Real-Time Systems

Lecture 7: DC Properties II

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Contents & Goals

Last Lecture:

- RDC in discrete time
- Started: Satisfiability and realisability from 0 is decidable for RDC in discrete time

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Facts: (un)decidability properties of DC in discrete/continuous time.
  - What's the idea of the considered (un)decidability proofs?
- **Content:**
  - Complete: Satisfiability and realisability from 0 is decidable for RDC in discrete time
  - Undecidable problems of DC in continuous time

RDC in Discrete Time Cont'd

Recall: Decidability of Satisfiability/Realisability from 0

Theorem 3.6.
The satisfiability problem for RDC with discrete time is decidable.

Theorem 3.9.
The realisability problem for RDC with discrete time is decidable.

Recall: Proof Sketch

RDC formula F, DT/ $\mathcal{I}$.

$$\begin{array}{ccc} F & \dashv & \mathcal{I}, [0,n] \\ \Big\downarrow \text{construct} & \Updownarrow \text{prove} & \Big\downarrow \text{construct} \\ \mathcal{L}(F) & \ni & w \in \Sigma^* \text{ word} \end{array}$$

regular language

finite words

Sketch: Proof of Theorem 3.6

- give a procedure to construct, given a formula F, a **regular** language $\mathcal{L}(F)$ such that

  $$\mathcal{I}, [0, n] \models F \text{ if and only if } w \in \mathcal{L}(F)$$

  where word w describes $\mathcal{I}$ on $[0, n]$
  (suitability of the procedure: **Lemma 3.4**)

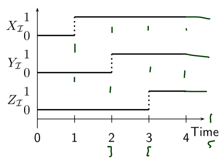
- then F is satisfiable in discrete time if and only if $\mathcal{L}(F)$ is not empty (**Lemma 3.5**)

- Theorem 3.6 follows because
  - $\mathcal{L}(F)$ can **effectively** be constructed,
  - the emptyness problem is **decidable** for regular languages.

Construction of $\mathcal{L}(F)$

- **Idea:**
  - alphabet $\Sigma(F)$ consists of basic conjuncts of the state variables in F,
  - a letter corresponds to an interpretation on an interval of length 1,
  - a word of length n describes an interpretation on interval $[0, n]$.
- **Example:** Assume F contains exactly state variables X, Y, Z, then

$$\Sigma(F) = \{X \wedge Y \wedge Z, X \wedge Y \wedge \neg Z, X \wedge \neg Y \wedge Z, X \wedge \neg Y \wedge \neg Z, \neg X \wedge Y \wedge Z, \neg X \wedge Y \wedge \neg Z, \neg X \wedge \neg Y \wedge Z, \neg X \wedge \neg Y \wedge \neg Z\}.$$



$w = (\neg X \wedge \neg Y \wedge \neg Z)$
$\cdot (X \wedge \neg Y \wedge \neg Z)$
$* \cdot (X \wedge Y \wedge \neg Z)$
$\cdot (X \wedge Y \wedge Z) \in \Sigma(F)^*$
$. (X \wedge Y \wedge Z)$

Construction of $\mathcal{L}(F)$ more Formally

$X \wedge \neg Y$
$\Leftrightarrow (X \wedge \neg Y \wedge Z) \vee (X \wedge \neg Y \wedge \neg Z)$

Definition 3.2. A word $w = a_1 \dots a_n \in \Sigma(F)^*$ with $n \geq 0$ **describes** a **discrete** interpretation $\mathcal{I}$ on $[0, n]$ if and only if

$$\forall\, j \in \{1, \dots, n\}\ \forall\, t \in\,]j-1, j[\, : \mathcal{I}[\![a_j]\!](t) = 1.$$

For $n = 0$ we put $w = \varepsilon$.

$DNF(X \wedge \neg Y) = \{(X \wedge \neg Y \wedge Z), (X \wedge \neg Y \wedge \neg Z)\}$

- Each state assertion P can be transformed into an equivalent **disjunctive normal form** $\bigvee_{i=1}^m a_i$ with $a_i \in \Sigma(F)$.
- Set $DNF(P) := \{a_1, \dots, a_m\}$ $(\subseteq \Sigma(F))$.
- Define $\mathcal{L}(F)$ inductively:

finite words, length at least one

$$\begin{aligned}
\mathcal{L}(\lceil P \rceil) &= DNF(P)^+, & &\text{(regular language)}\\
\mathcal{L}(\neg F_1) &= \Sigma(F)^* \setminus \mathcal{L}(F_1), & &\text{(again regular)}\\
\mathcal{L}(F_1 \vee F_2) &= \mathcal{L}(F_1) \cup \mathcal{L}(F_2), & &(-\ \cdot\ -)\\
\mathcal{L}(F_1\, ;\, F_2) &= \mathcal{L}(F_1) \cdot \mathcal{L}(F_2). & &(-\ \cdot\ -)
\end{aligned}$$

concatenate

Lemma 3.4

Lemma 3.4. For all RDC formulae F, discrete interpretations $\mathcal{I}$, $n \geq 0$, and all words $w \in \Sigma(F)^*$ which **describe** $\mathcal{I}$ on $[0, n]$,

$$\mathcal{I}, [0, n] \models F \text{ if and only if } w \in \mathcal{L}(F).$$

Proof: Structural induction

Base $F = \lceil P \rceil$: assume $w = a_1 \dots a_n$ describes $\mathcal{I}$ on $[0,n]$

$\mathcal{I}, [0,n] \models \lceil P \rceil \Leftrightarrow \mathcal{I}, [0,n] \models \lceil P \rceil$ and $n \geq 1$
$\Leftrightarrow n \geq 1$ and $\forall\, 1 \leq j \leq n \bullet \mathcal{I}, [j-1, j] \models \lceil P \rceil$
$\Leftrightarrow n \geq 1$ and $\forall\, 1 \leq j \leq n \bullet \mathcal{I}, [j-1, j] \models (\lceil P \rceil \wedge \lceil a_j \rceil)$ and $a_j \in DNF(P)$
"describes" $\Leftrightarrow n \geq 1$ and $\forall\, 1 \leq j \leq n \bullet a_j \in DNF(P)$ clear
$\Leftrightarrow w \in DNF(P)^+$
$\Leftrightarrow w \in \mathcal{L}(\lceil P \rceil)$

Steps:
- $\neg F_1$
- $F_1 \vee F_2$
- $F_1 ; F_2$

Sketch: Proof of Theorem 3.9

Theorem 3.9.
The realisability problem for RDC with discrete time is decidable.

- $kern(L)$ contains all words of L whose prefixes are again in L.
- If L is regular, then $kern(L)$ is also regular.
- $kern(\mathcal{L}(F))$ can effectively be constructed.
- We have

  Lemma 3.8. For all RDC formulae F, F is realisable from 0 in discrete time if and only if $kern(\mathcal{L}(F))$ is infinite.

- Infinity of regular languages is decidable.

(Variants of) RDC in Continuous Time

Recall: Restricted DC (RDC)

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1\, ;\, F_2$$

where P is a state assertion, but with **boolean** observables **only**.

From now on: "RDC + $\ell = x, \forall x$"

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1\, ;\, F_2 \mid \ell = 1 \mid \ell = x \mid \forall x \bullet F_1$$

Undecidability of Satisfiability/Realisability from 0

Theorem 3.10.
The realisability from 0 problem for DC with **continuous time** is undecidable, not even semi-decidable.

Theorem 3.11.
The satisfiability problem for DC with continuous time is undecidable.

Sketch: Proof of Theorem 3.10

Reduce divergence of **two-counter machines** to realisability from 0:

- Given a two-counter machine $\mathcal{M}$ with final state q_{fin},
- construct a DC formula $F(\mathcal{M}) := encoding(\mathcal{M})$
- such that

  $\mathcal{M}$ **diverges** **if and only if** the DC formula

  $$F(\mathcal{M}) \wedge \neg\Diamond\lceil q_{fin}\rceil$$

  is **realisable from 0**.
- If realisability from 0 was (semi-)decidable, divergence of two-counter machines would be (which it isn't).

Recall: Two-counter machines

A **two-counter** machine is a structure

$$\mathcal{M} = (\mathcal{Q}, q_0, q_{fin}, Prog)$$

where

- $\mathcal{Q}$ is a finite set of **states**,
- comprising the **initial state** q_0 and the **final state** q_{fin}
- $Prog$ is the **machine program**, i.e. a finite set of **commands** of the form

  start state of command

  $$q : inc_i : q' \quad \text{and} \quad q : dec_i : q', q'', \qquad i \in \{1, 2\}.$$

  $q : inc_1 : q'$
  $q : inc_2 : q'$

  $q, q', q'' \in \mathcal{Q}$

- We assume **deterministic** 2CM: for each $q \in \mathcal{Q}$, at most one command starts in q, and q_{fin} is the only state where no command starts.

2CM Configurations and Computations

current state
values of counters ?2

- a **configuration** of $\mathcal{M}$ is a triple $K = (q, n_1, n_2) \in \mathcal{Q} \times \mathbb{N}_0 \times \mathbb{N}_0$.
- The **transition relation** "$\vdash$" on configurations is defined as follows:

Command	Semantics: $K \vdash K'$
$q : inc_1 : q'$	$(q, n_1, n_2) \vdash (q', n_1 + 1, n_2)$
$q : dec_1 : q', q''$	$(q, 0, n_2) \vdash (q', 0, n_2)$
	$(q, n_1 + 1, n_2) \vdash (q'', n_1, n_2)$
$q : inc_2 : q'$	$(q, n_1, n_2) \vdash (q', n_1, n_2 + 1)$
$q : dec_2 : q', q''$	$(q, n_1, 0) \vdash (q', n_1, 0)$
	$(q, n_1, n_2 + 1) \vdash (q'', n_1, n_2)$

- The (!) **computation** of $\mathcal{M}$ is a finite sequence of the form ("$\mathcal{M}$ **halts**")

  $$K_0 = (q_0, 0, 0) \vdash K_1 \vdash K_2 \vdash \dots \vdash (q_{fin}, n_1, n_2)$$

  or an infinite sequence of the form ("$\mathcal{M}$ **diverges**")

  $$K_0 = (q_0, 0, 0) \vdash K_1 \vdash K_2 \vdash \dots$$

2CM Example

- $\mathcal{M} = (\mathcal{Q}, q_0, q_{fin}, Prog)$
- commands of the form $q : inc_i : q'$ and $q : dec_i : q', q''$, $i \in \{1, 2\}$
- configuration $K = (q, n_1, n_2) \in \mathcal{Q} \times \mathbb{N}_0 \times \mathbb{N}_0$.
-

Command	Semantics: $K \vdash K'$
$q : inc_1 : q'$	$(q, n_1, n_2) \vdash (q', n_1 + 1, n_2)$
$q : dec_1 : q', q''$	$(q, 0, n_2) \vdash (q', 0, n_2)$
	$(q, n_1 + 1, n_2) \vdash (q'', n_1, n_2)$
$q : inc_2 : q'$	$(q, n_1, n_2) \vdash (q', n_1, n_2 + 1)$
$q : dec_2 : q', q''$	$(q, n_1, 0) \vdash (q', n_1, 0)$
	$(q, n_1, n_2 + 1) \vdash (q'', n_1, n_2)$

$Q = \{q_0, q_1, q_{fin}\}$

$Prog = \{q_0 : inc_1 : q_1, \; q_1 : inc_1 : q_{fin}\}$

$(q_0, 0, 0) \vdash (q_1, 1, 0) \vdash (q_{fin}, 2, 0)$ ⊳ machine halts

$Q = \{q_0, q_{fin}\}$

$Prog = \{q_0 : inc_2 : q_0\}$

$(q_0, 0, 0) \vdash (q_0, 1, 0) \vdash (q_0, 2, 0) \vdash \dots$ ⊳ machine diverges

Reducing Divergence to DC realisability: Idea In Pictures

2CM $\mathcal{M}$ diverges

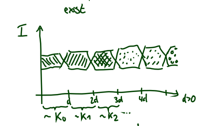
iff

exists $\pi : K_0 \vdash K_1 \vdash K_2 \dots$

q_{fin} does not occur

iff

exist



("$\mathcal{I}$ describes π")

and

$\mathcal{I} \models_0 F(\mathcal{M}) \wedge \neg\Diamond\lceil q_{fin}\rceil$

$F(\mathcal{M})$ intuitively requires:

- $[0, d]$ encodes $(q_0, 0, 0)$
- $[n \cdot d, (n+1) \cdot d]$ encodes a configuration
- $[n \cdot d, (n+1) \cdot d]$ and $[(n+1) \cdot d, (n+2) \cdot d]$ encode configurations which are in $\vdash$-relation
- if q_{fin} is reached, we stay there

Reducing Divergence to DC realisability: Idea

- A single configuration K of $\mathcal{M}$ can be encoded in an interval of length 4; being an encoding interval can be **characterised** by a DC formula.
- An interpretation on 'Time' encodes **the** computation of $\mathcal{M}$ if
  - each interval $[4n, 4(n+1)]$, $n \in \mathbb{N}_0$, **encodes** a configuration K_n,
  - each two subsequent intervals $[4n, 4(n+1)]$ and $[4(n+1), 4(n+2)]$, $n \in \mathbb{N}_0$, encode configurations $K_n \vdash K_{n+1}$ **in transition relation**.
- Being encoding of the run can be **characterised** by DC formula $F(\mathcal{M})$.
- Then $\mathcal{M}$ **diverges** if and only if $F(\mathcal{M}) \wedge \neg\Diamond\lceil q_{fin}\rceil$ is realisable from 0.

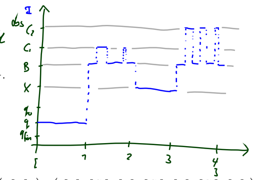
Encoding Configurations

- We use Obs $= \{\text{obs}\}$ with $\mathcal{D}(\text{obs}) = \mathcal{Q}_\mathcal{M} \,\dot\cup\, \{C_1, C_2, B, X\}$.

set of states of $\mathcal{M}$

disjoint union

abbrev. for $\lceil \text{obs} = q \rceil$



Examples:

- $K = (q, 2, 3)$

$$\begin{pmatrix} \lceil q \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B \rceil ; \lceil C_1 \rceil ; \lceil B \rceil ; \lceil C_1 \rceil ; \lceil B \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil X \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B \rceil ; \lceil C_2 \rceil ; \lceil B \rceil ; \lceil C_2 \rceil ; \lceil B \rceil ; \lceil C_2 \rceil ; \lceil B \rceil \\ \wedge \\ \ell = 1 \end{pmatrix}$$

- $K_0 = (q_0, 0, 0)$

$$\begin{pmatrix} \lceil q_0 \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil X \rceil \\ \wedge \\ \ell = 1 \end{pmatrix} ; \begin{pmatrix} \lceil B \rceil \\ \wedge \\ \ell = 1 \end{pmatrix}$$

or, using abbreviations, $\lceil q_0 \rceil^1 ; \lceil B \rceil^1 ; \lceil X \rceil^1 ; \lceil B \rceil^1$.

Construction of $F(\mathcal{M})$

In the following, we give DC formulae describing

- the initial configuration,
- the general form of configurations,
- the transitions between configurations,
- the handling of the final state.

$F(\mathcal{M})$ is the conjunction of all these formulae.

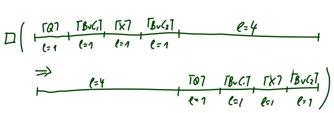
$$F(\mathcal{M}) = init \wedge keep \wedge \dots \wedge \bigwedge_{q : inc_i : q' \in Prog_\mathcal{M}} F(q : inc_i : q') \wedge \bigwedge_{q : dec_i : q', q'' \in Prog_\mathcal{M}} F(q : dec_i : q' ; q'')$$

Initial and General Configurations

$$init :\iff (\ell \geq 4 \implies \lceil q_0 \rceil^1 ; \lceil B \rceil^1 ; \lceil X \rceil^1 ; \lceil B \rceil^1 ; true)$$

$$keep :\iff \Box(\lceil Q \rceil^1 ; \lceil B \vee C_1 \rceil^1 ; \lceil X \rceil^1 ; \lceil B \vee C_2 \rceil^1 ; \ell = 4 \implies \ell = 4 ; \lceil Q \rceil^1 ; \lceil B \vee C_1 \rceil^1 ; \lceil X \rceil^1 ; \lceil B \vee C_2 \rceil^1)$$

where $Q := \neg(X \vee C_1 \vee C_2 \vee B)$.



Auxiliary Formula Pattern copy

formula

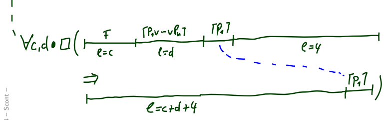
state assertions

$copy(F, \{P_1, \dots, P_n\}) :\iff$

$\quad \forall c, d \bullet \Box((F \wedge \ell = c) ; (\lceil P_1 \vee \dots \vee P_n \rceil \wedge \ell = d) ; \lceil P_1 \rceil ; \ell = 4$
$\qquad \implies \ell = c + d + 4 ; \lceil P_1 \rceil$

$\quad \wedge \dots$

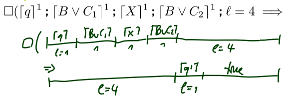
$\quad \wedge \forall c, d \bullet \Box((F \wedge \ell = c) ; (\lceil P_1 \vee \dots \vee P_n \rceil \wedge \ell = d) ; \lceil P_n \rceil ; \ell = 4$
$\qquad \implies \ell = c + d + 4 ; \lceil P_n \rceil$



$q : inc_1 : q'$ (Increment) $\in Prog_\mathcal{M}$

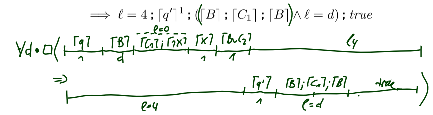
(i) Change state

$$\Box(\lceil q \rceil^1 ; \lceil B \vee C_1 \rceil^1 ; \lceil X \rceil^1 ; \lceil B \vee C_2 \rceil^1 ; \ell = 4 \implies \ell = 4 ; \lceil q' \rceil^1 ; true)$$



(ii) Increment counter

$$\forall d \bullet \Box(\lceil q \rceil^1 ; \lceil B \rceil^d ; (\ell = 0 \vee \lceil C_1 \rceil ; \lceil \neg X \rceil) ; \lceil X \rceil^1 ; \lceil B \vee C_2 \rceil^1 ; \ell = 4$$
$$\implies \ell = 4 ; \lceil q' \rceil^1 ; (\lceil B \rceil ; \lceil C_1 \rceil ; \lceil B \rceil \wedge \ell = d) ; true$$



$q : inc_1 : q'$ (Increment)

(i) Keep rest of first counter

$$copy(\overbrace{\lceil q \rceil^1 ; \lceil B \vee C_1 \rceil ; \lceil C_1 \rceil}^{F}, \overbrace{\{B, C_1\}}^{\{P_1, P_2\}})$$

(ii) Leave second counter unchanged

$$copy(\underbrace{\lceil q \rceil^1 ; \lceil B \vee C_1 \rceil ; \lceil X \rceil^1}_{F}, \underbrace{\{B, C_2\}}_{\{P_1, P_2\}})$$

$q : dec_1 : q', q''$ (Decrement)

(i) If zero

$$\Box(\lceil q \rceil^1 ; \lceil B \rceil^1 ; \lceil X \rceil^1 ; \lceil B \vee C_2 \rceil^1 ; \ell = 4 \implies \ell = 4 ; \lceil q' \rceil^1 ; \lceil B \rceil^1 ; true)$$

(ii) Decrement counter

$$\forall\, d \bullet \Box(\lceil q \rceil^1 ; (\lceil B \rceil ; \lceil C_1 \rceil \wedge \ell = d) ; \lceil B \rceil ; \lceil B \vee C_1 \rceil ; \lceil X \rceil^1 ; \lceil B \vee C_2 \rceil^1 ; \ell = 4$$
$$\implies \ell = 4 ; \lceil q'' \rceil^1 ; \lceil B \rceil^d ; true)$$

(iii) Keep rest of first counter

$$copy(\lceil q \rceil^1 ; \lceil B \rceil ; \lceil C_1 \rceil ; \lceil B_1 \rceil, \{B, C_1\})$$

(iv) Leave second counter unchanged

$$copy(\lceil q \rceil^1 ; \lceil B \vee C_1 \rceil ; \lceil X \rceil^1, \{B, C_2\})$$

Final State

$$copy(\lceil q_{fin} \rceil^1 ; \lceil B \vee C_1 \rceil^1 ; \lceil X \rceil ; \lceil B \vee C_2 \rceil^1, \{q_{fin}, B, X, C_1, C_2\})$$

Satisfiability

- Following [Chaochen and Hansen, 2004] we can observe that

  $\mathcal{M}$ **halts if and only if** the DC formula $F(\mathcal{M}) \wedge \Diamond\lceil q_{fin} \rceil$ is **satisfiable**.

  This yields

  > **Theorem 3.11.** The satisfiability problem for DC with continuous time is undecidable.

  (It is semi-decidable.)

- Furthermore, by taking the contraposition, we see

  $\mathcal{M}$ **diverges** **if and only if** $\mathcal{M}$ does not **halt**
  if and only if $F(\mathcal{M}) \wedge \neg\Diamond\lceil q_{fin} \rceil$ is **not** satisfiable.

- Thus whether a DC formula is **not satisfiable** is not decidable, not even semi-decidable.

Validity

- By Remark 2.13, F is valid iff $\neg F$ is not satisfiable, so

  > **Corollary 3.12.** The validity problem for DC with continuous time is undecidable, not even semi-decidable.

- This provides us with an alternative proof of Theorem 2.23 ("there is no sound and complete proof system for DC"):
  - **Suppose** there were such a calculus $\mathcal{C}$.
  - By Lemma 2.22 it is semi-decidable whether a given DC formula F is a theorem in $\mathcal{C}$.
  - By the soundness and completeness of $\mathcal{C}$, F is a theorem in $\mathcal{C}$ **if and only if** F is valid.
  - Thus it is semi-decidable whether F is valid. **Contradiction.**

Discussion

- Note: the DC fragment defined by the following grammar is **sufficient** for the reduction

  $$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 ; F_2 \mid \ell = 1 \mid \ell = x \mid \forall\, x \bullet F_1,$$

  P a state assertion, x a global variable.

- Formulae used in the reduction are abbreviations:

  $$\ell = 4 \iff \ell = 1 ; \ell = 1 ; \ell = 1 ; \ell = 1$$
  $$\ell \geq 4 \iff \ell = 4 ; true$$
  $$\ell = x + y + 4 \iff \ell = x ; \ell = y ; \ell = 4$$

- Length 1 is not necessary — we can use $\ell = z$ instead, with fresh z.
- This is RDC augmented by "$\ell = x$" and "$\forall\, x$", which we denote by **RDC** $+ \ell = x, \forall\, x$.

References

References

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[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.