Contents & Goals

Last Lecture:
- Started DC Syntax and Semantics: Symbols, State Assertions

This Lecture:
- Educational Objectives: Capabilities for following tasks/questions.
  - Read (and at best also write) Duration Calculus terms and formulae.
- Content:
  - Duration Calculus Terms
  - Duration Calculus Formulae
Duration Calculus Cont’d

Duration Calculus: Overview

We will introduce three (or five) syntactical “levels”:

(i) Symbols:

\[ f, g, \text{true}, \text{false}, =, <, \leq, \geq, x, y, z, X, Y, Z, d \]

(ii) State Assertions:

\[ P ::= 0 | 1 | X = d | \neg P_1 | P_1 \land P_2 \]

(iii) Terms:

\[ \theta ::= x | \ell | \int P | f(\theta_1, \ldots, \theta_n) \]

(iv) Formulae:

\[ F ::= p(\theta_1, \ldots, \theta_n) | \neg F_1 | F_1 \land F_2 | \forall x \bullet F_1 | F_1 ; F_2 \]

(v) Abbreviations:

\[ [], [P], [P]^t, [P] \leq t, \Diamond F, \Box F \]
Terms: Syntax

- **Duration terms** (DC terms or just terms) are defined by the following grammar:

  \[ \theta ::= x | \ell | \int P | f(\theta_1, \ldots, \theta_n) \]

  where \( x \) is a global variable, \( \ell \) and \( \int \) are special symbols, \( P \) is a state assertion, and \( f \) a function symbol (of arity \( n \)).

- \( \ell \) is called the **length operator**, \( \int \) is called the **integral operator**

- Notation: we may write function symbols in **infix notation** as usual, i.e. write \( \theta_1 + \theta_2 \) instead of \( +(\theta_1, \theta_2) \).

Definition 1. [Rigid]

A term **without** length and integral symbols is called **rigid**.

Example: \( x + (y-z) \cdot 3 + 2 \) is rigid

\( \ell \rightarrow 3 \) is not rigid

Terms: Semantics

- Closed **intervals** in the time domain

  \[ \text{Intv} := \{ [b, e] | b, e \in \text{Time and } b \leq e \} \]

  **Point intervals**: \([b, b]\)

- Let \( \text{GVar} \) be the set of global variables. A valuation of \( \text{GVar} \) is a function

  \[ V: \text{GVar} \rightarrow \mathbb{R} \]

  We use \( \text{Val} \) to denote the set of all valuations of \( \text{GVar} \), i.e. \( \text{Val} = (\text{GVar} \rightarrow \mathbb{R}) \).
Terms: Semantics

- The semantics of a term is a function
  \[ I[\theta] : \text{Val} \times \text{Intv} \rightarrow \mathbb{R} \]
  i.e. \( I[\theta](V, [b, c]) \) is the real number that \( \theta \) denotes under interpretation \( I \) and valuation \( V \) in the interval \([b, c]\).

- The value is defined inductively on the structure of \( \theta \):
  \[
  I[x](V, [b, c]) = V(x),
  I[\ell](V, [b, c]) = e - b,
  I[\int P](V, [b, c]) = \int_{b}^{e} P(t) \, dt,
  I[f(\theta_1, \ldots, \theta_n)](V, [b, c]) = \hat{f}(I[\theta_1](V, [b, c]), \ldots, I[\theta_n](V, [b, c]))
  \]

Terms: Example

\[ L : G \rightarrow T \]

\[ \theta = x \cdot \int L = (x, \int_{L}^{3.7}) \]

\[ \mathcal{I} \]

\[ 0 \quad 0.05 \quad 1 \quad 1.1 \quad 2 \quad 2.4 \quad 3 \quad 3.7 \quad 4 \]

\[ \mathcal{V}(x) = 20. \]

- \( I[\xi x 1(V_1, [b, c])] = V_1[x] = 20 \)
- \( I[\xi x L(V_1, [b, c])] = \int_{b}^{e} L_2(x) \, dx = \int_{0}^{3.7} L_2(x) \, dx = 4.25 \)
- \( I[\xi x 3(V_1, [b, c])] \) has a definite integral from 0 to 3.7, \( \int_{0}^{3.7} L_3(x) \, dx = 0 \)
Terms: Semantics Well-defined?

- So, $\int_b^e P_T(t) \, dt$ — but does the integral always exist?
- IOW: is there a $P_T$ which is not (Riemann-)integrable? Yes. For instance
  
  \[ P_T(t) = \begin{cases} 
  1 & \text{if } t \in \mathbb{Q} \\
  0 & \text{if } t \notin \mathbb{Q} 
  \end{cases} \]

- To exclude such functions, DC considers only interpretations $I$ satisfying the following condition of finite variability:
  
  For each state variable $X$ and each interval $[b, e]$ there is a finite partition of $[b, e]$ such that the interpretation $X_I$ is constant on each part.
  
  Thus on each interval $[b, e]$ the function $X_T$ has only finitely many points of discontinuity.

Terms: Remarks

“finitely many points do not matter.”

Remark 2.5. The semantics $I[\theta]$ of a term is insensitive against changes of the interpretation $I$ at individual time points.

Let $I_1, I_2$ be interpretations such that $I_1(x)(t) = I_2(x)(t)$ for all $x$ except for one $t \in \text{Time}$. Then $I_1[\theta](V, [b, e]) = I_2[\theta](V, [b, e])$.

Remark 2.6. The semantics $I[\theta](V, [b, e])$ of a rigid term does not depend on the interval $[b, e]$.
Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) Symbols:
\[ a \in \mathbb{R}, f, g, \quad \text{true, false, =, <, >, \leq, \geq, } \quad x, y, z, \quad X, Y, Z, \quad d \]

(ii) State Assertions:
\[ P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \land P_2 \]

(iii) Terms:
\[ \theta ::= x \mid \ell \mid \int P \mid f(\theta_1, \ldots, \theta_n) \]

(iv) Formulae:
\[ F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \land F_2 \mid \forall x \cdot F_1 \mid F_1 ; F_2 \]

(v) Abbreviations:
\[ [\mathbb{P}], [\mathbb{P}]^t, [\mathbb{P}]^{\leq t}, \quad \Diamond F, \quad \Box F \]

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Formulae: Syntax

- The set of DC formulae is defined by the following grammar:
  \[ F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \land F_2 \mid \forall x \cdot F_1 \mid F_1 ; F_2 \]
  where \( p \) is a predicate symbol, \( \theta_i \) a term, \( x \) a global variable.

- chop operator: ‘;’
- atomic formula: \( p(\theta_1, \ldots, \theta_n) \)
- rigid formula: all terms are rigid
- chop free: ‘;’ doesn’t occur
- usual notion of free and bound (global) variables

- Note: quantification only over \( \text{first-order} \) global variables, not over \( \text{second-order} \) state variables.
**Formulae: Priority Groups**

- To avoid parentheses, we define the following five priority groups from highest to lowest priority:
  - $\neg$ (negation)
  - $;$ (chop)
  - $\land, \lor$ (and/or)
  - $\Rightarrow, \Leftarrow$ (implication/equivalence)
  - $\exists, \forall$ (quantifiers)

**Examples:**
- $\neg F; F \lor H$
- $\forall x \bullet \{F \land G\}$
- $\exists z \bullet G$ or $\forall z \bullet G$ for some $z$ occurring in $\theta$
- $F \land (F \lor H)$
- $((\neg F) \lor F) \land H$
- $\neg (F \land F) \lor H$
- $\neg F; F \lor H$
- $\forall x \bullet \{F \land G\}$

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**Syntactic Substitution...**

...of a term $\theta$ for a variable $x$ in a formula $F$.

- We use $F[x := \theta]$ to denote the formula that results from performing the following steps:
  - (i) transform $F$ into $\tilde{F}$ by (consistently) renaming bound variables such that no free occurrence of $x$ in $\tilde{F}$ appears within a quantified subformula $\exists z \bullet G$ or $\forall z \bullet G$ for some $z$ occurring in $\theta$,
  - (ii) textually replace all free occurrences of $x$ in $\tilde{F}$ by $\theta$.

**Examples:**
- $F := (x \geq y \implies \exists z \bullet z \geq 0 \land x = y + z)$, $\theta_1 := \ell$, $\theta_2 := \ell + z$
- $F[x := \theta_1] = (\ell \geq y \implies \exists z \bullet z \geq 0 \land \ell = y + z)$
- $F[x := \theta_2] = (\ell \geq y \implies \exists z \bullet z \geq 0 \land \ell + z = y + z)$
Formulæ: Semantics

- The semantics of a formula is a function

\[ I[F] : \text{Val} \times \text{Intv} \to \{ \text{tt}, \text{ff} \} \]

i.e. \( I[F](\mathcal{V}, [b, e]) \) is the truth value of \( F \) under interpretation \( I \) and valuation \( \mathcal{V} \) in the interval \([b, e] \).

- This value is defined inductively on the structure of \( F \):

\[
I[p(\theta_1, \ldots, \theta_n)](\mathcal{V}, [b, e]) = 1 \text{ if } I[\theta_1](\mathcal{V}, [b, e]) = I[\theta_n](\mathcal{V}, [b, e]) = 1 \\
I[\neg F](\mathcal{V}, [b, e]) = \text{tt} \text{ iff } I[F](\mathcal{V}, [b, e]) = \text{ff} \\
I[F_1 \land F_2](\mathcal{V}, [b, e]) = \text{tt} \text{ iff } I[F_1](\mathcal{V}, [b, e]) = \text{tt} \text{ and } I[F_2](\mathcal{V}, [b, e]) = \text{tt} \\
I[\forall x \cdot F](\mathcal{V}, [b, e]) = \text{tt} \text{ iff for all } a \in \mathbb{R}, I[F[x := a]](\mathcal{V}, [b, e]) = \text{tt} \\
I[F_1 ; F_2](\mathcal{V}, [b, e]) = \text{tt} \text{ iff there is an } m \in [b, e] \text{ such that } I[F_1](\mathcal{V}, [b, m]) = I[F_2](\mathcal{V}, [m, e]) = \text{tt}.
\]

Formulæ: Example

\[ F := \begin{cases} \int L = 0 \quad & \int L = 1 \\ 0 & 1 \end{cases} \]

\[
\begin{array}{c|c|c|c|c}
0 & 1 & \cdots & 3 & 4 \\
\hline
\int L & 0 & 1 & \cdots & 3 \end{array}
\]

\[
I[F](\mathcal{V}, [0, 2]) = \text{tt}
\]

Proof:

\[
\begin{align*}
I[\int L = 0](\mathcal{V}, [0, 2]) &= 1 \text{ if } (0, 2) = 0 \\
I[\int L = 1](\mathcal{V}, [0, 2]) &= 1 \text{ if } (0, 2) = 1 \\
I[\int L = 1](\mathcal{V}, [3, 4]) &= 1 \text{ if } (3, 4) = 1 \\
I[\int L = 0](\mathcal{V}, [1, 3]) &= 1 \text{ if } (1, 3) = 0
\end{align*}
\]

- The chop point is not unique here. All \( m \in [0, 3] \) are proper chop points.
- \( \int L = 1, \int L = 1 \)
References