Nested Word Automata

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30.6.2014

Nested Words

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- This is the last list item

Structure of this talk

1. Motivation
2. Nested words
3. Nested word automata

Section 1

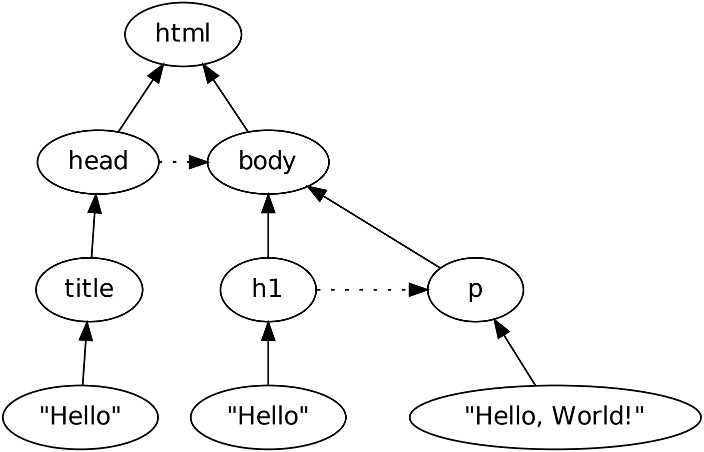
Motivation

Subsection 1

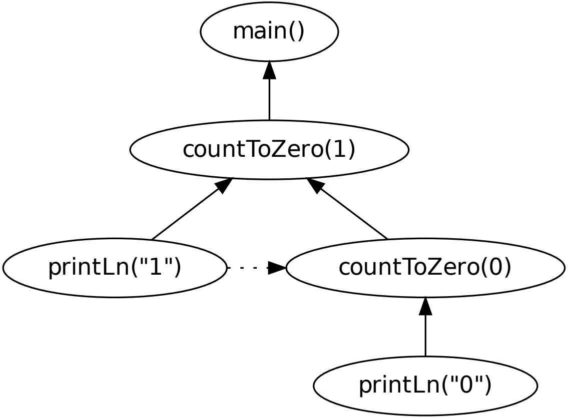
Data with both linear ordering and hierarchically nested matching

1. Document trees (e.g. HTML)
2. Executions of structured programs (with call-return semantics)

Document trees (e.g. HTML)



Executions of structured programs (with call-return semantics)



Subsection 2

Formal Languages

- Regular Languages
- Context-Free Languages

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Regular language over an alphabet Σ

- Most easily explained as generated by a regular expression (RE)
  - Example RE: `0|[123456789][0123456789]*`

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  - Typical implementation: DFA (Deterministic Finite Automaton)

“Problems” with Regular Languages

- Can’t express arbitrarily deep nesting

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  - terminal symbols Σ and non-terminal symbols V
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HTML : "<html>" BODY "</html>"
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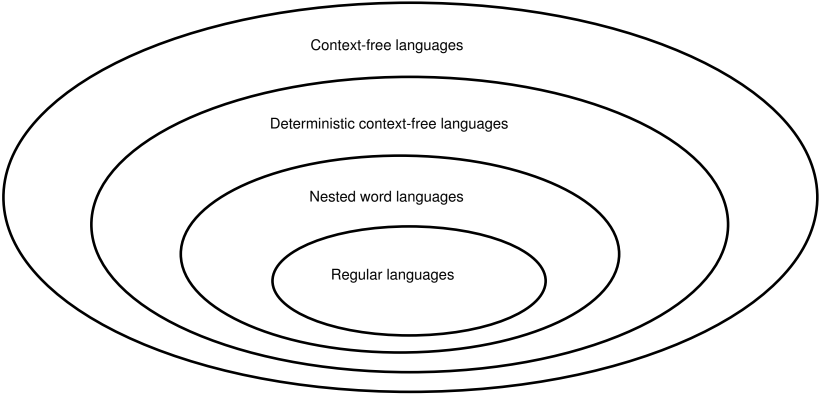
- Can’t decide inclusion
- Can’t decide equivalence
- Not determinizable (Deterministic Context-free languages are a strict subset of Context-free languages)

Nested words

- Nested words were constructed to overcome the limitations of Context-free and Regular languages
- The class of nested word languages lies properly between deterministic context-free languages and Regular languages



Section 2

Nested words

Nested words are ordinary words with extra information:

The nesting structure is explicitly contained in the input.

$\Rightarrow$ automata for nested words need not parse the nesting.

Definition: Nested word

- Later!
- For now: *well-matched* nested words

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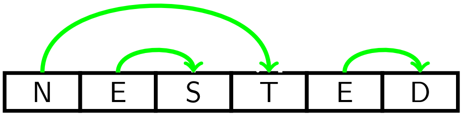
- $a_1 \dots a_n \in \Sigma^*$ is a word over Σ
- The *matching* $\rightsquigarrow$ matches "start tags" with their "end tags":
  - $\rightsquigarrow \subset [1..n] \times [1..n]$
  - Given $(i,j) \neq (k,l)$ elements of $\rightsquigarrow$, **either** $i < j < k < l$ **or** $i < k < l < j$

For $(i,j) \in \rightsquigarrow$, i is a *call position* and j is a *return position*

Well-matched



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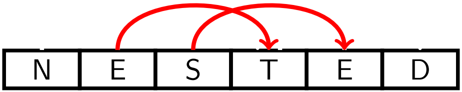

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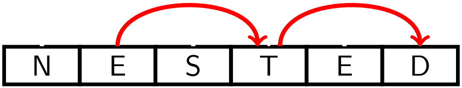

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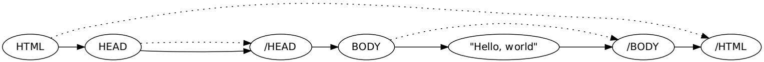

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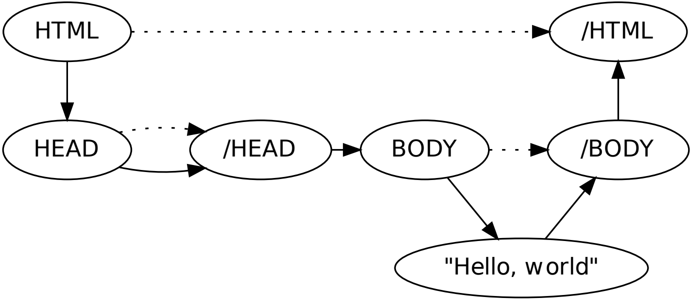
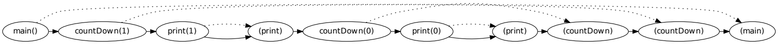


Example: Process trace



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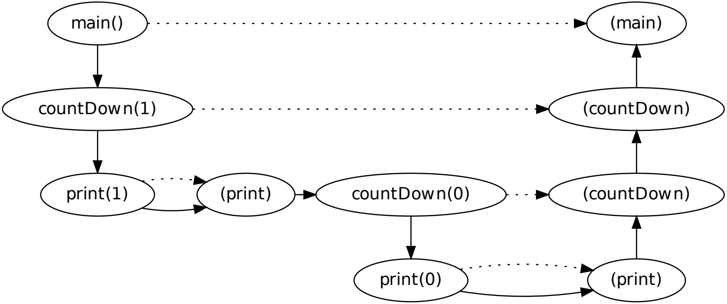

Section 3

Nested Word Automata (NWA)

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Nested word automata have much of the power of Pushdown Automata, but can take advantage of the fact that their inputs carry a “pre-parsed” hierarchical structure.

Definition: Deterministic Nested word automaton

Definition: A *deterministic nested word automaton* (*DNWA*) over an alphabet Σ is a structure

$$
\begin{array}{llll}
(\; Q, & Q_0, & Q_f & \text{// linear states, initial, accepting} \\
, \; P, & P_0, & P_f & \text{// hierarchical states, initial, accepting} \\
, \; \delta_c, & \delta_i, & \delta_r & \text{// transitions: call, internal, return} \\
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\begin{array}{lll}
\delta_c \subset & (\Sigma \times Q) & \mapsto (Q \times P) \\
\delta_i \subset & (\Sigma \times Q) & \mapsto Q \\
\delta_r \subset & (\Sigma \times Q \times P) & \mapsto Q
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Definition: DNWA: Run

The *run* of a DNWA over a nested word $(a_1..a_n, \rightsquigarrow)$ is defined as

- A sequence q_i for $i \in [1, n]$
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so that for $i \in [1, n]$ it holds that:

- if i is a call position, then $\delta_c(a_i, q_{i-1}) = (q_i, p_i)$
- else if i is an internal position, then $\delta_i(a_i, q_{i-1}) = q_i$
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The run is always uniquely and well-defined (after adding transitions to a black hole state where the transition functions are undefined)

Definition: DNWA: Acceptance

A DNWA accepts a nested word if the run over it ends in an accepting linear state:

Let A be a DNWA with accepting linear states Q_f, and let $(q_{1..n}, p_{1..m})$ be the run of A over a nested word w.
Then A *accepts* w **iff** $q_n \in Q_f$.

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\delta_r & = & \{\ (), q, () \mapsto q, \quad (], q, [) \mapsto q\ \} \\
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With these restrictions, processing a nested word takes place in fixed **linear time and space**.

This is the last slide.