Real-Time Systems

Lecture 05: Duration Calculus III

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Contents & Goals

Last Lecture:

- DC Syntax and Semantics: Terms, Formulae

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Read (and at best also write) Duration Calculus formulae – including abbreviations.
  - What is Validity/Satisfiability/Realisability for DC formulae?
  - How can we prove a design correct?
- **Content:**
  - Duration Calculus Abbreviations
  - Basic Properties
  - Validity, Satisfiability, Realisability
  - A correctness proof for a gas burner design

Duration Calculus Cont'd

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$$f, g, \quad true, false, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \dots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Formulae: Remarks

Remark 2.10. [*Rigid and chop-free*] Let F be a duration formula, $\mathcal{I}$ an interpretation, $\mathcal{V}$ a valuation, and $[b, e] \in$ Intv.

- If F is **rigid**, then

$$\forall [b', e'] \in \mathsf{Intv} : \mathcal{I}[\![F]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F]\!](\mathcal{V}, [b', e']).$$

- If F is **chop-free** or θ is **rigid**, then in the calculation of the semantics of F, every occurrence of θ denotes the same value.

";" does not occur in F

in F

e.g. $\underbrace{\textstyle\int(x)}_{\theta} > 3 \,;\, \underbrace{\textstyle\int(x)}_{\theta} > 5$

e.g. $\underbrace{\ell}_{\theta} > 0 \wedge \underbrace{\ell}_{\theta} > 1 \qquad \ell > 0 \,;\, \ell > 1$ not chop-free

Substitution Lemma

Lemma 2.11. [*Substitution*]
Consider a formula F, a global variable x, and a term θ such that F is **chop-free** or θ is **rigid**.

Then for all interpretations $\mathcal{I}$, valuations $\mathcal{V}$, and intervals $[b, e]$,

$$\mathcal{I}[\![\underbrace{F[x := \theta]}_{\text{syntactic modification of } F}]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F]\!](\underbrace{\mathcal{V}[x := d]}_{\text{semantical/ modification of assignment}}, [b, e])$$

where $d = \mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$.

Term $=(\ell, x)$

- $F := ((\ell = x) ; (\ell = x)) \Longrightarrow \ell = \underbrace{2 \cdot x}_{\text{Term}}, \quad \theta := \ell \qquad \mathcal{V}, [e,b] = [5, 11]$

- $\mathcal{I}[\![F[x := \theta]]\!](\mathcal{V}, [e,b]) = \mathcal{I}[\![\ell = \ell \,;\, \ell = \ell \Rightarrow \ell = 2 \cdot \ell]\!](\mathcal{V}, [e,b]) =$ ff if $e < b$

- $\mathcal{I}[\![F]\!](\mathcal{V}[x := 6], [e,b]) =$ tt , F is even valid
  $d = \mathcal{I}[\![\theta]\!](\mathcal{V}, [5, 11]) = 6$

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$$f, g, \quad true, false, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \int P \mid f(\theta_1, \dots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

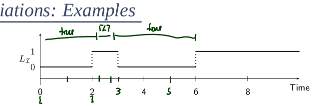
$$\lceil\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Duration Calculus Abbreviations

Abbreviations

- $\lceil\rceil := \ell = 0$ **(point interval)**
- $\lceil P \rceil := (\int P = \ell) \wedge \ell > 0$ **(almost everywhere)**
- $\lceil P \rceil^t := \lceil P \rceil \wedge \ell = t$ **(for time t)**
- $\lceil P \rceil^{\leq t} := \lceil P \rceil \wedge \ell \leq t$ **(up to time t)**
- $\Diamond F := true \,;\, F \,;\, true$ **(for some subinterval)**
- $\Box F := \neg \Diamond \neg F$ **(for all subintervals)**

Abbreviations: Examples



$\int \neg L = \ell \wedge \ell > 0$

$\mathcal{I}[\![(\int L) = 0]\!](\mathcal{V}, [0,2]) =$ tt

$\mathcal{I}[\![(\int L) = 1]\!](\mathcal{V}, [2,6]) =$ tt

$\mathcal{I}[\![\int L = 0 \,;\, \int L = 1]\!](\mathcal{V}, [0,6]) =$ tt

$\mathcal{I}[\![\lceil \neg L \rceil]\!](\mathcal{V}, [0,2]) =$ tt

$\mathcal{I}[\![\lceil L \rceil]\!](\mathcal{V}, [2,3]) =$ tt

$\mathcal{I}[\![\lceil \neg L \rceil \,;\, \lceil L \rceil]\!](\mathcal{V}, [0,3]) =$ tt ← unique chop point

$\mathcal{I}[\![\lceil \neg L \rceil \,;\, \lceil L \rceil \,;\, \lceil \neg L \rceil]\!](\mathcal{V}, [0,6]) =$ tt — $2 \leq m_1 < m_2 \leq 3$ are witness chop points

true ; ⌈L⌉ ; true

$\mathcal{I}[\![\Diamond \lceil L \rceil]\!](\mathcal{V}, [0,6]) =$ tt

$\mathcal{I}[\![\Diamond \lceil \neg L \rceil]\!](\mathcal{V}, [0,6]) =$ tt

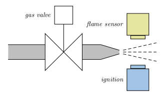
$\mathcal{I}[\![\Diamond \lceil \neg L \rceil^2]\!](\mathcal{V}, [0,6]) =$ tt

$\mathcal{I}[\![\lceil \neg L \rceil^2 \,;\, \lceil \neg L \rceil^1 \,;\, \lceil \neg L \rceil^3]\!](\mathcal{V}, [0,6]) =$ ff

$\mathcal{I}[\![\lceil \neg L \rceil^2 \,;\, \lceil L \rceil^1 \,;\, \lceil \neg L \rceil^3]\!](\mathcal{V}, [0,6]) =$ tt

Duration Calculus: Looking back

- Duration Calculus is an **interval logic**.
- Formulae are evaluated in an (**implicitly given**) interval.

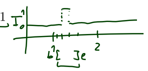


Back to our gas burner:

- $G, F, I, H, \quad \mathcal{D}(G) = \dots = \mathcal{D}(H) = \{0, 1\}$
- Define L as $G \wedge \neg F$.

Strangest operators:

- almost **everywhere** — Example: $\lceil G \rceil$
  (Holds in a given interval $[b, e]$ iff the gas valve is open almost everywhere.)
- **chop** — Example: $\mathcal{I}(\lceil \neg I \rceil \,;\, \lceil I \rceil \,;\, \lceil \neg I \rceil) \implies \ell \geq 1$
  (Ignition phases last at least one time unit.)



- **integral** — Example: $\ell \geq 60 \implies \int L \leq \frac{\ell}{20}$
  (At most 5% leakage time within intervals of at least 60 time units.)

DC Validity, Satisfiability, Realisability

Validity, Satisfiability, Realisability

Let $\mathcal{I}$ be an interpretation, $\mathcal{V}$ a valuation, $[b,e]$ an interval, and F a DC formula.

- $\mathcal{I}, \mathcal{V}, [b,e] \models F$ ("F **holds** in $\mathcal{I}$, $\mathcal{V}$, $[b,e]$") iff $\quad \mathcal{I}[\![F]\!](\mathcal{V},[b,e]) = \mathrm{tt}$.
- F is called **satisfiable** iff it holds in some $\mathcal{I}$, $\mathcal{V}$, $[b,e]$.
- $\mathcal{I}, \mathcal{V} \models F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F") iff $\quad \forall [b,e] \in \mathsf{Intv} : \mathcal{I}, \mathcal{V}, [b,e] \models F$.
- F is called **realisable** iff some $\mathcal{I}$ and $\mathcal{V}$ realise F.
- $\mathcal{I} \models F$ ("$\mathcal{I}$ **realises** F") iff $\quad \forall \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models F$.
- $\models F$ ("F is **valid**") iff $\quad \forall$ interpretation $\mathcal{I} : \mathcal{I} \models F$.

Validity vs. Satisfiability vs. Realisability

Remark 2.13. For all DC formulae F,

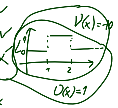
- F is satisfiable iff $\neg F$ is not valid,
  F is valid iff $\neg F$ is not satisfiable.
- If F is valid then F is realisable, but not vice versa.
- If F is realisable then F is satisfiable, but not vice versa.

Examples: Valid? Realisable? Satisfiable?

- $\mathcal{I}, \mathcal{V}, [b,e] \models F$ ("F **holds** in $\mathcal{I}$, $\mathcal{V}$, $[b,e]$") iff $\quad \mathcal{I}[\![F]\!](\mathcal{V},[b,e]) = \mathrm{tt}$.
- F is called **satisfiable** iff it holds in some $\mathcal{I}$, $\mathcal{V}$, $[b,e]$.
- $\mathcal{I}, \mathcal{V} \models F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F") iff $\quad \forall [b,e] \in \mathsf{Intv} : \mathcal{I}, \mathcal{V}, [b,e] \models F$.
- F is called **realisable** iff some $\mathcal{I}$ and $\mathcal{V}$ realise F.
- $\mathcal{I} \models F$ ("$\mathcal{I}$ **realises** F") iff $\quad \forall \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models F$.
- $\models F$ ("F is **valid**") iff $\quad \forall$ interpretation $\mathcal{I} : \mathcal{I} \models F$.

state ass.

	Satisfiable	Realisable	Valid
$\ell \geq 0$	✓	$\Leftarrow$ ✓	$\Leftarrow$ ✓
$\ell = \int 1$			✓
$(\ell = 30) \iff ((\ell = 10) ; (\ell = 20))$			✓
$((F\,;G)\,;H) \iff (F\,;(G\,;H))$			✓
$\int L \leq x$	✓	$\Leftarrow$ ✓	✗
$\ell = 2$	✓	✗	✗
$\ell < 0$	✗	$\Rightarrow$ ✗	$\Rightarrow$ ✗



Initial Values

- $\mathcal{I}, \mathcal{V} \models_0 F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F **from** 0") iff
  $\forall t \in \mathsf{Time} : \mathcal{I}, \mathcal{V}, [0,t] \models F$.
- F is called **realisable from** 0 iff some $\mathcal{I}$ and $\mathcal{V}$ realise F from 0.
- Intervals of the form $[0,t]$ are called **initial intervals**.
- $\mathcal{I} \models_0 F$ ("$\mathcal{I}$ **realises** F **from** 0") iff $\quad \forall \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models_0 F$.
- $\models_0 F$ ("F is **valid from** 0") iff $\quad \forall$ interpretation $\mathcal{I} : \mathcal{I} \models_0 F$.

Initial or not Initial...

For all interpretations $\mathcal{I}$, valuations $\mathcal{V}$, and DC formulae F,

(i) $\mathcal{I}, \mathcal{V} \models F$ implies $\mathcal{I}, \mathcal{V} \models_0 F$, but not vice versa,

(ii) if F is realisable then F is realisable from 0, but not vice versa,

(iii) F is valid iff F is valid from 0.

Specification and Semantics-based Correctness Proofs of Real-Time Systems with DC

Methodology: Ideal World...

(i) Choose a collection of **observables** 'Obs'.

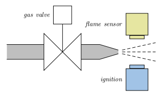
(ii) Provide the **requirement/specification** 'Spec' as a conjunction of DC formulae (over 'Obs').

(iii) Provide a description 'Ctrl' of the **controller** in form of a DC formula (over 'Obs').

(iv) We say 'Ctrl' is **correct** (wrt. 'Spec') iff

$$\models_0 \text{Ctrl} \implies \text{Spec}.$$

Gas Burner Revisited



(i) Choose **observables**:

- two boolean observables G and F (i.e. Obs $= \{G, F\}$, $\mathcal{D}(G) = \mathcal{D}(F) = \{0, 1\}$)
- $G = 1$: gas valve open (output)
- $F = 1$: have flame (input)
- define $L := G \wedge \neg F$ (leakage)

(ii) Provide the **requirement**:

$$\text{Req} : \iff \Box(\ell \geq 60 \implies \cancel{20}\int L \leq \tfrac{\ell}{20})$$

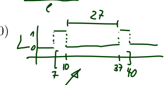
Gas Burner Revisited

(iii) Provide a description 'Ctrl' of the **controller** in form of a DC formula (over 'Obs'). Here, firstly consider a **design**:

$\mathcal{I}[\![F]\!](\mathcal{V},[7,40])$

$\Box(\lceil L \rceil ; \lceil \neg L \rceil ; \lceil L \rceil \Rightarrow \ell > 30)$

- Des-1 : $\iff \Box(\lceil L \rceil \implies \ell \leq 1)$
- Des-2 : $\iff \Box(\lceil L \rceil ; \lceil \neg L \rceil ; \lceil L \rceil \implies \ell > 30)$



(iv) Prove **correctness**:

- We want (or do we want $\models_0$...?):

$$\models (\text{Des-1} \wedge \text{Des-2} \implies \text{Req}) \quad \text{(Thm. 2.16)}$$

wrt following Des-2 intuition

Gas Burner Revisited

(iii) Provide a description 'Ctrl' of the **controller** in form of a DC formula (over 'Obs'). Here, firstly consider a **design**:

- Des-1 : $\iff \Box(\lceil L \rceil \implies \ell \leq 1)$
- Des-2 : $\iff \Box(\lceil L \rceil ; \lceil \neg L \rceil ; \lceil L \rceil \implies \ell > 30)$

(iv) Prove **correctness**:

- We want (or do we want $\models_0$...?):

$$\models (\text{Des-1} \wedge \text{Des-2} \implies \text{Req}) \quad \text{(Thm. 2.16)}$$

- We do show

$$\models \text{Req-1} \implies \text{Req} \quad \text{(Lem. 2.17)}$$

with the simplified requirement

$$\text{Req-1} := \Box(\ell \leq 30 \implies \int L \leq 1),$$

- and we show

$$\models (\text{Des-1} \wedge \text{Des-2}) \implies \text{Req-1}. \quad \text{(Lem. 2.19)}$$

References

References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.