have the following $T_{\mathbb{Q}}$-formulae:

$$(x \geq 0 \rightarrow y > z) \land (x + y \geq z \rightarrow y \leq z) \land (y \geq 0 \rightarrow x \geq 0) \land x + y \geq z$$

Converting to CNF and restricting to $\leq$:

$$\begin{aligned}(\neg(0 \leq x) \lor \neg(y \leq z)) \land (\neg(z \leq x + y) \lor (y \leq z))\\ \land(\neg(0 \leq y) \lor (0 \leq x)) \land (z \leq x + y)\end{aligned}$$

Now, introduce boolean variables for each atom:

$$P_1 : 0 \leq x \qquad P_2 : y \leq z$$
$$P_3 : z \leq x + y \qquad P_4 : 0 \leq y$$

Gives a propositional formula:

$$(\neg P_1 \lor \neg P_2) \land (\neg P_3 \lor P_2) \land (\neg P_4 \lor P_1) \land P_3$$

# DPLL-Algorithm

The core feature of the DPLL-algorithm is Unit Propagation.

$$(\neg P_1 \vee \neg P_2) \wedge (\neg P_3 \vee P_2) \wedge (\neg P_4 \vee P_1) \wedge P_3$$

The clause $P_3$ is a unit clause; set $P_3$ to $\top$.
Then $\neg P_3 \vee P_2$ is a unit clause; set $P_2$ to $\top$.
Then $\neg P_1 \vee \neg P_2$ is a unit clause; set $P_1$ to $\bot$.
Then $\neg P_4 \vee P_1$ is a unit clause; set $P_4$ to $\bot$.

Only solution is $P_3 \wedge P_2 \wedge \neg P_1 \wedge \neg P_4$.

Only solution is $P_3 \land P_2 \land \neg P_1 \land \neg P_4$.

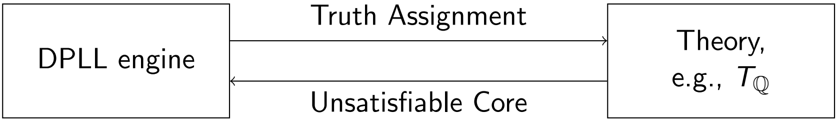
$$P_1 : 0 \leq x \qquad\qquad P_2 : y \leq z$$
$$P_3 : z \leq x + y \qquad\qquad P_4 : 0 \leq y$$

This gives the conjunctive $T_{\mathbb{Q}}$-formula

$$z \leq x + y \land y \leq z \land x < 0 \land y < 0.$$

# DPLL(T): DPLL Modulo Theory

The DPLL/CDCL algorithm is combined with a Decision Procedures for a Theory




DPLL takes the propositional core of a formula,
assigns truth-values to atoms.

Theory takes a conjunctive formula (conjunction of literals),
returns a minimal unsatisfiable core.

# Minimal Unsatisfiable Core

Suppose we have a decision procedure for a conjunctive theory, e.g., Simplex Algorithm for $T_{\mathbb{Q}}$.

Given an unsatisfiable conjunction of literals $\ell_1 \wedge \cdots \wedge \ell_n$.
Find a subset $\text{UnsatCore} = \{\ell_{i_1}, \ldots, \ell_{i_m}\}$, such that

- $\ell_{i_1} \wedge \ldots \wedge \ell_{i_m}$ is unsatisfiable.
- For each subset of UnsatCore the conjunction is satisfiable.

Possible approach: check for each literal whether it can be omitted.
$\longrightarrow$ $n$ calls to decision procedure.

Most decision procedures can give small unsatisfiable cores for free.

# Unsatisfiable Core and Conflict Clause

Theory returns an unsatisfiable core:

- a conjunction of literals from current truth assignment
- that is unsatisfible.

DPLL learns conflict clauses, a disjunction of literals

- that are implied by the formula
- and in conflict to current truth assignment.

Thus the negation of an unsatisfiable core is a conflict clause.

We describe DPLL(T) by a set of rules modifying a configuration.
A configuration is a triple

$$\langle M, F, C \rangle ,$$

where

- $M$ (model) is a sequence of literals (that are currently set to true) interspersed with backtracking points denoted by $\square$.
- $F$ (formula) is a formula in CNF,
  i. e., a set of clauses where each clause is a set of literals.
- $C$ (conflict) is either $\top$ or a conflict clause (a set of literals).
  A conflict clause $C$ is a clause with $F \Rightarrow C$ and $M \not\models C$.
  Thus, a conflict clause shows $M \not\models F$.

# Rules for CDCL (Conflict Driven Clause Learning)

| | | |
|---|---|---|
| Decide | $\dfrac{\langle M, F, \top\rangle}{\langle M \cdot \Box \cdot \ell, F, \top\rangle}$ | where $\ell \in lit(F)$, $\ell, \overline{\ell}$ $\not\text{in}$ $M$ |
| Propagate | $\dfrac{\langle M, F, \top\rangle}{\langle M \cdot \ell, F, \top\rangle}$ | where $\{\ell_1, \ldots, \ell_k, \ell\} \in F$ and $\overline{\ell_1}, \ldots, \overline{\ell_k}$ in $M$, $\ell, \overline{\ell}$ $\not\text{in}$ $M$. |
| Conflict | $\dfrac{\langle M, F, \top\rangle}{\langle M, F, \{\ell_1, \ldots, \ell_k\}\rangle}$ | where $\{\ell_1, \ldots, \ell_k\} \in F$ and $\overline{\ell_1}, \ldots, \overline{\ell_k}$ in $M$. |
| Explain | $\dfrac{\langle M, F, C \cup \{\ell\}\rangle}{\langle M, F, C \cup \{\ell_1, \ldots, \ell_k\}\rangle}$ | where $\ell \notin C$, $\{\ell_1, \ldots, \ell_k, \overline{\ell}\} \in F$, and $\overline{\ell_1}, \ldots, \overline{\ell_k} \prec \overline{\ell}$ in $M$. |
| Learn | $\dfrac{\langle M, F, C\rangle}{\langle M, F \cup \{C\}, C\rangle}$ | where $C \neq \top$, $C \notin F$. |
| Back | $\dfrac{\langle M, F, \{\ell_1, \ldots, \ell_k, \ell\}\rangle}{\langle M' \cdot \ell, F, \top\rangle}$ | where $\{\ell_1, \ldots, \ell_k, \ell\} \in F$, $M = M' \cdot \Box \cdots \overline{\ell} \cdots$, and $\overline{\ell_1}, \ldots, \overline{\ell_k}$ in $M'$. |

The DPLL part only needs one new rule:

$$\text{TConflict} \quad \frac{\langle M, F, \top \rangle}{\langle M, F, C \rangle} \quad \begin{array}{l}\text{where } M \text{ is unsatisfiable in the theory} \\ \text{and } \neg C \text{ an unsatisfiable core of } M.\end{array}$$

# Example: DPLL(T)

$$F : \; y \geq 1 \land (x \geq 0 \rightarrow y \leq 0) \land (x \leq 1 \rightarrow y \leq 0)$$

Atomic propositions:

$$P_1 : y \geq 1 \qquad\qquad P_2 : x \geq 0$$
$$P_3 : y \leq 0 \qquad\qquad P_4 : x \leq 1$$

Propositional core of $F$ in CNF:

$$F_0 : (P_1) \land (\neg P_2 \lor P_3) \land (\neg P_4 \lor P_3)$$

# Running DPLL(T)

$F_0 \ : \ \{\{P_1\}, \{\bar{P}_2, P_3\}, \{\bar{P}_4, P_3\}\}$
$P_1 \ : \ y \geq 1 \quad P_2 \ : \ x \geq 0 \quad P_3 \ : \ y \leq 0 \quad P_4 \ : \ x \leq 1$

$$\begin{aligned}
&\langle \epsilon, F_0, \top\rangle \xrightarrow{\text{Propagate}} \langle P_1, F_0, \top\rangle \xrightarrow{\text{Decide}} \langle P_1 \Box P_3, F_0, \top\rangle \xrightarrow{\text{TConflict}} \\
&\langle P_1 \Box P_3, F_0, \{\bar{P}_1, \bar{P}_3\}\rangle \xrightarrow{\text{Learn}} \langle P_1 \Box P_3, F_1, \{\bar{P}_1, \bar{P}_3\}\rangle \xrightarrow{\text{Back}} \langle P_1 \bar{P}_3, F_1, \top\rangle \xrightarrow{\text{Propagate}} \\
&\langle P_1 \bar{P}_3 \bar{P}_2, F_1, \top\rangle \xrightarrow{\text{Propagate}} \langle P_1 \bar{P}_3 \bar{P}_2 \bar{P}_4, F_1, \top\rangle \xrightarrow{\text{TConflict}} \\
&\langle P_1 \bar{P}_3 \bar{P}_2 \bar{P}_4, F_1, \{P_2, P_4\}\rangle \xrightarrow{\text{Explain}} \langle P_1 \bar{P}_3 \bar{P}_2 \bar{P}_4, F_1, \{P_2, P_3\}\rangle \xrightarrow{\text{Explain}} \\
&\langle P_1 \bar{P}_3 \bar{P}_2 \bar{P}_4, F_1, \{P_3\}\rangle \xrightarrow{\text{Explain}} \langle P_1 \bar{P}_3 \bar{P}_2 \bar{P}_4, F_1, \{\bar{P}_1\}\rangle \xrightarrow{\text{Explain}} \langle P_1 \bar{P}_3 \bar{P}_2 \bar{P}_4, F_1, \emptyset\rangle \xrightarrow{\text{Learn}} \\
&\langle P_1 \bar{P}_3 \bar{P}_2 \bar{P}_4, F_1 \cup \{\emptyset\}, \emptyset\rangle
\end{aligned}$$

where $F_1 := F_0 \cup \{\{\bar{P}_1, \bar{P}_3\}\}$

No further step is possible; the formula $F$ is unsatisfiable.

**Theorem (Correctness of DPLL(T))**

*Let $F$ be a $\Sigma$-formula and $F'$ its propositional core. Let*

$$\langle \epsilon, F', \top \rangle = \langle M_0, F_0, C_0 \rangle \longrightarrow \ldots \longrightarrow \langle M_n, F_n, C_n \rangle$$

*be a maximal sequence of rule application of DPLL(T).*

*Then $F$ is $T$-satisfiable iff $C_n$ is $\top$.*

Before proving the theorem, we note some important invariants:

- $M_i$ never contains a literal more than once.
- $M_i$ never contains $\ell$ and $\overline{\ell}$.
- Every $\square$ in $M_i$ is followed immediately by a literal.
- If $C_i = \{\ell_1, \ldots, \ell_k\}$ then $\overline{\ell_1}, \ldots, \overline{\ell_k}$ in $M$.
- $C_i$ is always implied by $F_i$ (or the theory).
- $F$ is equivalent to $F_i$ for all steps $i$ of the computation.
- If a literal $\ell$ in $M$ is not immediately preceded by $\square$, then $F$ contains a clause $\{\ell, \ell_1, \ldots, \ell_k\}$ and $\overline{\ell_1}, \ldots, \overline{\ell_k} \prec \ell$ in $M$.

# Correctness proof

Proof: If the sequence ends with $\langle M_n, F_n, \top\rangle$ and there is no rule applicable, then:

- Since Decide is not applicable, all literals of $F_n$ appear in $M_n$ either positively or negatively.
- Since Conflict is not applicable, for each clause at least one literal appears in $M_n$ positively.
- Since TConflict is not applicable, the conjunction of truth assignments of $M_n$ is satisfiable by a model $I$.

Thus, $I$ is a model for $F_n$, which is equivalent to $F$.

If the sequence ends with $\langle M_n, F_n, C_n\rangle$ with $C_n \neq \top$.
Assume $C_n = \{\ell_1, \dots, \ell_k, \ell\} \neq \emptyset$. W.l.o.g., $\overline{\ell}_1, \dots, \overline{\ell}_k \prec \ell$. Then:

- Since Learn is not applicable, $C_n \in F_n$.
- Since Explain is not applicable $\overline{\ell}$ must be immediately preceded by $\Box$.
- However, then Back is applicable, contradiction!

Therefore, the assumption was wrong and $C_n = \emptyset (= \bot)$.
Since $F$ implies $C_n$, $F$ is not satisfiable.