

Real-Time Systems
Lecture 15: The Universality Problem for TBA
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Contents & Goals

- Last Lecture:**
 - Timed Words and Languages [Alur and Dill, 1994]
- This Lecture:**
 - Educational Objectives:** Capabilities for following tasks/questions:
 - What's a TBA and what's the difference to (extended) TAZ?
 - What's undecidable for timed (Büchi) automata?
 - What's the idea of the proof?
- Content:**
 - Timed Büchi Automata and timed regular languages [Alur and Dill, 1994].
 - The Universality Problem is undecidable for TBA [Alur and Dill, 1994]
 - Why this is unfortunate.
 - Timed regular languages are not everything.

Timed Büchi Automata
Alur and Dill, 1994

Recall: Timed Languages

- Definition.** A time sequence $\tau = \tau_1, \tau_2, \dots$ is an infinite sequence of time values $\tau_i \in \mathbb{R}_+^+$, satisfying the following constraints:
- (i) **Monotonicity:** τ increases strictly monotonically, i.e. $\tau_i < \tau_{i+1}$ for all $i \geq 1$.
 - (ii) **Progress:** For every $t \in \mathbb{R}_+^+$, there is some $i \geq 1$ such that $\tau_i > t$.
- Definition.** A time word over an alphabet Σ is a pair (σ, τ) where
- $\sigma = \sigma_1, \sigma_2, \dots \in \Sigma^\omega$ is an infinite word over Σ , and
 - τ is a time sequence.
- Definition.** A timed language over an alphabet Σ is a set of timed words over Σ .

Recall:

Example: Timed Language

- Timed word over alphabet Σ , a pair (σ, τ) where
- $\sigma = \sigma_1, \sigma_2, \dots$ is an infinite word over Σ , and
- τ is a time sequence (strictly (i) monotonic, non-zero)

a could be "system hangs"
 b could be "system finishes (app)!"

$L_{TBA} = \{ (ab)^n \mid \exists t \forall j \geq 1: (\tau_j < \tau_{j+1} + 2) \}$

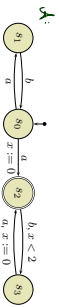
Handwritten notes:
 "finite prefix from about middle" (points to a^n)
 "infinite suffix from after the a before" (points to b^ω)
 "but most (not including) 2 may occur" (points to the 2 in the constraint)

Timed Büchi Automata

- Definition.** The set $\Phi(X)$ of clock constraints over X is defined inductively by
- $$\delta ::= x \leq c \mid c \leq x \mid \neg \delta \mid \delta_1 \wedge \delta_2$$
- where $x \in X$ and $c \in \mathbb{Q}$ is a rational constant.
- Definition.** A timed Büchi automaton (TBA) \mathcal{A} is a tuple $(\Sigma, S, S_0, X, E, F)$, where
- Σ is an alphabet,
 - S is a finite set of states, $S_0 \subseteq S$ is a set of start states,
 - X is a finite set of clocks, and
 - $E \subseteq S \times S \times \Sigma^+ \times 2^X \times \Phi(X)$ gives the set of transitions.
- An edge $(s, s', \alpha, \lambda, \delta)$ represents a transition from state s to state s' on input symbol α . The set $\lambda \subseteq X$ gives the clocks to be reset with this transition, and δ is a clock constraint over X .
- $F \subseteq S$ is a set of accepting states.

Example: Language of a TBA

$L(A) = \{(\sigma, \tau) \mid A \text{ has an accepting run over } (\sigma, \tau)\}$.



Claim:

$L(A) = L_{rec} = \{(\langle ab \rangle^x, \tau) \mid \exists! \forall j \geq 1: (\tau_j < \tau_{j-1} + 2)\}$

• Let $\sigma \in L(A)$: Pick some $(\sigma, \tau) \in L(A)$. Construct an accepting run of A .
 • Let $\sigma \notin L(A)$: Pick some $(\sigma, \tau) \notin L(A)$. Then there is an accepting run (σ, τ) over (σ, τ) .

Question: Is L_{rec} timed regular or not?

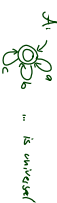
The Universality Problem is Undecidable for TBA

[Aur and Dil, 1994]

The Universality Problem

- Given: A TBA A over alphabet Σ .
 - Question: Does A accept all timed words over Σ^* ?
- In other words: Is $L(A) = \{(\sigma, \tau) \mid \sigma \in \Sigma^*, \tau \text{ time sequence}\}$.

$\Sigma = \{a, b, \epsilon\}$



The Universality Problem

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Theorem 5.2. The problem of deciding whether a timed automaton over alphabet Σ accepts all timed words over Σ^* is Π_1^1 -hard.

(The class Π_1^1 consists of highly undecidable problems, including some nonarithmetical sets for an exposition of the analytical hierarchy consult, for instance Rogers, 1987.)

Recall: With classical Buchi Automata (undtimed) this is different:

- Let B be a Buchi Automaton over Σ . $L(B) = \emptyset$ is **coNP-complete in Σ^*** .
- Let B be a Buchi Automaton over Σ . $L(B) = \emptyset$ is **coNP-complete in Σ^*** .
- B' such that $L(B') = \overline{L(B)}$ is effectively computable.
- Language emptiness is decidable for Buchi Automata.

Proof Idea



Theorem 5.2. The problem of deciding whether a timed automaton over alphabet Σ accepts all timed words over Σ^* is Π_1^1 -hard.

Proof Idea:

- Consider a language L_{rec} which consists of the recurring computations of a 2-counter machine M .
- Construct a TBA A from M which accepts the complement of L_{rec} , i.e. with $L(A) = \overline{L_{rec}}$.

- Then A is universal if and only if L_{rec} is empty...
- ... which is the case if and only if M doesn't have a recurring computation.

Once Again: Two Counter Machines (Different Flavour)

- A two-counter machine M
- has two counters C, D and
- a finite program consisting of n instructions.
- An instruction increments or decrements one of the counters, or jumps, here even non-deterministically.
- A configuration of M is a triple (i, c, d) : program counter $i \in \{1, \dots, n\}$, values $c, d \in \mathbb{N}_0$ of C and D .

A computation of M is an infinite consecutive sequence

$(1, 0, 0) = (c_0, c_0, d_0), (c_1, c_1, d_1), (c_2, c_2, d_2), \dots$

that is, $(i_{j+1}, c_{j+1}, d_{j+1})$ is a result executing instruction i_j at (i_j, c_j, d_j) .
 $\langle 1, 0, 0 \rangle \langle 2, 0, 0 \rangle \langle 3, 1, 0 \rangle \dots$

A computation of M is called **recurring** iff $i_j = 1$ for infinitely many $j \in \mathbb{N}_0$.

Step 1: The Language of Recurring Computations

- Let M be a ZCM with y instructions.

Wanted: A timed language L_{timed} (over some alphabet) representing exactly the recurring computations of M .
(In particular $\text{str } L_{\text{timed}} = \emptyset$ if and only if M has no recurring computation.)

- Choose $\Sigma = \{b_1, \dots, b_n, a_1, a_2\}$ as alphabet.
- We represent a configuration $\beta \in \mathcal{C}(\mathcal{Q})$ of M by the sequence

$$b_1 a_1 \dots a_n b_1 a_1 \dots a_n = b_1 a_1^n a_2^n$$

c times d times

Step 1: The Language of Recurring Computations

Let L_{timed} be the set of the timed words (σ, τ) with

- σ is of the form $b_1 a_1^n a_2^n b_1 a_1^n a_2^n \dots$
 - $(t_1, c_1, d_1), (t_2, c_2, d_2), \dots$ is a recurring computation of M .
 - For all $j \in \mathbb{N}_0$,
 - the time of b_j is j .
 - if $c_{j+1} = c_j$,
 - for every a_1 at time t in the interval $[j, j+1]$ there is an a_1 at time $t+1$.
 - if $c_{j+1} = c_j + 1$,
 - for every a_1 at time t in the interval $[j+1, j+2]$, except for the last one, there is an a_1 at time $t-1$.
 - if $c_{j+1} = c_j - 1$,
 - for every a_1 at time t in the interval $[j, j+1]$, except for the last one, there is an a_1 at time $t+1$.
- And analogously for the a_2 's

Step 2: Construct "Observer" for L_{timed}

Wanted: A TBA \mathcal{A} such that

$$L(\mathcal{A}) = L_{\text{timed}}$$

i.e., \mathcal{A} accepts a timed word (σ, τ) if and only if $(\sigma, \tau) \notin L_{\text{timed}}$.

Approach: What are the reasons for a timed word **not to be** in L_{timed} ?

- Recall:** (σ, τ) is in L_{timed} if and only if:
- $\sigma = b_1 a_1^n a_2^n b_1 a_1^n a_2^n \dots$
 - $(t_1, c_1, d_1), (t_2, c_2, d_2), \dots$ is a recurring computation of M .
 - the time of b_j is j .
 - if $c_{j+1} = c_j (= c_j + 1, = c_j - 1), \dots$

Step 2: Construct "Observer" for L_{timed}

Wanted: A TBA \mathcal{A} such that

$$L(\mathcal{A}) = L_{\text{timed}}$$

i.e., \mathcal{A} accepts a timed word (σ, τ) if and only if $(\sigma, \tau) \notin L_{\text{timed}}$.

Approach: What are the reasons for a timed word **not to be** in L_{timed} ?

- (i) The b_j at time $j \in \mathbb{N}$ is missing, or there is a spurious b_k at time $k \in [j, j+1]$.
- (ii) The prefix of the timed word with times $0 \leq t < 1$ doesn't encode $(1, 0, 0)$.
- (iii) The timed word is not recurring, i.e. it has only finitely many b_k .
- (iv) The configuration encoded in $[j+1, j+2]$ doesn't faithfully represent the effect of instruction b_j on the configuration encoded in $[j, j+1]$.

Plan: Construct a TBA \mathcal{A}_0 for case (i), a TBA \mathcal{A}_{rec} for case (ii), a TBA $\mathcal{A}_{\text{recur}}$ for case (iii), and one TBA \mathcal{A}_1 for each instruction for case (iv).

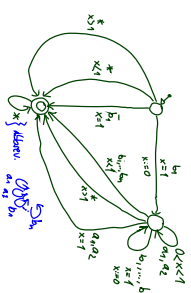
Then set

$$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_{\text{rec}} \cup \mathcal{A}_{\text{recur}} \cup \bigcup_{1 \leq k \leq n} \mathcal{A}_k$$

Step 2.(i): Construct \mathcal{A}_0

(i) The b_k at time $j \in \mathbb{N}$ is missing, or there is a spurious b_k at time $t \in [j, j+1]$.

[Alur and Dill, 1994]: "It is easy to construct such a timed automaton."

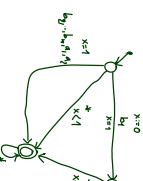


Step 2.(ii): Construct \mathcal{A}_{rec}

(ii) The prefix of the timed word with times $0 \leq t < 1$ doesn't encode $(1, 0, 0)$.

- It accepts

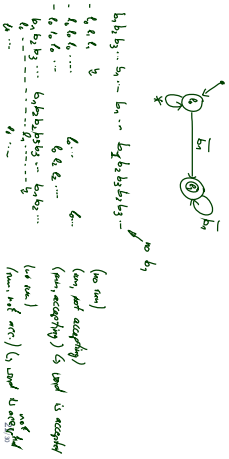
$$\{(\sigma, \tau) \in \mathbb{N} \mid (\sigma_0 \neq b_1) \vee (\tau_0 \neq 0) \vee (\tau_1 \neq 1)\}$$



Step 2.(iii): Construct A_{never}

(iii) The timed word is not recurring, i.e. it has only finitely many b_i .

- A_{never} accepts words with only finitely many b_i .



Consequences: Language Inclusion

- Given:** Two TBAs A_1 and A_2 over alphabet B .
- Question:** Is $L(A_1) \subseteq L(A_2)$?

Possible applications of a decision procedure:

- Characterise the allowed behaviour as A_2 and model the design as A_1 .
- Automatically check whether the behaviour of the design is a subset of the allowed behaviour.

- If **Language Inclusion** was decidable, then we could use it to decide universality of A by checking

$$L(A_{\text{never}}) \subseteq L(A)$$

where A_{never} is any universal TBA (which is easy to construct)

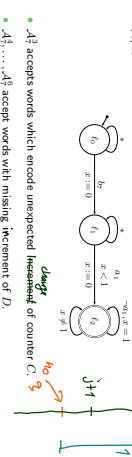
Step 2.(iv): Construct A_1

(iv) The configuration encoded in $[j+1, j+2]$ doesn't faithfully represent the effect of instruction b_j on the configuration encoded in $[j, j+1]$.

Example: assume instruction Z is:

Increment counter D and jump non-deterministically to instruction 3 or 5. Once again, stepsize: $A_1^3 \cup \dots \cup A_1^5$

- A_1^3 accepts words with b_j at time j but neither b_0 nor b_2 at time $j+1$. "Easy to construct."
- A_1^5 is



- A_1^3 accepts words which encode unexpected increment of counter C .
- A_1^5, \dots, A_1^6 accept words with missing increment of D .

Consequences: Complementation

- Given:** A timed regular language W over B (that is, there is a TBA A such that $L(A) = W$).
- Question:** Is \overline{W} timed regular?

Possible applications of a decision procedure:

- Characterise the allowed behaviour as A_2 and model the design as A_1 .
- Automatically construct A_2 with $L(A_2) = \overline{L(A_1)}$ and check

$$L(A_1) \cap L(A_2) = \emptyset$$

that is, whether the design has any non-allowed behaviour.

- Taking for granted that:
 - The intersection automaton is effectively computable.
 - The emptiness problem for Buchi automata is decidable.
- (Proof by construction of region automaton [Alur and Dill, 1994])

Alur, Arul...?

Consequences: Complementation

- Given:** A timed regular language W over B (that is, there is a TBA A such that $L(A) = W$).
- Question:** Is \overline{W} timed regular?

- If the class of timed regular languages were closed under **complementation**, "the complement of the inclusion problem is recursively enumerable. This contradicts the Π_1^1 -hardness of the inclusion problem." [Alur and Dill, 1994]

A non-complementable TBA A



$$L(A) = \{(a^i, (t_i)_{i \in \mathbb{N}}) \mid \exists t \in \mathbb{N}_0 \exists j > i : (t_j = t_i + 1)\}$$

Complement language:

$$\overline{L(A)} = \{(a^i, (t_i)_{i \in \mathbb{N}}) \mid \text{no two } a \text{ are separated by distance } 1\}$$

Beyond Timed Regular

$$x + y \leq x' + y'$$

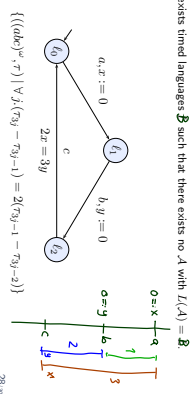
With clock constraints of the form

we can describe timed languages which are not timed regular.

In other words: $\text{LTL} \not\subseteq \text{TL}$

- There are strictly timed languages than timed regular languages
- There exists timed languages \mathcal{B} such that there exists no \mathcal{A} with $L(\mathcal{A}) = \mathcal{B}$

Example:



Beyond Timed Regular

References

[Alur and Dill, 1994] Alur, R. and Dill, D. L. (1994). A theory of timed automata. *Theoretical Computer Science*, 120(2):183-235.

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.

References