Nested Word Automata

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Nested Words
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- Theoretically and practically pleasant model for the representation of data with both:
  - a linear ordering
  - a hierarchically nested matching
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- Applications in software verification and document processing
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- This is the last list item
Structure of this talk

1. Motivation
2. Nested words
3. Nested word automata
Section 1

Motivation
Subsection 1

Data with both linear ordering and hierarchically nested matching

1. Document trees (e.g. HTML)
2. Executions of structured programs (with call-return semantics)
Document trees (e.g. HTML)
Executions of structured programs (with call-return semantics)

main()
countToZero(1)
countToZero(0)
printLn("1")
printLn("0")
Subsection 2

Formal Languages

- Regular Languages
- Context-Free Languages
Regular Languages

Regular language over an alphabet $\Sigma$
- Most easily explained as generated by a regular expression (RE)
  - Example RE: $0| [123456789] [0123456789]^*$
Regular Languages

Regular language over an alphabet $\Sigma$

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  - Example RE: 0| [123456789] [0123456789] *
  - Typical implementation: DFA (Deterministic Finite Automaton)
“Problems” with Regular Languages

- Can’t express arbitrarily deep nesting
Context-free Languages

Context-free language over $\Sigma$

- Superset of Regular Languages

Example for real world usage:

- HTML: `<html>` BODY `</html>`
- BODY: `<body>` CONTENT `</html>`
- CONTENT: "Hello, world!" | "Hallo, Welt!"

Typical implementation: Pushdown Automaton
Context-free Languages

Context-free language over $\Sigma$

- Superset of Regular Languages
- Most easily explained as generated by a Context-free Grammar (CFG)
  - terminal symbols $\Sigma$ and non-terminal symbols $V$
  - start symbol $S \in V$
  - Productions $\subset V \times (V \cup \Sigma)^*$
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“Problems” with Context-free Languages

- Not closed under intersection
- Not closed under complementation
- Not closed under difference
- Can’t decide inclusion
- Can’t decide equivalence
- Not determinizable (Deterministic Context-free languages are a strict subset of Context-free languages)
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Nested words

- Nested words were constructed to overcome the limitations of Context-free and Regular languages.
- The class of nested word languages lies properly between deterministic context-free languages and Regular languages.
Section 2

Nested words
Nested words are ordinary words with extra information:
The nesting structure is explicitly contained in the input.
\[\Rightarrow\] automata for nested words need not parse the nesting.
Definition: Nested word

▶ Later!
▶ For now: well-matched nested words
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Definition: Well-matched nested word

A well-matched nested word over an alphabet $\Sigma$ is a pair $(a_1 \ldots a_n, \rightarrow)$

- $a_1 \ldots a_n \in \Sigma^*$ is a word over $\Sigma$
- The matching $\rightarrow$ matches “start tags” with their “end tags”:
  - $\rightarrow \subseteq [1..n] \times [1..n]$
  - Given $(i, j) \neq (k, l)$ elements of $\rightarrow$, either $i < j < k < l$ or $i < k < l < j$

For $(i, j) \in \rightarrow$, $i$ is a call position and $j$ is a return position
Well-matched
Not well-matched
Not well-matched
Example: Simple HTML tree
Example: Simple HTML tree
Example: Process trace
Example: Process trace

main()
countDown(1)
print(1)
countDown(0)
print(0)
Section 3

Nested Word Automata (NWA)
A Nested Word Automaton takes a nested word as input and (as automatons do) accepts or rejects it.
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Nested word automata have much of the power of Pushdown Automata, but can take advantage of the fact that their inputs carry a “pre-parsed” hierarchical structure.
Definition: A deterministic nested word automaton (DNWA) over an alphabet $\Sigma$ is a structure

$$( Q, Q_0, Q_f \quad // \text{linear states, initial, accepting}$$

$$( P, P_0, P_f \quad // \text{hierarchical states, initial, accepting}$$

$$( \delta_c, \delta_i, \delta_r \quad // \text{transitions: call, internal, return}$$

)$

where $Q$ and $P$ are sets of symbols,
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(Q, \quad Q_0, \quad Q_f \quad \text{// linear states, initial, accepting}
, \quad P, \quad P_0, \quad P_f \quad \text{// hierarchical states, initial, accepting}
, \quad \delta_c, \quad \delta_i, \quad \delta_r \quad \text{// transitions: call, internal, return}
)
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where $Q$ and $P$ are sets of symbols, $Q_0 \in Q$, $P_0 \in P$, $Q_f \subset Q$, $P_f \subset P$.
Definition: Deterministic Nested word automaton

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\[(Q, Q_0, Q_f, P, P_0, P_f, \delta_c, \delta_i, \delta_r)\]

where \(Q\) and \(P\) are sets of symbols, \(Q_0 \in Q\), \(P_0 \in P\), \(Q_f \subset Q\), \(P_f \subset P\), and the three \(\delta\) are transition functions

\[
\begin{align*}
\delta_c & \subset (\Sigma \times Q) \mapsto (Q \times P) \\
\delta_i & \subset (\Sigma \times Q) \mapsto Q \\
\delta_r & \subset (\Sigma \times Q \times P) \mapsto Q
\end{align*}
\]
Definition: DNWA: Run

The run of a DNWA over a nested word \((a_1..a_n, \rightsquigarrow)\) is defined as

- A sequence \(q_i\) for \(i \in [1, n]\)
- And a sequence \(p_i\) for all call positions \(i\)

Informally: \(q_i\) is the linear trace and \(p_i\) the hierarchical trace.

The run is always uniquely and well-defined (after adding transitions to a black hole state where the transition functions are undefined).
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so that for \(i \in [1, n]\) it holds that:

- if \(i\) is a call position, then \(\delta_c(a_i, q_{i-1}) = (q_i, p_i)\)
- else if \(i\) is an internal position, then \(\delta_i(a_i, q_{i-1}) = q_i\)
- else if \(i\) is a return position (let \(h\) be its corresponding call position), then \(\delta_r(a_i, q_{i-1}, p_h) = q_i\)

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Definition: DNWA: Acceptance

A DNWA accepts a nested word if the run over it ends in an accepting linear state:

Let $A$ be a DNWA with accepting linear states $Q_f$, and let $(q_1 \ldots n, p_1 \ldots m)$ be the run of $A$ over a nested word $w$. Then $A$ accepts $w$ iff $q_n \in Q_f$. 
Example: Nested word automaton

Task: Given $\Sigma = \{[, (, ),]\}$, build an acceptor for the language of properly balanced parentheses.
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Task: Given $\Sigma = \{[ , ( , ] , ) \}$, build an acceptor for the language of properly balanced parentheses.

$Q = \{q\}$

$Q_0 = q$

$Q_f = \{q\}$
Example: Nested word automaton

Task: Given $\Sigma = \{[, (, ], )\}$, build an acceptor for the language of properly balanced parentheses.

$$
Q = \{q\} \\
Q_0 = q \\
Q_f = \{q\} \\
P = \Sigma \cup \{\perp\} \\
P_0 = \perp \\
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Example: Nested word automaton

Task: Given $\Sigma = \{[, (, ], ))\}$, build an acceptor for the language of properly balanced parentheses.

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P & = \Sigma \cup \{\perp\} \\
P_0 & = \perp \\
P_f & = \{\perp\} \\
\delta_c & = \{(q, () \mapsto (q,), (q, [) \mapsto (q, [))\} \\
\delta_r & = \{(), q, () \mapsto q, (], q, [) \mapsto q\} \\
\delta_i & = \emptyset
\end{align*}
\]
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The last example is actually a showcase example for Pushdown automata.
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- Push: exactly one element per call position.
- Pop: exactly one element per return position.

Reading the stack only when popping (returning).

With these restrictions, processing a nested word takes place in fixed linear time and space.
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