Tutorials for Decision Procedures
Exercise sheet 3

Exercise 1: Prenex Normal Form
Transform the following formula into prenex normal form:

\( \forall z. \left( (\forall x. q(x, z)) \rightarrow p(x, g(y, z)) \right) \land \neg(\forall z. \neg(\forall x. q(f(x, y), z))) \)

Exercise 2: Syntax and Semantics of FOL
(a) Assume that \(a, b\) are constant symbols, \(f\) resp. \(g\) are function symbols of arity one resp. two, \(p\) is a predicate symbol of arity two, and \(x\) is a variable. For each of the following strings determine whether it is a term, an atom, a literal, or a formula. Note that it can be more than one of these, or none if it is syntactically incorrect.

(i) \(f(a)\)
(ii) \(g(f(a), f)\)
(iii) \(p(f(a), x)\)
(iv) \(g(x, f(x))\)
(v) \(\neg p(a, b)\)
(vi) \(\exists a.p(a, b)\)
(vii) \(\exists x.p(x, f(a))\)
(viii) \(p(x, p(x, x))\)
(ix) \(p(a, b) \lor p(b, a)\)
(x) \(p \land \exists x.p(x, x)\)
(xi) \(\neg \exists x.p(a, b)\)
(xii) \(\forall x.\exists x.p(x, x)\)

(b) For each of the following formulae give a satisfying interpretation. Assume that equals and \(p\) are binary predicates, \(f\) is a unary function, and \(add\) is a binary function.

(i) \(equals(add(2, 2), 5)\)
(ii) \(\forall x. p(x, x)\)
(iii) \(\exists y. \forall x.p(x, y)\)
(iv) \(\forall x. (p(x, f(x)) \land \neg p(f(x), x))\)

Exercise 3: Semantic Tableaux for FOL
Use the semantic tableaux method to prove the validity of the following formulae.

(a) \((\forall x. (p(x) \rightarrow q(a))) \land (\exists x. p(x)) \rightarrow q(a)\)
(b) \((\forall x. p(f(x))) \land (\forall y. (q(y) \rightarrow \neg p(f(y)))) \rightarrow \neg q(b)\)
(c) \((\forall x, y. (p(x, y) \lor p(y, x))) \rightarrow \forall z.p(z, z)\)
(d) \(\forall y. \exists x. (p(x) \rightarrow p(y))\)
(e) \(\exists x. \forall y. (p(x) \rightarrow p(y))\)