# Real-Time Systems

## Lecture 12: Location Reachability (or: The Region Automaton)

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## Contents & Goals

**Last Lecture:**

- Networks of Timed Automata
- Uppaal Demo

**This Lecture:**

- **Educational Objectives:** Capabilities for following tasks/questions.
  - What are decidable problems of TA?
  - How can we show this? What are the essential premises of decidability?
  - What is a region? What is the region automaton of this TA?
  - What's the time abstract system of a TA? Why did we consider this?
  - What can you say about the complexity of Region-automaton based reachability analysis?
- **Content:**
  - ~~Timed Transition System of network of timed automata~~
  - Location Reachability Problem
  - Constructive, region-based decidability proof



## The Location Reachability Problem



## The Location Reachability Problem

**Given:** A timed automaton $\mathcal{A}$ and one of its control locations $\ell$.
**Question:** Is $\ell$ **reachable**?

That is, is there a transition sequence of the form

$$\langle \ell_{ini}, \nu_0 \rangle \xrightarrow{\lambda_1} \langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle \xrightarrow{\lambda_3} \dots \xrightarrow{\lambda_n} \langle \ell_n, \nu_n \rangle \, ~~= \langle \ell, \nu \rangle~~, \ell_n = \ell$$

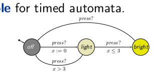
in the labelled transition system $\mathcal{T}(\mathcal{A})$?

- **Note:** Decidability is not **soo** obvious, recall that
  - clocks range over real numbers, thus infinitely many configurations,
  - at each configuration, uncountably many transitions $\xrightarrow{t}$ may originate
- **Consequence:** The timed automata as we consider them here **cannot** encode a 2-counter machine, and they are strictly less expressive than DC.



## Decidability of The Location Reachability Problem

**Claim:** (**Theorem 4.33**)
The location reachability problem is **decidable** for timed automata.



**Approach:** Constructive proof.

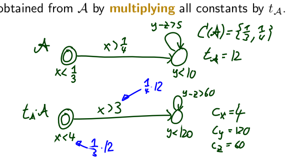
- Observe: clock constraints are **simple** — w.l.o.g. assume constants $c \in \mathbb{N}_0$.
- **Def. 4.19**: **time-abstract transition system** $\mathcal{U}(\mathcal{A})$ — abstracts from uncountably many delay transitions, still infinite-state.
- **Lem. 4.20**: location reachability of $\mathcal{A}$ is **preserved** in $\mathcal{U}(\mathcal{A})$.
- **Def. 4.29**: **region automaton** $\mathcal{R}(\mathcal{A})$ — equivalent configurations collapse into regions
- **Lem. 4.32**: location reachability of $\mathcal{U}(\mathcal{A})$ is **preserved** in $\mathcal{R}(\mathcal{A})$.
- **Lem. 4.28**: $\mathcal{R}(\mathcal{A})$ is **finite**.



## Without Loss of Generality: Natural Constants

**Recall:** Simple clock constraints are $\varphi ::= x \sim c \mid x - y \sim c \mid \varphi \wedge \varphi$ with $x, y \in X$, $c \in \mathbb{Q}_0^+$, and $\sim \in \{<, >, \leq, \geq\}$.

- Let $C(\mathcal{A}) = \{c \in \mathbb{Q}_0^+ \mid c \text{ appears in } \mathcal{A}\}$ — $C(\mathcal{A})$ is **finite**! (Why?)
- Let $t_\mathcal{A}$ be the **least common multiple of the denominators** in $C(\mathcal{A})$.
- Let $t_\mathcal{A} \cdot \mathcal{A}$ be the TA obtained from $\mathcal{A}$ by **multiplying** all constants by $t_\mathcal{A}$.



## Without Loss of Generality: Natural Constants

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- Let $C(\mathcal{A}) = \{c \in \mathbb{Q}_0^+ \mid c \text{ appears in } \mathcal{A}\}$ — $C(\mathcal{A})$ is **finite**! (Why?)
- Let $t_{\mathcal{A}}$ be the **least common multiple of the denominators** in $C(\mathcal{A})$.
- Let $t_{\mathcal{A}} \cdot \mathcal{A}$ be the TA obtained from $\mathcal{A}$ by **multiplying** all constants by $t_{\mathcal{A}}$.
- Then:
  - $C(t_{\mathcal{A}} \cdot \mathcal{A}) \subset \mathbb{N}_0$.
  - A location $\ell$ is reachable in $t_{\mathcal{A}} \cdot \mathcal{A}$ if and only if $\ell$ is reachable in $\mathcal{A}$.
- That is: we can **without loss of generality** in the following consider only timed automata $\mathcal{A}$ with $C(\mathcal{A}) \subset \mathbb{N}_0$.

**Definition.** Let $x$ be a clock of timed automaton $\mathcal{A}$ (with $C(\mathcal{A}) \subset \mathbb{N}_0$). We denote by $c_x \in \mathbb{N}_0$ the **largest time constant** $c$ that appears together with $x$ in a constraint of $\mathcal{A}$.



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**Approach:** Constructive proof.

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- ✘ **Def. 4.19**: **time-abstract transition system** $\mathcal{U}(\mathcal{A})$ — abstracts from uncountably many delay transitions, still infinite-state.
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## Helper: Relational Composition

**Recall**: $\mathcal{T}(\mathcal{A}) = (Conf(\mathcal{A}), \mathrm{Time} \cup B_{?!}, \{\xrightarrow{\lambda} \mid \lambda \in \mathrm{Time} \cup B_{?!}\}, C_{ini})$

- Note: The $\xrightarrow{\lambda}$ are binary relations on configurations.

**Definition.** Let $\mathcal{A}$ be a TA. For all $\langle \ell_1, \nu_1 \rangle, \langle \ell_2, \nu_2 \rangle \in Conf(\mathcal{A})$,

$$\langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_1} \circ \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle$$

if and only if there **exists some** $\langle \ell', \nu' \rangle \in Conf(\mathcal{A})$ such that

$$\langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_1} \langle \ell', \nu' \rangle \text{ and } \langle \ell', \nu' \rangle \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle.$$



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$$\langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_1} \langle \ell', \nu' \rangle \text{ and } \langle \ell', \nu' \rangle \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle.$$

**Remark.** The following property of **time additivity** holds.

$$\forall t_1, t_2 \in \mathrm{Time} : \xrightarrow{t_1} \circ \xrightarrow{t_2} \; = \; \xrightarrow{t_1 + t_2}$$



## Time-abstract Transition System

**Definition 4.19.** [*Time-abstract transition system*]
Let $\mathcal{A}$ be a timed automaton.
The **time-abstract transition system** $\mathcal{U}(\mathcal{A})$ is obtained from $\mathcal{T}(\mathcal{A})$ (Def. 4.4) by taking

$$\mathcal{U}(\mathcal{A}) = (Conf(\mathcal{A}), B_{?!}, \{\stackrel{\alpha}{\Longrightarrow} \mid \alpha \in B_{?!}\}, C_{ini})$$

where

$$\stackrel{\alpha}{\Longrightarrow} \subseteq Conf(\mathcal{A}) \times Conf(\mathcal{A})$$

is defined as follows: Let $\langle \ell, \nu \rangle, \langle \ell', \nu' \rangle \in Conf(\mathcal{A})$ be configurations of $\mathcal{A}$ and $\alpha \in B_{?!}$ an action. Then

$$\langle \ell, \nu \rangle \stackrel{\alpha}{\Longrightarrow} \langle \ell', \nu' \rangle$$

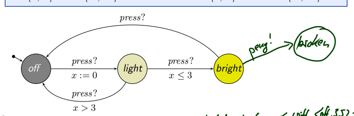
if and only if there exists $t \in \mathrm{Time}$ such that

$$\langle \ell, \nu \rangle \xrightarrow{t} \circ \xrightarrow{\alpha} \langle \ell', \nu' \rangle.$$



## Example

$\langle \ell, \nu \rangle \stackrel{\alpha}{\Longrightarrow} \langle \ell', \nu' \rangle$ iff $\exists\, t \in \mathrm{Time} \bullet \langle \ell, \nu \rangle \xrightarrow{t} \circ \xrightarrow{\alpha} \langle \ell', \nu' \rangle$




$\langle off, x=3 \rangle \stackrel{?}{\Longrightarrow} \langle off, x=3.5 \rangle$ — NO, $t = 0.5$ is a delay candidate, but no $\alpha$ with $\langle off, 3.5 \rangle \xrightarrow{\alpha} \langle off, 3.5 \rangle$

$\langle off, x=4 \rangle \stackrel{press?}{\Longrightarrow} \langle light, x=0 \rangle$ — YES, any $t \in \mathbb{R}_0^+$ works, $\alpha = press?$

$\langle off, x=4 \rangle \stackrel{?}{\Longrightarrow} \langle light, x=5 \rangle$ — NO, $\langle off, x=t \rangle \xrightarrow{t'} \circ \xrightarrow{\alpha} \langle light, x=t'' \rangle$ implies $\alpha = press?$ and $t'' = 0$

$\langle off, x=7 \rangle \stackrel{!}{\Longrightarrow} \langle bright, x=3 \rangle$ — NO, cannot go from off to bright with one action transition

$\langle bright, x=13 \rangle \stackrel{peng!}{\Longrightarrow} \langle broken, x=13 \rangle$ — YES, $t=0$ and $\alpha = peng!$

$\langle broken, x=13 \rangle \stackrel{?}{\Longrightarrow} \langle broken, x=27 \rangle$ — NO, no outgoing edge from broken

# Location Reachability is preserved in $\mathcal{U}(\mathcal{A})$

**Lemma 4.20.** For all locations $\ell$ of a given timed automaton $\mathcal{A}$ the following holds:

$\ell$ is reachable in $\mathcal{T}(\mathcal{A})$ if and only if $\ell$ is reachable in $\mathcal{U}(\mathcal{A})$.

**Proof:**

"⇐": easy

"⇒": $\ell$ reachable in $\mathcal{T}(\mathcal{A})$ ⇔ there is $\langle \ell_0, \nu_0\rangle \xrightarrow{t_{0_1}} \langle \ell_0, \nu_{0_1}\rangle \to \cdots \xrightarrow{t_{0_{n_0}}} \langle \ell_0, \nu_{0_{n_0}}\rangle \xrightarrow{\alpha_1} \langle \ell_1, \nu_1\rangle \xrightarrow{t_{1_1}} \langle \ell_1, \nu_{1_1}\rangle \to \cdots \xrightarrow{t_{n_1}} \langle \ell_{n-1}, \nu_{n-1,1}\rangle \to \cdots \langle \ell_{n-1}, \nu_{n-1,n_{n-1}}\rangle \xrightarrow{\alpha_n} \langle \ell_n, \nu_n\rangle$, $\ell_n = \ell$, $n_i \in \mathbb{N}_0$

$n_0 \geq 0$, i.e. sequence could be empty; $t_1 = \sum_{i=1}^{n_0} t_{0_i}$

$\Rightarrow \langle \ell_0, \nu_0\rangle \xRightarrow{\alpha_1} \langle \ell_1, \nu_1\rangle \cdots \xRightarrow{\alpha_n} \langle \ell_n, \nu_n\rangle$, $\ell_n = \ell$

by $\xrightarrow{t_1} \circ \xrightarrow{\alpha_1}$





# Decidability of The Location Reachability Problem

**Claim:** (**Theorem 4.33**)

The location reachability problem is **decidable** for timed automata.

**Approach:** Constructive proof.

- ✔ Observe: clock constraints are **simple** — w.l.o.g. assume constants $c \in \mathbb{N}_0$.
- ✔ **Def. 4.19**: **time-abstract transition system** $\mathcal{U}(\mathcal{A})$ — abstracts from uncountably many delay transitions, still infinite-state.
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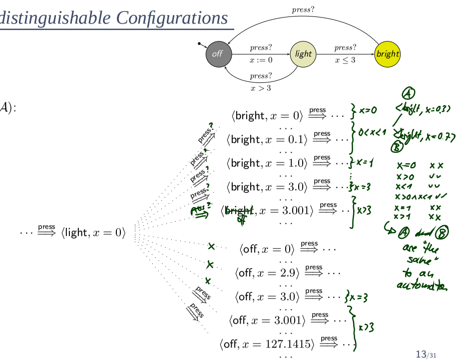
# Indistinguishable Configurations

$\mathcal{U}(\mathcal{A})$:








# Distinguishing Clock Valuations: One Clock

- Assume $\mathcal{A}$ with only a single clock, i.e. $X = \{x\}$ (**recall**: $C(\mathcal{A}) \subset \mathbb{N}$.)
  - $\mathcal{A}$ **could detect**, for a given $\nu$, whether $\nu(x) \in \{0, \ldots, c_x\}$. e.g. $x \leq 3 \wedge x \geq 3$
  - $\mathcal{A}$ **cannot distinguish** $\nu_1$ and $\nu_2$ if $\nu_i(x) \in (k, k+1)$, $i = 1, 2$, and $k \in \{0, \ldots, c_x - 1\}$. e.g. $x > 1 \wedge x < 2$
  - $\mathcal{A}$ **cannot distinguish** $\nu_1$ and $\nu_2$ if $\nu_i(x) > c_x$, $i = 1, 2$. e.g. $x > c_x$
- If $c_x \geq 1$, there are $(2c_x + 2)$ **equivalence classes**:

$$\{\{0\}, (0, 1), \{1\}, (1, 2), \ldots, \{c_x\}, (c_x, \infty)\}$$

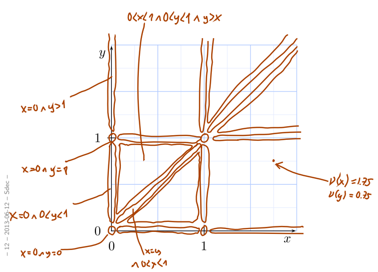
If $\nu_1(x)$ and $\nu_2(x)$ are in the **same** equivalence class, then $\nu_1$ and $\nu_2$ are **indistinguishable** by $\mathcal{A}$.





# Distinguishing Clock Valuations: Two Clocks

- $X = \{x, y\}$, $c_x = 1$, $c_y = 1$.








# Helper: Floor and Fraction

- **Recall:**

Each $q \in \mathbb{R}_0^+$ can be split into
- **floor** $\lfloor q \rfloor \in \mathbb{N}_0$ and
- **fraction** $frac(q) \in [0, 1)$

such that

$$q = \lfloor q \rfloor + frac(q).$$

## An Equivalence-Relation on Valuations

**Definition.** Let $X$ be a set of clocks, $c_x \in \mathbb{N}_0$ for each clock $x \in X$, and $\nu_1, \nu_2$ clock valuations of $X$.
We set $\nu_1 \cong \nu_2$ iff the following **four** conditions are satisfied.

**(1)** For all $x \in X$,

$$\lfloor \nu_1(x) \rfloor = \lfloor \nu_2(x) \rfloor \text{ or both } \nu_1(x) > c_x \text{ and } \nu_2(x) > c_x.$$

**(2)** For all $x \in X$ with $\nu_1(x) \leq c_x$,

$$frac(\nu_1(x)) = 0 \text{ if and only if } frac(\nu_2(x)) = 0.$$

**(3)** For all $x, y \in X$,

$$\lfloor \nu_1(x) - \nu_1(y) \rfloor = \lfloor \nu_2(x) - \nu_2(y) \rfloor$$
$$\text{or both } |\nu_1(x) - \nu_1(y)| > c \text{ and } |\nu_2(x) - \nu_2(y)| > c.$$

**(4)** For all $x, y \in X$ with $-c \leq \nu_1(x) - \nu_1(y) \leq c$,

$$frac(\nu_1(x) - \nu_1(y)) = 0 \text{ if and only if } frac(\nu_2(x) - \nu_2(y)) = 0.$$

Where $c = \max\{c_x, c_y\}$.



## Example: Regions

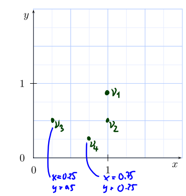
**(1)** $\forall x \in X : \lfloor \nu_1(x) \rfloor = \lfloor \nu_2(x) \rfloor \vee (\nu_1(x) > c_x \wedge \nu_2(x) > c_x)$

**(2)** $\forall x \in X : \nu_1(x) \leq c_x$
$\implies (frac(\nu_1(x)) = 0 \iff frac(\nu_2(x)) = 0)$

**(3)** $\forall x, y \in X : \lfloor \nu_1(x) - \nu_1(y) \rfloor = \lfloor \nu_2(x) - \nu_2(y) \rfloor$
$\vee (|\nu_1(x) - \nu_1(y)| > c \wedge |\nu_2(x) - \nu_2(y)| > c)$

**(4)** $\forall x, y \in X : -c \leq \nu_1(x) - \nu_1(y) \leq c \implies$
$(frac(\nu_1(x) - \nu_1(y)) = 0 \iff frac(\nu_2(x) - \nu_2(y)) = 0)$

$c_x = 1$
$c_y = 1$




- $\nu_1 \cong \nu_2$:
  (1) ✓✓
  (2) ✓✓
  (3) ✓
  (4) ✓
- $\nu_3 \not\cong \nu_4$:
  (1) ✓✓
  (2) ✓✓
  (3) $\nu_3(x) - \nu_3(y) = -0.25$ $\lfloor -0.25 \rfloor = -1$
  $\nu_4(x) - \nu_4(y) = 0.5$ $\lfloor 0.5 \rfloor = 0$ $\neq$



## Regions

**Proposition.** $\cong$ is an **equivalence relation**.

**Definition 4.27.** For a given valuation $\nu$ we denote by $[\nu]$ the equivalence class of $\nu$. We call equivalence classes of $\cong$ **regions**.

$\{\nu' \mid \nu' \cong \nu\}$



## The Region Automaton

**Definition 4.29.** [*Region Automaton*] The **region automaton** $\mathcal{R}(\mathcal{A})$ of the timed automaton $\mathcal{A}$ is the labelled transition system

$$\mathcal{R}(\mathcal{A}) = (Conf(\mathcal{R}(\mathcal{A})), B_{?!}, \{\xrightarrow{\alpha}_{R(\mathcal{A})} \mid \alpha \in B_{?!}\}, C_{ini})$$

where

- $Conf(\mathcal{R}(\mathcal{A})) = \{\langle \ell, [\nu] \rangle \mid \ell \in L, \nu : X \to \text{Time}, \nu \models I(\ell)\}$,
- for each $\alpha \in B_{?!}$,

  $$\langle \ell, [\nu] \rangle \xrightarrow{\alpha}_{R(\mathcal{A})} \langle \ell', [\nu'] \rangle \text{ if and only if } \langle \ell, \nu \rangle \stackrel{\alpha}{\Longrightarrow} \langle \ell', \nu' \rangle$$

  in $\mathcal{U}(\mathcal{A})$, and
- $C_{ini} = \{\langle \ell_{ini}, [\nu_{ini}] \rangle\} \cap Conf(\mathcal{R}(\mathcal{A}))$ with $\nu_{ini}(X) = \{0\}$.

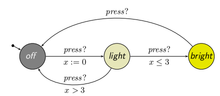
representative of region $[\nu]$

representative of region $[\nu']$

**Proposition.** The transition relation of $\mathcal{R}(\mathcal{A})$ is **well-defined**, that is, independent of the choice of the representative $\nu$ of a region $[\nu]$.



## Example: Region Automaton




$\mathcal{U}(\mathcal{A})$:

$\cdots \stackrel{press}{\Longrightarrow} \langle \text{light}, x = 0 \rangle$

$\stackrel{press}{\Longrightarrow} \langle \text{bright}, x = 0 \rangle \stackrel{press}{\Longrightarrow} \cdots$
$\stackrel{press}{\Longrightarrow} \langle \text{bright}, x = 0.1 \rangle \stackrel{press}{\Longrightarrow} \cdots$
$\stackrel{press}{\Longrightarrow} \langle \text{bright}, x = 1.0 \rangle \stackrel{press}{\Longrightarrow} \cdots$
$\stackrel{press}{\Longrightarrow} \langle \text{bright}, x = 1.31415 \rangle \stackrel{press}{\Longrightarrow} \cdots$
$\stackrel{press}{\Longrightarrow} \langle \text{off}, x = 2.0 \rangle \stackrel{press}{\Longrightarrow} \cdots$
$\stackrel{press}{\Longrightarrow} \langle \text{off}, x = 3.0 \rangle \stackrel{press}{\Longrightarrow} \cdots$
$\stackrel{press}{\Longrightarrow} \langle \text{off}, x = 127 \rangle \stackrel{press}{\Longrightarrow} \cdots$



## Remark

**Remark 4.30.** That a configuration $\langle \ell, [\nu] \rangle$ is reachable in $\mathcal{R}(\mathcal{A})$ represents the fact, that all $\langle \ell, \nu \rangle$ are reachable.

IAW: in $\mathcal{A}$, we can observe $\nu$ when location $\ell$ has **just been entered**.

The clock values reachable by staying/letting time pass in $\ell$ are **not explicitly** represented by the regions of $\mathcal{R}(\mathcal{A})$.

## *Decidability of The Location Reachability Problem*

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**Approach:** Constructive proof.

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## *Region Automaton Properties*

**Lemma 4.32.** [*Correctness*] For all locations $\ell$ of a given timed automaton $\mathcal{A}$ the following holds:

$\ell$ is reachable in $\mathcal{U}(\mathcal{A})$ if and only if $\ell$ is reachable in $\mathcal{R}(\mathcal{A})$.

For the **Proof**:

Definition 4.21. [*Bisimulation*] An equivalence relation $\sim$ on valuations is a **(strong) bisimulation** if and only if, whenever

$$\nu_1 \sim \nu_2 \text{ and } \langle \ell, \nu_1 \rangle \stackrel{\alpha}{\Longrightarrow} \langle \ell', \nu_1' \rangle$$

then there exists $\nu_2'$ with $\nu_1' \sim \nu_2'$ and $\langle \ell, \nu_2 \rangle \stackrel{\alpha}{\Longrightarrow} \langle \ell', \nu_2' \rangle$.

**Lemma 4.26.** [*Bisimulation*] $\cong$ is a **strong bisimulation.**



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## *The Number of Regions*

**Lemma 4.28.** Let $X$ be a set of clocks, $c_x \in \mathbb{N}_0$ the maximal constant for each $x \in X$, and $c = \max\{c_x \mid x \in X\}$. Then

$$(2c+2)^{|X|} \cdot (4c+3)^{\frac{1}{2}|X| \cdot (|X|-1)}$$

is an **upper bound** on the **number of regions**.

**Proof**: [Olderog and Dierks, 2008]



## *Observations Regarding the Number of Regions*

- Lemma 4.28 **in particular** tells us that each timed automaton (in our definition) has **finitely** many regions.
- Note: the upper bound is a **worst case**, not an **exact bound**.



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## Putting It All Together

Let $\mathcal{A} = (L, B, X, I, E, \ell_{ini})$ be a timed automaton, $\ell \in L$ a location.

- $\mathcal{R}(\mathcal{A})$ can be constructed effectively.
- There are finitely many locations in $L$ (by definition).
- There are finitely many regions by Lemma 4.28.
- So $Conf(\mathcal{R}(\mathcal{A}))$ is finite (by construction).
- It is decidable whether ($C_{init}$ of $\mathcal{R}(\mathcal{A})$ is empty) or whether there exists a sequence

$$\langle \ell_{ini}, [\nu_{ini}] \rangle \xrightarrow{\alpha}_{R(\mathcal{A})} \langle \ell_1, [\nu_1] \rangle \xrightarrow{\alpha}_{R(\mathcal{A})} \dots \xrightarrow{\alpha}_{R(\mathcal{A})} \langle \ell_n, [\nu_n] \rangle$$

such that $\ell_n = \ell$ (reachability in graphs).



## Putting It All Together

Let $\mathcal{A} = (L, B, X, I, E, \ell_{ini})$ be a timed automaton, $\ell \in L$ a location.

- $\mathcal{R}(\mathcal{A})$ can be constructed effectively.
- There are finitely many locations in $L$ (by definition).
- There are finitely many regions by Lemma 4.28.
- So $Conf(\mathcal{R}(\mathcal{A}))$ is finite (by construction).
- It is decidable whether ($C_{init}$ of $\mathcal{R}(\mathcal{A})$ is empty) or whether there exists a sequence

$$\langle \ell_{ini}, [\nu_{ini}] \rangle \xrightarrow{\alpha}_{R(\mathcal{A})} \langle \ell_1, [\nu_1] \rangle \xrightarrow{\alpha}_{R(\mathcal{A})} \dots \xrightarrow{\alpha}_{R(\mathcal{A})} \langle \ell_n, [\nu_n] \rangle$$

such that $\ell_n = \ell$ (reachability in graphs).

So we have

**Theorem 4.33.** [*Decidability*]
The location reachability problem for timed automata is **decidable**.



## The Constraint Reachability Problem

- **Given:** A timed automaton $\mathcal{A}$, one of its control locations $\ell$, and a clock constraint $\varphi$.
- **Question:** Is a configuration $\langle \ell, \nu \rangle$ **reachable** where $\nu \models \varphi$, i.e. is there a transition sequence of the form

$$\langle \ell_{ini}, \nu_{ini} \rangle \xrightarrow{\lambda_1} \langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle \xrightarrow{\lambda_3} \dots \xrightarrow{\lambda_n} \langle \ell_n, \nu_n \rangle = \langle \ell, \nu \rangle$$

in the labelled transition system $\mathcal{T}(\mathcal{A})$ with $\nu \models \varphi$?

- **Note:** we just observed that $\mathcal{R}(\mathcal{A})$ loses some information about the clock valuations that are possible in/from a region.



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- **Note:** we just observed that $\mathcal{R}(\mathcal{A})$ loses some information about the clock valuations that are possible in/from a region.

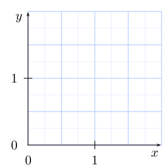
**Theorem 4.34.** The constraint reachability problem for timed automata is decidable.



## The Delay Operation

- Let $[\nu]$ be a clock region.
- We set

$$delay[\nu] = \{\nu' + t \mid \nu' \cong \nu \text{ and } t \in \text{Time}\}.$$



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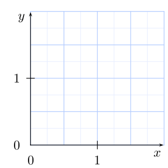


- **Note:** $delay[\nu]$ can be represented as a **finite** union of regions.
  **For example**, with our two-clock example we have

$delay[x = y = 0] =$

*References*





## References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.

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