

Real-Time Systems

Lecture 15: The Universality Problem for TBA

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Contents & Goals

Last Lecture:

- Timed Words and Languages [Alur and Dill, 1994]

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
 - What's a TBA and what's the difference to (extended) TA?
 - What's undecidable for timed (Büchi) automata?
 - What's the idea of the proof?
- **Content:**
 - Timed Büchi Automata and timed regular languages [Alur and Dill, 1994].
 - The Universality Problem is undecidable for TBA [Alur and Dill, 1994]
 - Why this is unfortunate.
 - Timed regular languages are not everything.

Timed Büchi Automata

[Alur and Dill, 1994]

Recall: Timed Languages

Definition. A **time sequence** $\tau = \tau_1, \tau_2, \dots$ is an infinite sequence of time values $\tau_i \in \mathbb{R}_0^+$, satisfying the following constraints:

(i) **Monotonicity:**

τ increases **strictly** monotonically, i.e. $\tau_i < \tau_{i+1}$ for all $i \geq 1$.

(ii) **Progress:** For every $t \in \mathbb{R}_0^+$, there is some $i \geq 1$ such that $\tau_i > t$.

Definition. A **timed word** over an alphabet Σ is a pair (σ, τ) where

- $\sigma = \sigma_1, \sigma_2, \dots \in \Sigma^\omega$ is an infinite word over Σ , and
- τ is a time sequence.

Definition. A **timed language** over an alphabet Σ is a set of timed words over Σ .

Recall:

Example: Timed Language

Timed word over alphabet Σ : a pair (σ, τ) where

- $\sigma = \sigma_1, \sigma_2, \dots$ is an infinite word over Σ , and
- τ is a time sequence (strictly (!) monotonic, non-Zeno).

a could be 'system beeps'
b could be 'system flashes light'

$$L_{crt} = \{((ab)^\omega, \tau) \mid \exists i \forall j \geq i : (\tau_{2j} < \tau_{2j-1} + 2)\}$$

↑
timestamp
of a 'b'
↑
timestamp
of the 'a' before

10.0
a b a b a b ... a b a b a b ...
 $\tau_1 \tau_2 \tau_3 \tau_4 \tau_5 \tau_6 \dots \tau_{27} \tau_{28} \tau_{29} \tau_{30} \tau_{31} \tau_{32}$

finite prefix
where timestamps
don't matter

↑
from
here
on...

$$\tau_{30} < \tau_{29} + 2$$

... but most (not including) 2 time units
after the a before

Definition. The set $\Phi(X)$ of **clock constraints** over X is defined inductively by

$$\delta ::= x \leq c \mid c \leq x \mid \neg\delta \mid \delta_1 \wedge \delta_2$$

where $x \in X$ and $c \in \mathbb{Q}$ is a rational constant.

Definition. A **timed Büchi automaton** (TBA) \mathcal{A} is a tuple $(\Sigma, S, S_0, X, E, F)$, where

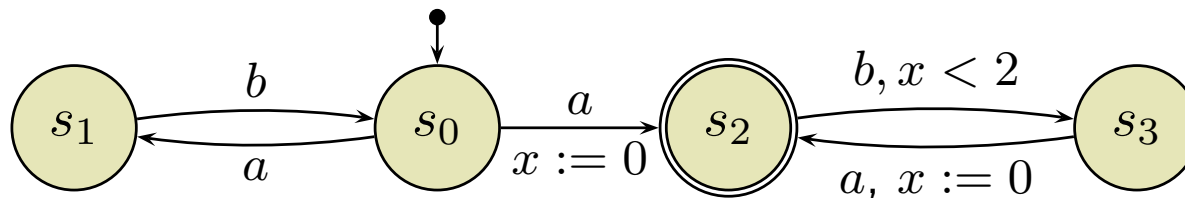
- Σ is an alphabet,
- S is a finite set of states, $S_0 \subseteq S$ is a set of start states,
- X is a finite set of clocks, and
- $E \subseteq S \times S \times \Sigma \times 2^X \times \Phi(X)$ gives the set of transitions.

An edge $(s, s', a, \lambda, \delta)$ represents a transition from state s to state s' on input symbol a . The set $\lambda \subseteq X$ gives the clocks to be reset with this transition, and δ is a clock constraint over X .

- $F \subseteq S$ is a set of **accepting states**.

Example: TBA

$$\mathcal{A} = (\Sigma, S, S_0, X, E, F)$$
$$(s, s', a, \lambda, \delta) \in E$$



$$\Sigma = \{a, b\}$$

$$S = \{s_0, s_1, s_3\}$$

$$S_0 = \{s_0\}$$

$$X = \{x\}$$

$$E = \{(s_2, s_3, b, \emptyset, x < 2), \dots\}$$

$$F = \{s_2\}$$

(Accepting) TBA Runs

Definition. A **run** r , denoted by $(\bar{s}, \bar{\nu})$, of a TBA $(\Sigma, S, S_0, X, E, F)$ over a timed word (σ, τ) is an **infinite** sequence of the form

$$r : \langle s_0, \nu_0 \rangle \xrightarrow[\tau_1]{\sigma_1} \langle s_1, \nu_1 \rangle \xrightarrow[\tau_2]{\sigma_2} \langle s_2, \nu_2 \rangle \xrightarrow[\tau_3]{\sigma_3} \dots$$

with $s_i \in S$ and $\nu_i : X \rightarrow \mathbb{R}_0^+$, satisfying the following requirements:

- **Initiation:** $s_0 \in S_0$ and $\nu(x) = 0$ for all $x \in X$.
- **Consecution:** for all $i \geq 1$, there is an edge in E of the form $(s_{i-1}, s_i, \sigma_i, \lambda_i, \delta_i)$ such that
 - $(\nu_{i-1} + (\tau_i - \tau_{i-1}))$ satisfies δ_i and
 - $\nu_i = (\nu_{i-1} + (\tau_i - \tau_{i-1}))[\lambda_i := 0]$.

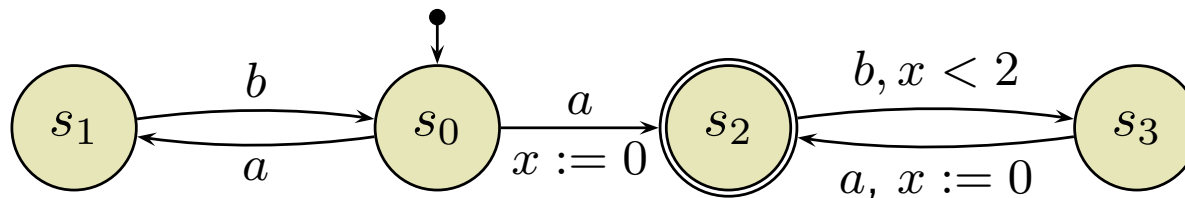
time shift (as before)

delay between τ_{i-1} and τ_i

Example: TBA

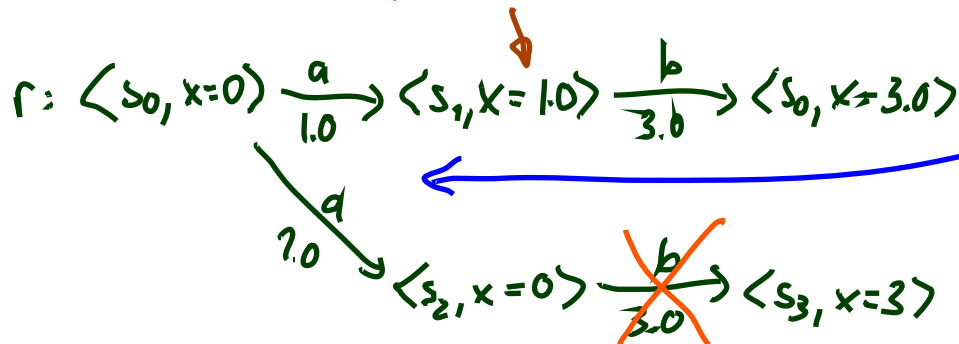
$$\mathcal{A} = (\Sigma, S, S_0, X, E, F)$$

$$(s, s', a, \lambda, \delta) \in E$$



$$(\sigma, \tau) = \begin{matrix} & a & b & a & b & a & \dots \\ \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = & 1.0 & 3.0 & 4.5 & 10.0 & 17.0 & \dots \end{matrix}$$

$$\{x > 0\} + (1.0 - 0)$$



TBA are non-det. in general

automaton "gets stuck" here — this is not a word

(Accepting) TBA Runs

Definition. A **run** r , denoted by $(\bar{s}, \bar{\nu})$, of a TBA $(\Sigma, S, S_0, X, E, F)$ over a timed word (σ, τ) is an **infinite** sequence of the form

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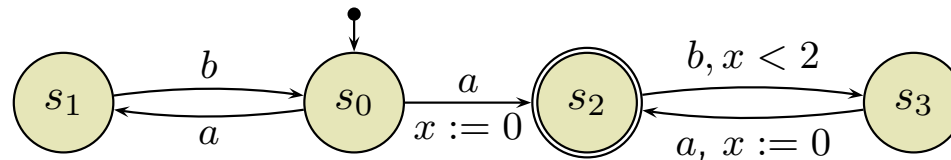
- **Initiation:** $s_0 \in S_0$ and $\nu(x) = 0$ for all $x \in X$.
- **Consecution:** for all $i \geq 1$, there is an edge in E of the form $(s_{i-1}, s_i, \sigma_i, \lambda_i, \delta_i)$ such that
 - $(\nu_{i-1} + (\tau_i - \tau_{i-1}))$ satisfies δ_i and
 - $\nu_i = (\nu_{i-1} + (\tau_i - \tau_{i-1}))[\lambda_i := 0]$.

The set $\text{inf}(r) \subseteq S$ consists of those states $s \in S$ such that $s = s_i$ for infinitely many $i \geq 0$.

Definition. A run $r = (\bar{s}, \bar{\nu})$ of a TBA over timed word (σ, τ) is called (an) **accepting** (run) if and only if $\text{inf}(r) \cap F \neq \emptyset$.

Example: (Accepting) Runs

$r : \langle s_0, \nu_0 \rangle \xrightarrow[\tau_1]{\sigma_1} \langle s_1, \nu_1 \rangle \xrightarrow[\tau_2]{\sigma_2} \langle s_2, \nu_2 \rangle \xrightarrow[\tau_3]{\sigma_3} \dots$ initial and $(s_{i-1}, s_i, \sigma_i, \lambda_i, \delta_i) \in E$, s.t.
 $(\nu_{i-1} + (\tau_i - \tau_{i-1})) \models \delta_i, \nu_i = (\nu_{i-1} + (\tau_i - \tau_{i-1}))[\lambda_i := 0]$. Accepting iff $\text{inf}(r) \cap F \neq \emptyset$.



Timed word: $(a, 1), (b, 2), (a, 3), (b, 4), (a, 5), (b, 6), \dots$

- Can we construct **any run**? Is it accepting?

$\langle s_0, 0 \rangle \xrightarrow{\frac{a}{1}} \langle s_2, 0 \rangle \xrightarrow{\frac{b}{2}} \langle s_3, 1 \rangle \xrightarrow{\frac{a}{3}} \langle s_2, 0 \rangle \xrightarrow{\frac{b}{4}} \langle s_3, 1 \rangle \dots$

$\text{inf}(r) = \{s_2, s_3\}$
 $\text{inf}(r) \cap F = \{s_2, s_3\} \cap \{s_2\} = \{s_2\} \neq \emptyset$

- Can we construct a **non-run** (get stuck)?

NO

- Can we construct a **(non-)accepting run**?

$r' : \langle s_0, 0 \rangle \xrightarrow{\frac{a}{1}} \langle s_1, 1 \rangle \xrightarrow{\frac{b}{2}} \langle s_0, 2 \rangle \xrightarrow{\frac{a}{3}} \langle s_1, 3 \rangle \dots$

$\text{inf}(r') = \{s_0, s_1\}$

The Language of a TBA

Definition. For a TBA \mathcal{A} , the **language** $L(\mathcal{A})$ of timed words it accepts is defined to be the set

$$\{(\sigma, \tau) \mid \mathcal{A} \text{ has an accepting run over } (\sigma, \tau)\}.$$

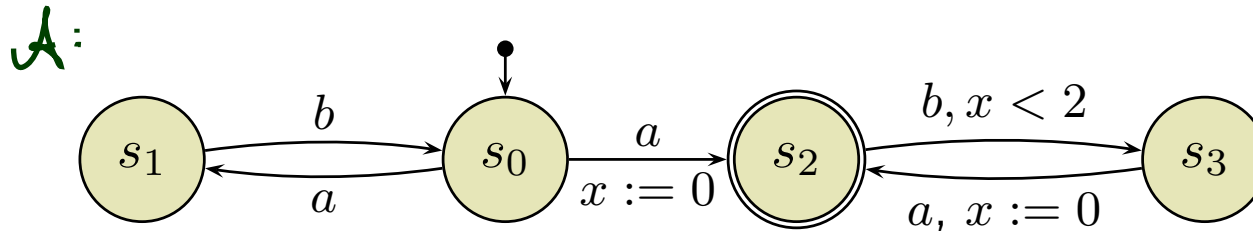
For short: $L(\mathcal{A})$ is the **language of** \mathcal{A} .



Definition. A timed language L is a **timed regular language** if and only if $L = L(\mathcal{A})$ for **some** TBA \mathcal{A} .

Example: Language of a TBA

$$L(\mathcal{A}) = \{(\sigma, \tau) \mid \mathcal{A} \text{ has an accepting run over } (\sigma, \tau)\}.$$



Claim:

$$L(\mathcal{A}) = L_{crt} (= \{((ab)^\omega, \tau) \mid \exists i \forall j \geq i : (\tau_{2j} < \tau_{2j-1} + 2)\})$$

- $L_{crt} \subseteq L(\mathcal{A})$: pick some $(\sigma, \tau) \in L_{crt}$. Construct an accepting run of \mathcal{A} .
- $L(\mathcal{A}) \subseteq L_{crt}$: pick some $(\sigma, \tau) \in L(\mathcal{A})$. Then there is an accepting run (s, ν) over (σ, τ) .

Question: Is L_{crt} timed regular or not?

The Universality Problem is Undecidable for TBA

[Alur and Dill, 1994]

The Universality Problem

- **Given:** A TBA \mathcal{A} over alphabet Σ .
- **Question:** Does \mathcal{A} accept all timed words over Σ ?

In other words: Is $L(\mathcal{A}) = \{(\sigma, \tau) \mid \sigma \in \Sigma^\omega, \tau \text{ time sequence}\}$.

$$\Sigma = \{a, b, c\}$$



The Universality Problem

- **Given:** A TBA \mathcal{A} over alphabet Σ .
- **Question:** Does \mathcal{A} accept all timed words over Σ ?

In other words: Is $L(\mathcal{A}) = \{(\sigma, \tau) \mid \sigma \in \Sigma^\omega, \tau \text{ time sequence}\}$.

Theorem 5.2. The problem of deciding whether a timed automaton over alphabet Σ accepts all timed words over Σ is Π_1^1 -hard.

(“The class Π_1^1 consists of highly undecidable problems, including some nonarithmetical sets (for an exposition of the analytical hierarchy consult, for instance [Rogers, 1967].)”)

Recall: With classical Büchi Automata (untimed), this is different:

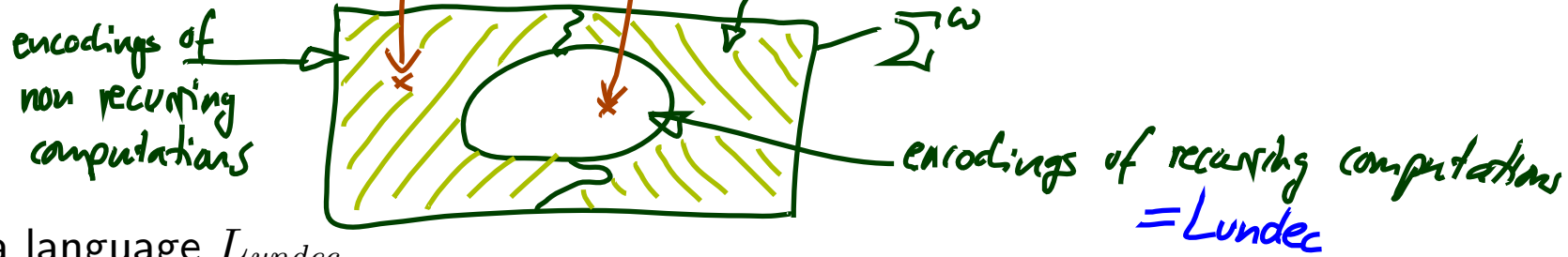
- Let \mathcal{B} be a Büchi Automaton over Σ . *complement in Σ^ω*
- \mathcal{B} is universal if and only if $\overline{L(\mathcal{B})} = \emptyset$.
- \mathcal{B}' such that $L(\mathcal{B}') = \overline{L(\mathcal{B})}$ is effectively computable.
- Language emptiness is decidable for Büchi Automata.

Proof Idea



Theorem 5.2. The problem of deciding whether a timed automaton over alphabet Σ accepts all timed words over Σ is Π_1^1 -hard.

Proof Idea:



- Consider a language L_{undec} which consists of the **recurring** computations of a **2-counter machine** M .
- Construct a TBA \mathcal{A} from M which accepts the complement of L_{undec} , i.e. with

$$L(\mathcal{A}) = \overline{L_{undec}}.$$

- Then \mathcal{A} is universal if and only if L_{undec} is empty...
 ... which is the case if and only if M **doesn't have** a recurring computation.

Once Again: Two Counter Machines (Different Flavour)

A **two-counter machine** M

- has two **counters** C, D and
- a finite **program** consisting of n instructions.
- An **instruction increments or decrements** one of the counters, or **jumps**, here even non-deterministically.

e.g. 1: inc D ; goto 2
2: inc C ; goto 1,3
3: dec D ; if ($D=0$) goto 1 else goto 4
4: inc D ; goto 1,2

- A **configuration** of M is a triple $\langle i, c, d \rangle$:

program counter $i \in \{1, \dots, n\}$, values $c, d \in \mathbb{N}_0$ of C and D .

- A **computation** of M is an infinite consecutive sequence

$$\langle 1, 0, 0 \rangle = \langle i_0, c_0, d_0 \rangle, \langle i_1, c_1, d_1 \rangle, \langle i_2, c_2, d_2 \rangle, \dots$$

that is, $\langle i_{j+1}, c_{j+1}, d_{j+1} \rangle$ is a result executing instruction i_j at $\langle i_j, c_j, d_j \rangle$.

$$\langle 1, 0, 0 \rangle, \langle 2, 0, 1 \rangle, \langle 3, 1, 1 \rangle, \dots$$

A computation of M is called **recurring** iff $i_j = 1$ for infinitely many $j \in \mathbb{N}_0$.

Step 1: The Language of Recurring Computations

- Let M be a 2CM with \underline{n} instructions.

Wanted: A timed language L_{undec} (over some alphabet) representing exactly the recurring computations of M .

(In particular s.t. $L_{undec} = \emptyset$ if and only if M has no recurring computation.)

- Choose $\Sigma = \{b_1, \dots, \underline{b_n}, a_1, a_2\}$ as alphabet.
- We represent a configuration $\langle i, c, d \rangle$ of M by the sequence

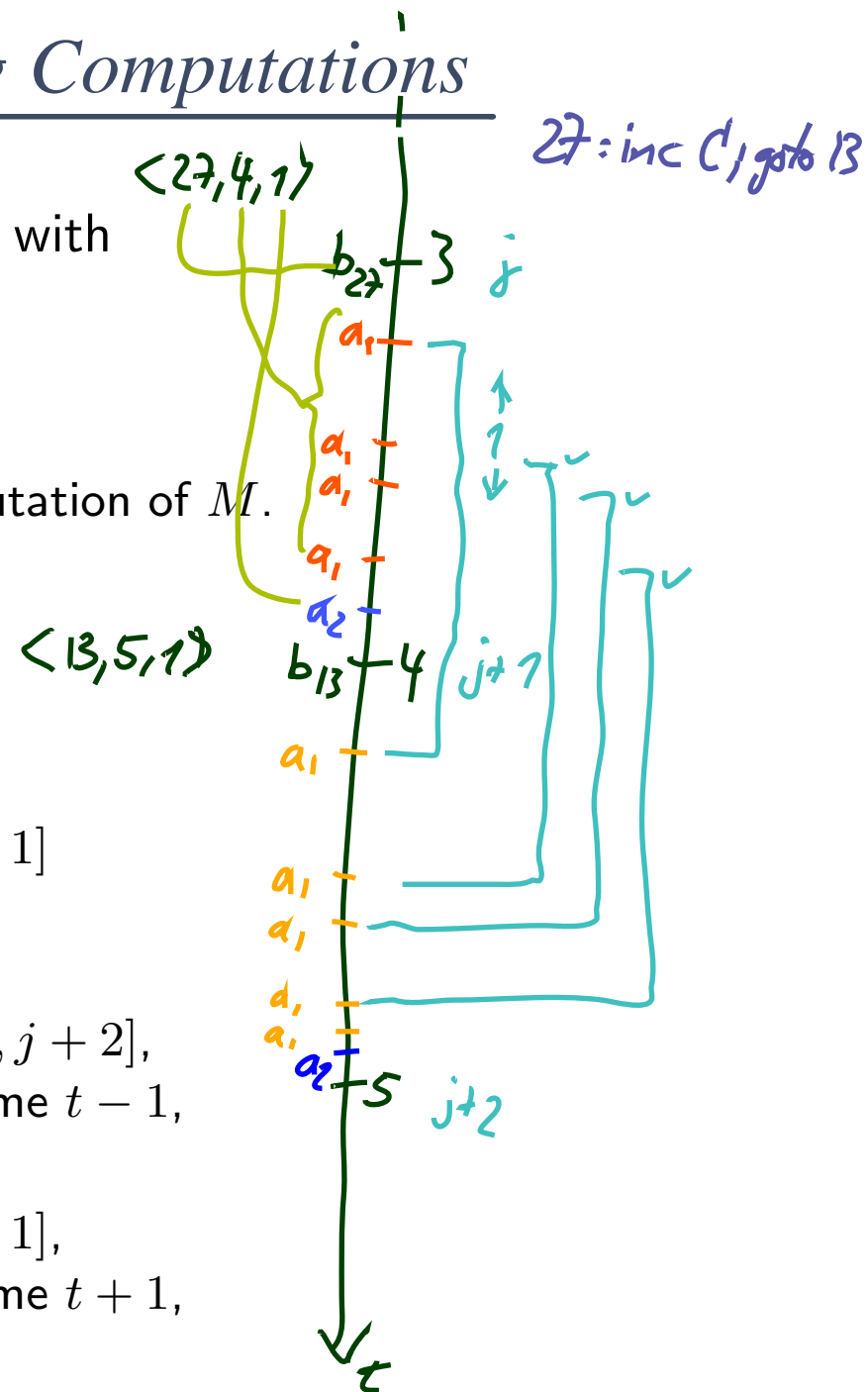
$$b_i \underbrace{a_1 \dots a_1}_{c \text{ times}} \underbrace{a_2 \dots a_2}_{d \text{ times}} = b_1 a_1^c a_2^d$$

Step 1: The Language of Recurring Computations

Let L_{undec} be the set of the timed words (σ, τ) with

- σ is of the form $b_{i_1} a_1^{c_1} a_2^{d_1} b_{i_2} a_1^{c_2} a_2^{d_2} \dots$
- $\langle i_1, c_1, d_1 \rangle, \langle i_2, c_2, d_2 \rangle, \dots$ is a recurring computation of M .
- For all $j \in \mathbb{N}_0$,
 - the time of b_{i_j} is j .
 - if $c_{j+1} = c_j$:
for every a_1 at time t in the interval $[j, j+1]$
there is an a_1 at time $t+1$,
 - if $c_{j+1} = c_j + 1$:
for every a_1 at time t in the interval $[j+1, j+2]$,
except for the last one, there is an a_1 at time $t-1$,
 - if $c_{j+1} = c_j - 1$:
for every a_1 at time t in the interval $[j, j+1]$,
except for the last one, there is an a_1 at time $t+1$,

And analogously for the a_2 's.



Step 2: Construct “Observer” for $\overline{L_{undec}}$

Wanted: A TBA \mathcal{A} such that

$$L(\mathcal{A}) = \overline{L_{undec}},$$

i.e., \mathcal{A} accepts a timed word (σ, τ) if and only if $(\sigma, \tau) \notin L_{undec}$.

Approach: What are the reasons for a timed word **not to be** in L_{undec} ?

Recall: (σ, τ) is **in** L_{undec} if and only if:

- $\sigma = b_{i_1} a_1^{c_1} a_2^{d_1} b_{i_2} a_1^{c_2} a_2^{d_2}$
- $\langle i_1, c_1, d_1 \rangle, \langle i_2, c_2, d_2 \rangle, \dots$
is a recurring computation of M .
- the time of b_{i_j} is j ,
- if $c_{j+1} = c_j$ ($= c_j + 1, = c_j - 1$): ...

Step 2: Construct “Observer” for $\overline{L_{undec}}$

Wanted: A TBA \mathcal{A} such that

$$L(\mathcal{A}) = \overline{L_{undec}},$$

i.e., \mathcal{A} accepts a timed word (σ, τ) if and only if $(\sigma, \tau) \notin L_{undec}$.

Approach: What are the reasons for a timed word **not to be** in L_{undec} ?

- (i) The b_i at time $j \in \mathbb{N}$ is missing, or there is a spurious b_i at time $t \in]j, j + 1[$.
- (ii) The prefix of the timed word with times $0 \leq t < 1$ doesn't encode $\langle 1, 0, 0 \rangle$.
- (iii) The timed word is not recurring, i.e. it has only finitely many b_i .
- (iv) The configuration encoded in $[j + 1, j + 2[$ doesn't faithfully represent the effect of instruction b_j on the configuration encoded in $[j, j + 1[$.

Plan: Construct a TBA \mathcal{A}_0 for case (i), a TBA \mathcal{A}_{init} for case (ii), a TBA \mathcal{A}_{recur} for case (iii), and one TBA \mathcal{A}_i for each instruction for case (iv).

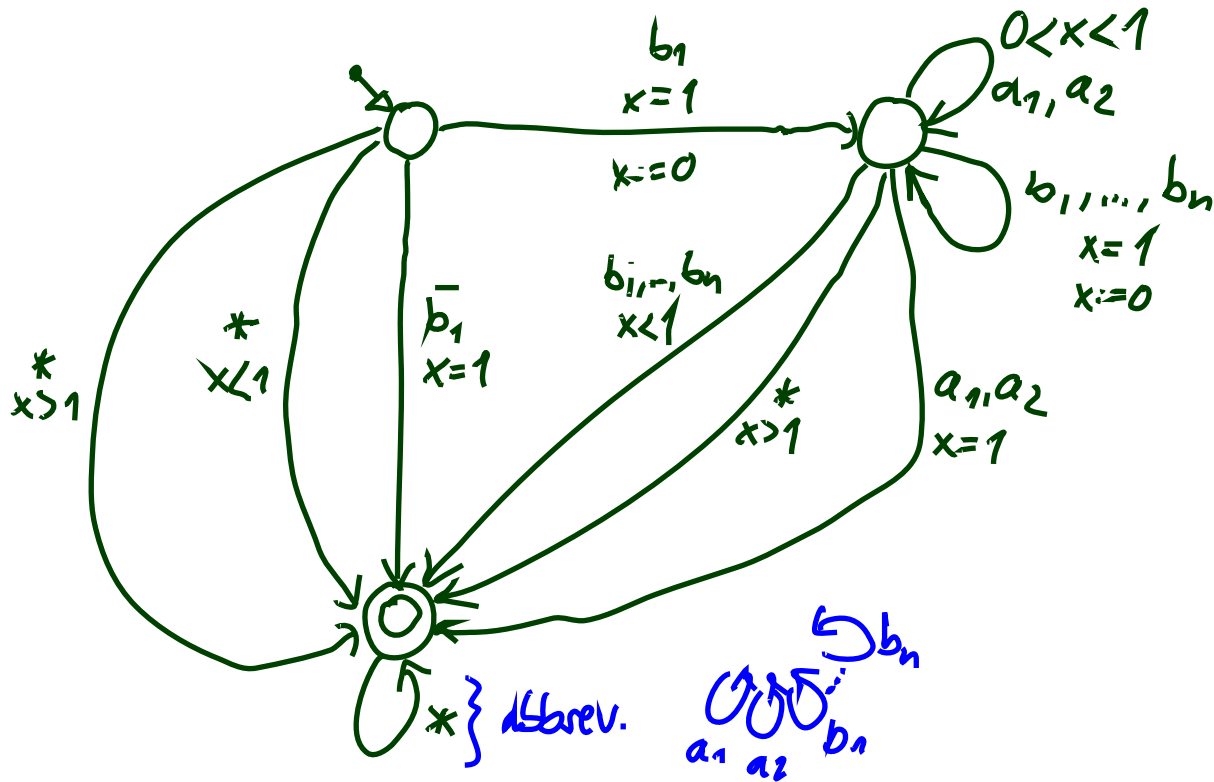
Then set

$$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_{init} \cup \mathcal{A}_{recur} \cup \bigcup_{1 \leq i \leq n} \mathcal{A}_i$$

Step 2.(i): Construct A_0

(i) The b_i at time $j \in \mathbb{N}$ is missing, or there is a spurious b_i at time $t \in]j, j + 1[$.

[Alur and Dill, 1994]: “It is easy to construct such a timed automaton.”

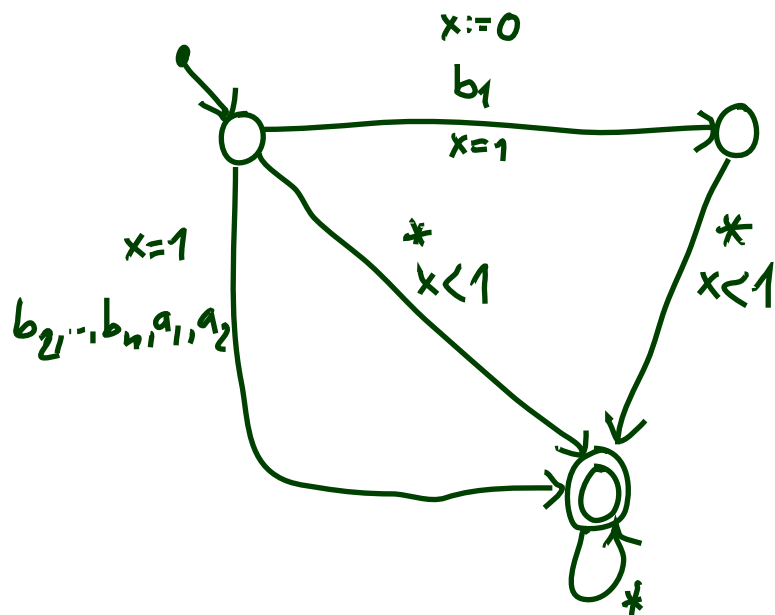


Step 2.(ii): Construct \mathcal{A}_{init}

(ii) The prefix of the timed word with times $0 \leq t < 1$ doesn't encode $\langle 1, 0, 0 \rangle$.

- It accepts

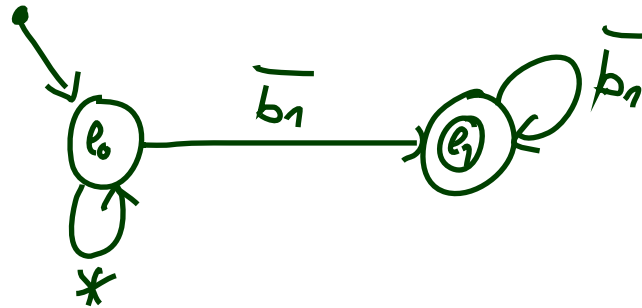
$$\{(\sigma_j, \tau_j)_{j \in \mathbb{N}_0} \mid (\sigma_0 \neq b_1) \vee (\tau_0 \neq 0) \vee (\tau_1 \neq 1)\}.$$



Step 2.(iii): Construct A_{recur}

(iii) The timed word is not recurring, i.e. it has only finitely many b_i .

- A_{recur} accepts words with only finitely many b_i .



$b_1 b_2 b_3 \dots b_n \dots b_1 \dots b_1 \dots b_1 b_2 b_3 b_2 b_3 \dots$

- $l_0 l_1 l_1 \dots$

- $l_0 l_0 l_0 \dots$

- $l_0 l_0 l_0 \dots$

$b_1 b_2 b_3 \dots b_1 b_2 b_2 b_3 b_3 \dots b_1 b_2 \dots$

$l_0 \dots \dots \dots l_1 \dots \dots \dots l_2$

$l_0 \dots \dots \dots l_0 \dots \dots$

(no run)

(run, not accepting)

(run, accepting) \hookrightarrow word is accepted

(no run)

(run, not acc.) \hookrightarrow word is not accepted

Step 2.(iv): Construct \mathcal{A}_i

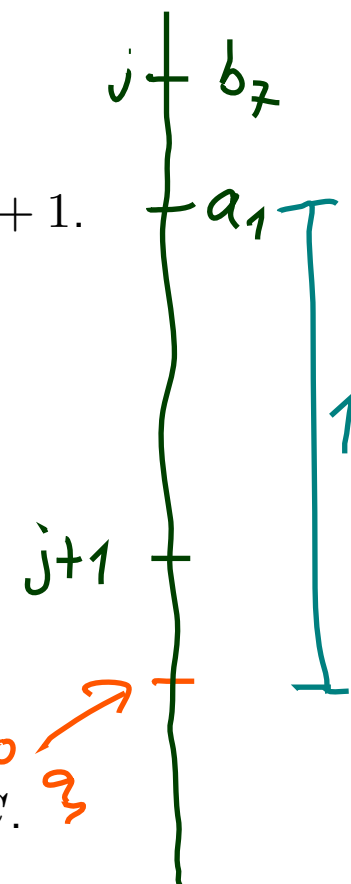
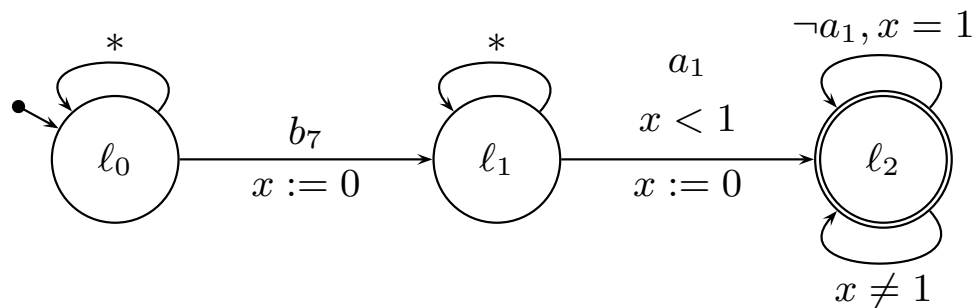
(iv) The configuration encoded in $[j + 1, j + 2[$ doesn't faithfully represent the effect of instruction b_i on the configuration encoded in $[j, j + 1[$.

Example: assume instruction 7 is:

Increment counter D and jump non-deterministically to instruction 3 or 5.

Once again: stepwise. \mathcal{A}_7 is $\mathcal{A}_7^1 \cup \dots \cup \mathcal{A}_7^6$.

- \mathcal{A}_7^1 accepts words with b_7 at time j but neither b_3 nor b_5 at time $j + 1$.
"Easy to construct."
- \mathcal{A}_7^2 is



- \mathcal{A}_7^3 accepts words which encode unexpected ~~increment~~ ^{change} of counter C .
- $\mathcal{A}_7^4, \dots, \mathcal{A}_7^6$ accept words with missing ~~in~~ ^{change} increment of D .

Aha, And...?

Consequences: Language Inclusion

- **Given:** Two TBAs \mathcal{A}_1 and \mathcal{A}_2 over alphabet B .
- **Question:** Is $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$?

Possible applications of a decision procedure:

- Characterise the allowed behaviour as \mathcal{A}_2 and model the design as \mathcal{A}_1 .
- Automatically check whether the behaviour of the design is a subset of the allowed behaviour.
- If **language inclusion** was decidable, then we could use it to decide universality of \mathcal{A} by checking

$$\mathcal{L}(\mathcal{A}_{univ}) \subseteq \mathcal{L}(\mathcal{A})$$

where \mathcal{A}_{univ} is **any** universal TBA (which is easy to construct).

Consequences: Complementation

- **Given:** A timed regular language W over B (that is, there is a TBA \mathcal{A} such that $\mathcal{L}(\mathcal{A}) = W$).
- **Question:** Is \overline{W} timed regular?

Possible applications of a decision procedure:

- Characterise the allowed behaviour as \mathcal{A}_2 and model the design as \mathcal{A}_1 .
- Automatically construct \mathcal{A}_3 with $L(\mathcal{A}_3) = \overline{L(\mathcal{A}_2)}$ and check

$$L(\mathcal{A}_1) \cap L(\mathcal{A}_3) = \emptyset,$$

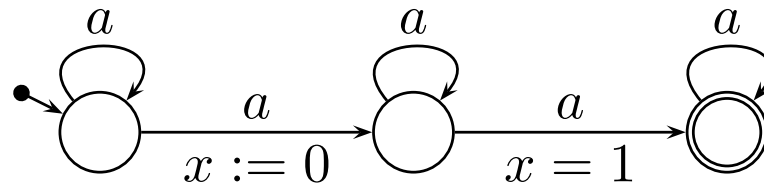
that is, whether the design has any non-allowed behaviour.

- Taking for granted that:
 - The intersection automaton is effectively computable.
 - The emptiness problem for Büchi automata **is decidable**.
(Proof by construction of region automaton [Alur and Dill, 1994].)

Consequences: Complementation

- **Given:** A timed regular language W over B (that is, there is a TBA \mathcal{A} such that $\mathcal{L}(\mathcal{A}) = W$).
- **Question:** Is \overline{W} timed regular?
- If the class of timed regular languages were closed under **complementation**, “the complement of the inclusion problem is recursively enumerable. This contradicts the Π_1^1 -hardness of the inclusion problem.” [Alur and Dill, 1994]

A non-complementable TBA \mathcal{A} :



$$\mathcal{L}(\mathcal{A}) = \{(a^\omega, (t_i)_{i \in \mathbb{N}_0}) \mid \exists i \in \mathbb{N}_0 \exists j > i : (t_j = t_i + 1)\}$$

Complement language:

$$\overline{\mathcal{L}(\mathcal{A})} = \{(a^\omega, (t_i)_{i \in \mathbb{N}_0}) \mid \text{no two } a \text{ are separated by distance } 1\}.$$

Beyond Timed Regular

Beyond Timed Regular

With clock constraints of the form

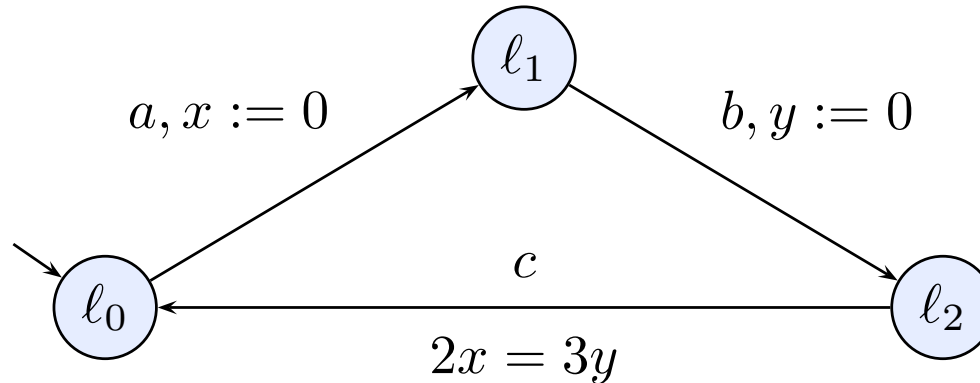
$$x + y \leq x' + y'$$

we can describe timed languages which are not timed regular.

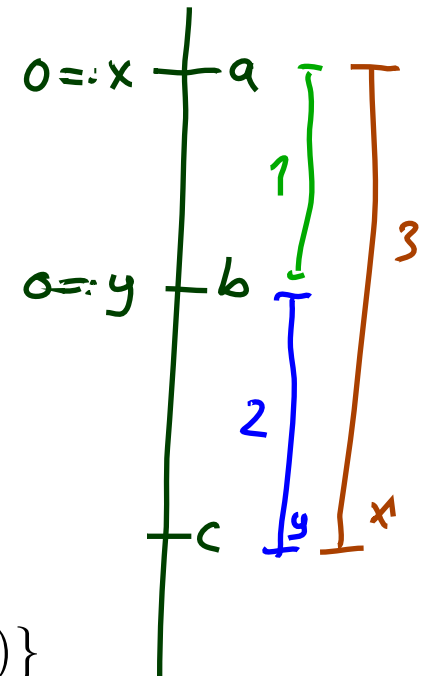
In other words: *more*

- There are strictly timed languages than timed regular languages.
- There exists timed languages \mathcal{B} such that there exists no \mathcal{A} with $L(\mathcal{A}) = \mathcal{B}$.

Example:



$$\{((abc)^\omega, \tau) \mid \forall j. (\tau_{3j} - \tau_{3j-1}) = 2(\tau_{3j-1} - \tau_{3j-2})\}$$



References

References

- [Alur and Dill, 1994] Alur, R. and Dill, D. L. (1994). A theory of timed automata. *Theoretical Computer Science*, 126(2):183–235.
- [Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.