Real-Time Systems

Lecture 7: DC Properties II

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Dr. Bernd Westphal

Albert-Ludwigs-Universität Freiburg, Germany

Contents & Goals

Last Lecture:

- RDC in discrete time
- Started: Satisfiability and realisability from 0 is decidable for RDC in discrete time

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Facts: (un)decidability properties of DC in discrete/continuous time.
  - What's the idea of the considered (un)decidability proofs?
- **Content:**
  - Complete: Satisfiability and realisability from 0 is decidable for RDC in discrete time
  - Undecidable problems of DC in continuous time

RDC in Discrete Time Cont'd

Recall: Decidability of Satisfiability/Realisability from 0

Theorem 3.6.
The satisfiability problem for RDC with discrete time is decidable.

Theorem 3.9.
The realisability problem for RDC with discrete time is decidable.

Recall: Proof Sketch

RDC formula F, DT/ $\mathcal{I}$.

$F \quad \Dashv \quad \mathcal{I}, [0,n]$

construct $\Updownarrow$ prove construct

finite words

regular language $\quad \mathcal{L}(F) \quad \ni \quad w \in \Sigma^*$ word

Sketch: Proof of Theorem 3.6

- give a procedure to construct, given a formula F, a **regular** language $\mathcal{L}(F)$ such that

$$\mathcal{I}, [0, n] \models F \text{ if and only if } w \in \mathcal{L}(F)$$

  where word w describes $\mathcal{I}$ on $[0, n]$
  (suitability of the procedure: **Lemma 3.4**)

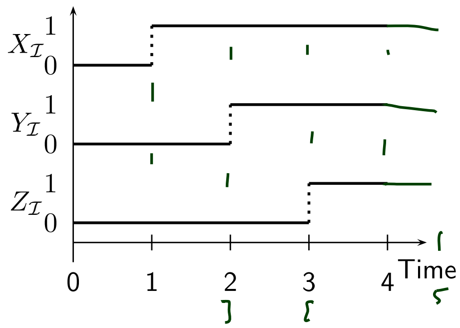
- then F is satisfiable in discrete time if and only if $\mathcal{L}(F)$ is not empty (**Lemma 3.5**)

- Theorem 3.6 follows because
  - $\mathcal{L}(F)$ can **effectively** be constructed,
  - the emptyness problem is **decidable** for regular languages.

Construction of $\mathcal{L}(F)$

- **Idea:**
  - alphabet $\Sigma(F)$ consists of basic conjuncts of the state variables in F,
  - a letter corresponds to an interpretation on an interval of length 1,
  - a word of length n describes an interpretation on interval $[0, n]$.
- **Example:** Assume F contains exactly state variables X, Y, Z, then

$$\Sigma(F) = \{X \wedge Y \wedge Z, X \wedge Y \wedge \neg Z, X \wedge \neg Y \wedge Z, X \wedge \neg Y \wedge \neg Z, \neg X \wedge Y \wedge Z, \neg X \wedge Y \wedge \neg Z, \neg X \wedge \neg Y \wedge Z, \neg X \wedge \neg Y \wedge \neg Z\}.$$



$$\begin{aligned} w = {} & (\neg X \wedge \neg Y \wedge \neg Z) \\ & \cdot (X \wedge \neg Y \wedge \neg Z) \\ * \; & \cdot (X \wedge Y \wedge \neg Z) \\ & \cdot (X \wedge Y \wedge Z) \in \Sigma(F)^* \\ & . (X \wedge Y \wedge Z) \end{aligned}$$

Construction of $\mathcal{L}(F)$ more Formally

$X \wedge \neg Y$
$\Leftrightarrow (X \wedge \neg Y \wedge Z) \vee (X \wedge \neg Y \wedge \neg Z)$

Definition 3.2. A word $w = a_1 \ldots a_n \in \Sigma(F)^*$ with $n \geq 0$ **describes** a **discrete** interpretation $\mathcal{I}$ on $[0, n]$ if and only if

$$\forall j \in \{1, \ldots, n\} \; \forall t \in]j-1, j[: \mathcal{I}[\![a_j]\!](t) = 1.$$

For $n = 0$ we put $w = \varepsilon$.

$DNF(X \wedge \neg Y) = \{(X \wedge \neg Y \wedge Z), (X \wedge \neg Y \wedge \neg Z)\}$

- Each state assertion P can be transformed into an equivalent **disjunctive normal form** $\bigvee_{i=1}^{m} a_i$ with $a_i \in \Sigma(F)$.
- Set $DNF(P) := \{a_1, \ldots, a_m\}$ $(\subseteq \Sigma(F))$.
- Define $\mathcal{L}(F)$ inductively:

finite words, length at least one

$$\begin{aligned} \mathcal{L}(\lceil P \rceil) &= DNF(P)^+, && \text{(regular language)} \\ \mathcal{L}(\neg F_1) &= \Sigma(F)^* \setminus \mathcal{L}(F_1) && \text{(again regular)} \\ \mathcal{L}(F_1 \vee F_2) &= \mathcal{L}(F_1) \cup \mathcal{L}(F_2), && (\text{— " —}) \\ \mathcal{L}(F_1 \, ; F_2) &= \mathcal{L}(F_1) \, . \, \mathcal{L}(F_2). && (\text{— " —}) \end{aligned}$$

concatenate

Lemma 3.4

Lemma 3.4. For all RDC formulae F, discrete interpretations $\mathcal{I}$, $n \geq 0$, and all words $w \in \Sigma(F)^*$ which **describe** $\mathcal{I}$ on $[0, n]$,

$$\mathcal{I}, [0, n] \models F \text{ if and only if } w \in \mathcal{L}(F).$$

Proof: Structural induction

Base $F = \lceil P \rceil$: assume $w = a_1 \dots a_n$ describes $\mathcal{I}$ on $[0,n]$

$\mathcal{I},[0,n] \models \lceil P \rceil \iff \mathcal{I},[0,n] \models \lceil P \rceil$ and $n \geq 1$

$\iff n \geq 1$ and $\forall\, 1 \leq j \leq n \bullet \mathcal{I},[j-1,j] \models \lceil P \rceil$

$\iff n \geq 1$ and $\forall\, 1 \leq j \leq n \bullet \mathcal{I},[j-1,j] \models (\lceil P \rceil \wedge \lceil a_j \rceil)$ and $a_j \in DNF(P)$

"describes"

$\iff n \geq 1$ and $\forall\, 1 \leq j \leq n \bullet a_j \in DNF(P)$ clear

$\iff w \in DNF(P)^+$

$\iff w \in \mathcal{L}(\lceil P \rceil)$

Steps:
- $\neg F_1$
- $F_1 \vee F_2$
- $F_1 ; F_2$

Sketch: Proof of Theorem 3.9

Theorem 3.9.
The realisability problem for RDC with discrete time is decidable.

- $kern(L)$ contains all words of L whose prefixes are again in L.
- If L is regular, then $kern(L)$ is also regular.
- $kern(\mathcal{L}(F))$ can effectively be constructed.
- We have

  Lemma 3.8. For all RDC formulae F, F is realisable from 0 in discrete time if and only if $kern(\mathcal{L}(F))$ is infinite.

- Infinity of regular languages is decidable.

(Variants of) RDC in Continuous Time

Recall: Restricted DC (RDC)

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \,;\, F_2$$

where P is a state assertion, but with **boolean** observables **only**.

From now on: "RDC + $\ell = x, \forall x$"

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \,;\, F_2 \mid \ell = 1 \mid \ell = x \mid \forall x \bullet F_1$$

Undecidability of Satisfiability/Realisability from 0

Theorem 3.10.
The realisability from 0 problem for DC with **continuous time** is undecidable, not even semi-decidable.

Theorem 3.11.
The satisfiability problem for DC with continuous time is undecidable.

Sketch: Proof of Theorem 3.10

Reduce divergence of **two-counter machines** to realisability from 0:

- Given a two-counter machine $\mathcal{M}$ with final state q_{fin},
- construct a DC formula $F(\mathcal{M}) := encoding(\mathcal{M})$
- such that

  $\mathcal{M}$ **diverges** **if and only if** the DC formula

  $$F(\mathcal{M}) \wedge \neg\Diamond\lceil q_{fin}\rceil$$

  is **realisable from 0**.

- If realisability from 0 was (semi-)decidable,
  divergence of two-counter machines would be (which it isn't).

Recall: Two-counter machines

A **two-counter** machine is a structure

$$\mathcal{M} = (\mathcal{Q}, q_0, q_{fin}, Prog)$$

where

start state of command

- $\mathcal{Q}$ is a finite set of **states**,
- comprising the **initial state** q_0 and the **final state** q_{fin}
- $Prog$ is the **machine program**, i.e. a finite set of **commands** of the form

$$q : inc_1 : q' \quad \text{and} \quad q : dec_i : q', q'', \qquad i \in \{1, 2\}.$$

$q : inc_2 : q'$

$q, q', q'' \in \mathcal{Q}$

- We assume **deterministic** 2CM: for each $q \in \mathcal{Q}$, at most one command starts in q, and q_{fin} is the only state where no command starts.

2CM Configurations and Computations

current state

values of counters 1,2

- a **configuration** of $\mathcal{M}$ is a triple $K = (q, n_1, n_2) \in \mathcal{Q} \times \mathbb{N}_0 \times \mathbb{N}_0$.
- The **transition relation** "$\vdash$" on configurations is defined as follows:

Command	Semantics: $K \vdash K'$
$q : inc_1 : q'$	$(q, n_1, n_2) \vdash (q', n_1 + 1, n_2)$
$q : dec_1 : q', q''$	$(q, 0, n_2) \vdash (q', 0, n_2)$
	$(q, n_1 + 1, n_2) \vdash (q'', n_1, n_2)$
$q : inc_2 : q'$	$(q, n_1, n_2) \vdash (q', n_1, n_2 + 1)$
$q : dec_2 : q', q''$	$(q, n_1, 0) \vdash (q', n_1, 0)$
	$(q, n_1, n_2 + 1) \vdash (q'', n_1, n_2)$

- The (!) **computation** of $\mathcal{M}$ is a finite sequence of the form ("$\mathcal{M}$ **halts**")

$$K_0 = (q_0, 0, 0) \vdash K_1 \vdash K_2 \vdash \cdots \vdash (q_{fin}, n_1, n_2)$$

or an infinite sequence of the form ("$\mathcal{M}$ **diverges**")

$$K_0 = (q_0, 0, 0) \vdash K_1 \vdash K_2 \vdash \ldots$$

2CM Example

- $\mathcal{M} = (\mathcal{Q}, q_0, q_{fin}, Prog)$
- commands of the form $q : inc_i : q'$ and $q : dec_i : q', q''$, $i \in \{1, 2\}$
- configuration $K = (q, n_1, n_2) \in \mathcal{Q} \times \mathbb{N}_0 \times \mathbb{N}_0$.
-

Command	Semantics: $K \vdash K'$
$q : inc_1 : q'$	$(q, n_1, n_2) \vdash (q', n_1 + 1, n_2)$
$q : dec_1 : q', q''$	$(q, 0, n_2) \vdash (q', 0, n_2)$
	$(q, n_1 + 1, n_2) \vdash (q'', n_1, n_2)$
$q : inc_2 : q'$	$(q, n_1, n_2) \vdash (q', n_1, n_2 + 1)$
$q : dec_2 : q', q''$	$(q, n_1, 0) \vdash (q', n_1, 0)$
	$(q, n_1, n_2 + 1) \vdash (q'', n_1, n_2)$

$Q = \{q_0, q_1, q_{fin}\}$

$Prog = \{q_0 : inc_1 : q_1,$
$q_1 : inc_1 : q_{fin}\}$

$(q_0, 0, 0)$
$\vdash$
$(q_1, 1, 0)$
$\vdash$
$(q_{fin}, 2, 0)$

$\hookrightarrow$ machine halts

$Q = \{q_0, q_{fin}\}$

$Prog = \{q_0 : inc_2 : q_0\}$

$(q_0, 0, 0)$
$\vdash$
$(q_0, 1, 0)$
$\vdash$
$(q_0, 2, 0)$
$\vdash$
$\vdots$

$\hookrightarrow$ machine diverges

Reducing Divergence to DC realisability: Idea In Pictures

2CM $\mathcal{M}$ diverges

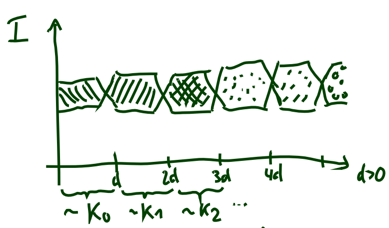
iff

exists $\pi : K_0 \vdash K_1 \vdash K_2 \ldots$

q_{fin} does not occur

iff

exist



("I describes π")

and

$\mathcal{I} \models_0 F(\mathcal{M}) \wedge \neg \Diamond \lceil q_{fin} \rceil$

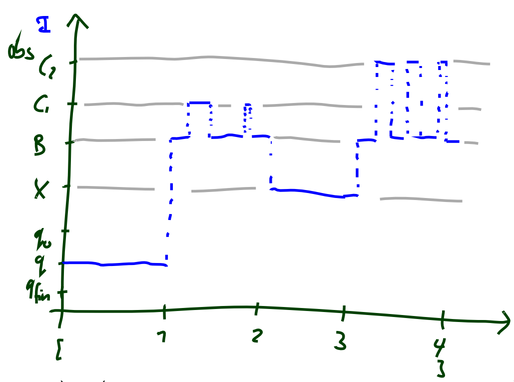
$F(\mathcal{M})$ intuitively requires:

- $[0, d]$ encodes $(q_0, 0, 0)$
- $[n \cdot d, (n+1) \cdot d]$ encodes a configuration
- $[n \cdot d, (n+1) \cdot d]$ and $[(n+1) \cdot d, (n+2) \cdot d]$ encode configurations which are in $\vdash$-relation
- if q_{fin} is reached, we stay there

Reducing Divergence to DC realisability: Idea

- A single configuration K of $\mathcal{M}$ can be encoded in an interval of length 4; being an encoding interval can be **characterised** by a DC formula.
- An interpretation on 'Time' encodes **the** computation of $\mathcal{M}$ if
  - each interval $[4n, 4(n+1)]$, $n \in \mathbb{N}_0$, **encodes** a configuration K_n,
  - each two subsequent intervals $[4n, 4(n+1)]$ and $[4(n+1), 4(n+2)]$, $n \in \mathbb{N}_0$, encode configurations $K_n \vdash K_{n+1}$ **in transition relation**.
- Being encoding of the run can be **characterised** by DC formula $F(\mathcal{M})$.
- Then $\mathcal{M}$ **diverges** if and only if $F(\mathcal{M}) \land \neg\Diamond\lceil q_{fin}\rceil$ is realisable from 0.

Encoding Configurations



set of states of $\mathcal{M}$

- We use Obs $= \{\text{obs}\}$ with $\mathcal{D}(\text{obs}) = \mathcal{Q}_{\mathcal{M}} \,\dot\cup\, \{C_1, C_2, B, X\}$.

disjoint union

abbrev. for $\lceil obs = q\rceil$

Examples:

- $K = (q, 2, 3)$

$$\begin{pmatrix} \lceil q\rceil \\ \land \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B\rceil ; \lceil C_1\rceil ; \lceil B\rceil ; \lceil C_1\rceil ; \lceil B\rceil \\ \land \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil X\rceil \\ \land \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B\rceil ; \lceil C_2\rceil ; \lceil B\rceil ; \lceil C_2\rceil ; \lceil B\rceil ; \lceil C_2\rceil ; \lceil B\rceil \\ \land \\ \ell = 1 \end{pmatrix}$$

- $K_0 = (q_0, 0, 0)$

$$\begin{pmatrix} \lceil q_0\rceil \\ \land \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B\rceil \\ \land \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil X\rceil \\ \land \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B\rceil \\ \land \\ \ell = 1 \end{pmatrix}$$

or, using abbreviations, $\lceil q_0\rceil^1 ; \lceil B\rceil^1 ; \lceil X\rceil^1 ; \lceil B\rceil^1$.

Construction of $F(\mathcal{M})$

In the following, we give DC formulae describing

- the initial configuration,
- the general form of configurations,
- the transitions between configurations,
- the handling of the final state.

$F(\mathcal{M})$ is the conjunction of all these formulae.

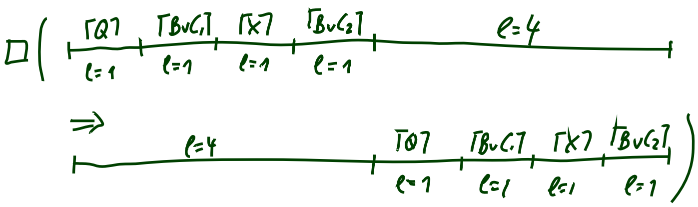
$$F(\mathcal{M}) = init \wedge keep \wedge \dots \wedge \bigwedge_{q:inc_i:q' \in Prog_{\mathcal{M}}} F(q:inc_i:q') \wedge \bigwedge_{q:dec_i:q',q'' \in Prog_{\mathcal{M}}} F(q:dec_i:q',q'')$$

Initial and General Configurations

$$init :\Longleftrightarrow (\ell \geq 4 \implies \lceil q_0 \rceil^1 ; \lceil B \rceil^1 ; \lceil X \rceil^1 ; \lceil B \rceil^1 ; true)$$

$$keep :\Longleftrightarrow \Box(\lceil Q \rceil^1 ; \lceil B \vee C_1 \rceil^1 ; \lceil X \rceil^1 ; \lceil B \vee C_2 \rceil^1 ; \ell = 4$$
$$\implies \ell = 4 ; \lceil Q \rceil^1 ; \lceil B \vee C_1 \rceil^1 ; \lceil X \rceil^1 ; \lceil B \vee C_2 \rceil^1)$$

where $Q := \neg(X \vee C_1 \vee C_2 \vee B)$.

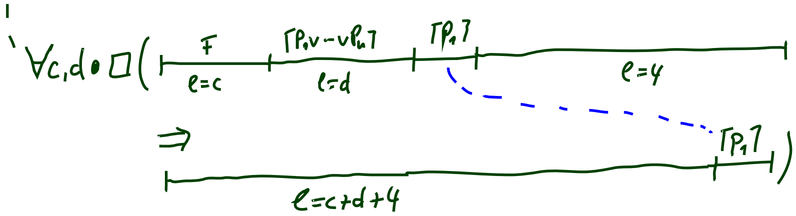


Auxiliary Formula Pattern *copy*

formula   state assertions

$$copy(F, \{P_1, \dots, P_n\}) :\iff$$

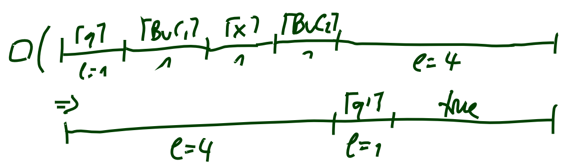
$$\forall\, c, d \bullet \Box((F \wedge \ell = c) \,;\, (\lceil P_1 \vee \dots \vee P_n \rceil \wedge \ell = d) \,;\, \lceil P_1 \rceil \,;\, \ell = 4$$
$$\implies \ell = c + d + 4 \,;\, \lceil P_1 \rceil$$
$$\wedge \dots$$
$$\wedge\, \forall\, c, d \bullet \Box((F \wedge \ell = c) \,;\, (\lceil P_1 \vee \dots \vee P_n \rceil \wedge \ell = d) \,;\, \lceil P_n \rceil \,;\, \ell = 4$$
$$\implies \ell = c + d + 4 \,;\, \lceil P_n \rceil$$



q : inc_1 : q' (Increment) $\in Prog_{\mathcal{M}}$

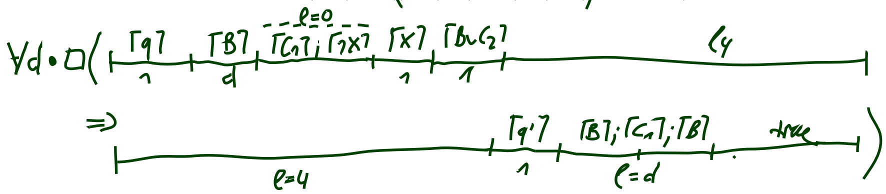
(i) Change state

$$\Box(\lceil q \rceil^1 \,;\, \lceil B \vee C_1 \rceil^1 \,;\, \lceil X \rceil^1 \,;\, \lceil B \vee C_2 \rceil^1 \,;\, \ell = 4 \implies \ell = 4 \,;\, \lceil q' \rceil^1 \,;\, true)$$



(ii) Increment counter

$$\forall\, d \bullet \Box(\lceil q \rceil^1 \,;\, \lceil B \rceil^d \,;\, (\ell = 0 \vee \lceil C_1 \rceil \,;\, \lceil \neg X \rceil) \,;\, \lceil X \rceil^1 \,;\, \lceil B \vee C_2 \rceil^1 \,;\, \ell = 4$$
$$\implies \ell = 4 \,;\, \lceil q' \rceil^1 \,;\, ((\lceil B \rceil \,;\, \lceil C_1 \rceil \,;\, \lceil B \rceil) \wedge \ell = d) \,;\, true$$



$q : inc_1 : q'$ (Increment)

(i) Keep rest of first counter

$$copy(\overbrace{\lceil q \rceil^1 \,;\, \lceil B \vee C_1 \rceil \,;\, \lceil C_1 \rceil}^{\mathcal{F}}, \overbrace{\{B, C_1\}}^{\{P_1, P_2\}})$$

(ii) Leave second counter unchanged

$$copy(\underbrace{\lceil q \rceil^1 \,;\, \lceil B \vee C_1 \rceil \,;\, \lceil X \rceil^1}_{\mathcal{F}}, \underbrace{\{B, C_2\}}_{\{P_1, P_2\}})$$

$q : dec_1 : q', q''$ (Decrement)

(i) If zero

$$\Box(\lceil q \rceil^1 \,;\, \lceil B \rceil^1 \,;\, \lceil X \rceil^1 \,;\, \lceil B \vee C_2 \rceil^1 \,;\, \ell = 4 \implies \ell = 4 \,;\, \lceil q' \rceil^1 \,;\, \lceil B \rceil^1 \,;\, true)$$

(ii) Decrement counter

$$\forall\, d \bullet \Box(\lceil q \rceil^1 \,;\, (\lceil B \rceil \,;\, \lceil C_1 \rceil \wedge \ell = d) \,;\, \lceil B \rceil \,;\, \lceil B \vee C_1 \rceil \,;\, \lceil X \rceil^1 \,;\, \lceil B \vee C_2 \rceil^1 \,;\, \ell = 4$$
$$\implies \ell = 4 \,;\, \lceil q'' \rceil^1 \,;\, \lceil B \rceil^d \,;\, true)$$

(iii) Keep rest of first counter

$$copy(\lceil q \rceil^1 \,;\, \lceil B \rceil \,;\, \lceil C_1 \rceil \,;\, \lceil B_1 \rceil, \{B, C_1\})$$

(iv) Leave second counter unchanged

$$copy(\lceil q \rceil^1 \,;\, \lceil B \vee C_1 \rceil \,;\, \lceil X \rceil^1, \{B, C_2\})$$

Final State

$$copy(\lceil q_{fin} \rceil^1 \,;\, \lceil B \vee C_1 \rceil^1 \,;\, \lceil X \rceil \,;\, \lceil B \vee C_2 \rceil^1, \{q_{fin}, B, X, C_1, C_2\})$$

Satisfiability

- Following [Chaochen and Hansen, 2004] we can observe that

  $\mathcal{M}$ **halts if and only if** the DC formula $F(\mathcal{M}) \wedge \Diamond\lceil q_{fin} \rceil$ is **satisfiable**.

  This yields

  > **Theorem 3.11.** The satisfiability problem for DC with continuous time is undecidable.

  (It is semi-decidable.)

- Furthermore, by taking the contraposition, we see

  $\mathcal{M}$ **diverges** **if and only if** $\mathcal{M}$ does not **halt**

  if and only if $F(\mathcal{M}) \wedge \neg\Diamond\lceil q_{fin} \rceil$ is **not** satisfiable.

- Thus whether a DC formula is **not satisfiable** is not decidable, not even semi-decidable.

Validity

- By Remark 2.13, F is valid iff $\neg F$ is not satisfiable, so

> **Corollary 3.12.** The validity problem for DC with continuous time is undecidable, not even semi-decidable.

- This provides us with an alternative proof of Theorem 2.23 ("there is no sound and complete proof system for DC"):
  - **Suppose** there were such a calculus $\mathcal{C}$.
  - By Lemma 2.22 it is semi-decidable whether a given DC formula F is a theorem in $\mathcal{C}$.
  - By the soundness and completeness of $\mathcal{C}$, F is a theorem in $\mathcal{C}$ **if and only if** F is valid.
  - Thus it is semi-decidable whether F is valid. **Contradiction.**

Discussion

- Note: the DC fragment defined by the following grammar is **sufficient** for the reduction

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \,;\, F_2 \mid \ell = 1 \mid \ell = x \mid \forall\, x \bullet F_1,$$

  P a state assertion, x a global variable.

- Formulae used in the reduction are abbreviations:

$$\begin{aligned} \ell = 4 &\iff \ell = 1 \,;\, \ell = 1 \,;\, \ell = 1 \,;\, \ell = 1 \\ \ell \geq 4 &\iff \ell = 4 \,;\, \mathit{true} \\ \ell = x + y + 4 &\iff \ell = x \,;\, \ell = y \,;\, \ell = 4 \end{aligned}$$

- Length 1 is not necessary — we can use $\ell = z$ instead, with fresh z.

- This is RDC augmented by "$\ell = x$" and "$\forall\, x$", which we denote by **RDC** $+\ \ell = x, \forall\, x$.

References

References

[Chaochen and Hansen, 2004] Chaochen, Z. and Hansen, M. R. (2004). *Duration Calculus: A Formal Approach to Real-Time Systems*. Monographs in Theoretical Computer Science. Springer-Verlag. An EATCS Series.

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.