Star height of regular languages

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Overview

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  - Problem: $\{a^{1000}\}$ more complicated than $\{a^n | n \in \mathbb{N}_0\}$

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The language described by a regular expression e is denoted by $\mathcal{L}(e)$.

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$$\mathrm{h}(L) := \min(\{\mathrm{h}(e) \mid \mathcal{L}(e) = L\}).$$

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- Operates on generalized automaton $\mathcal{A}$, i.e. an automaton whose edges are labeled with regular expressions
- Computes regular expression e with $\mathcal{L}(e) = \mathcal{L}(\mathcal{A})$

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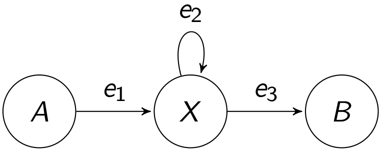

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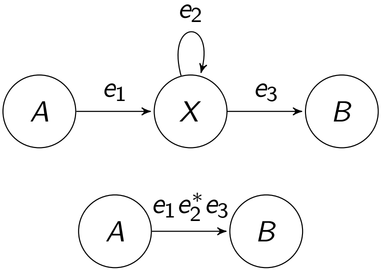

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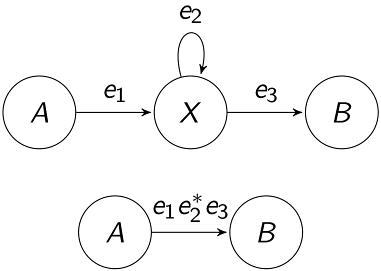

4. Join all expressions on edges from i to t using '$+$'

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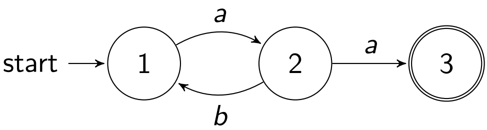

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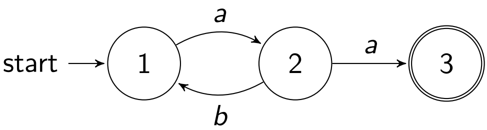

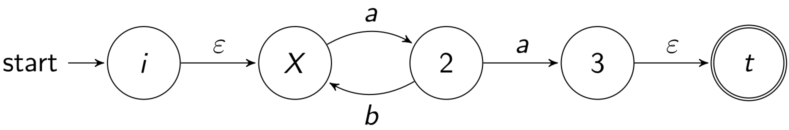

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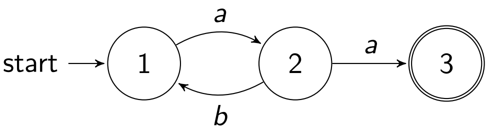

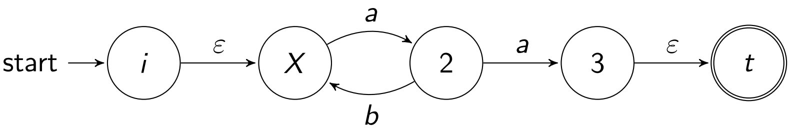

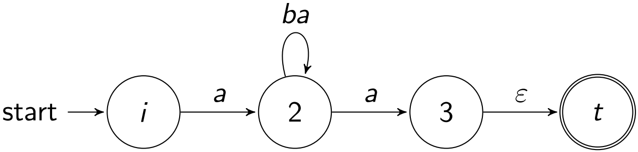

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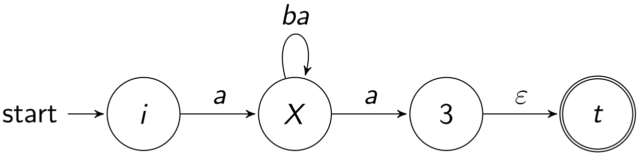

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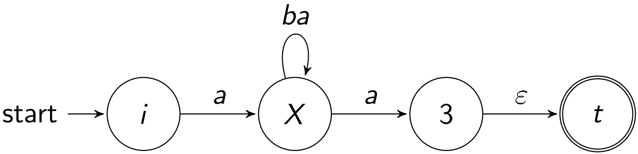

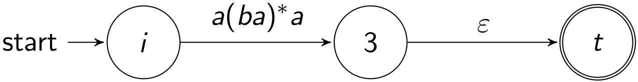

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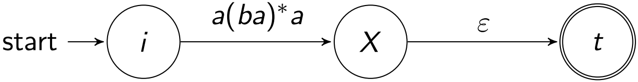

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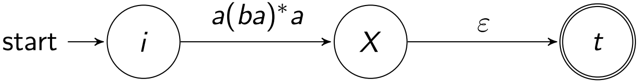

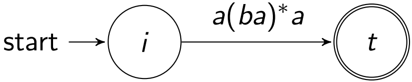

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- a *ball*, if V is strongly connected and has at least one edge.

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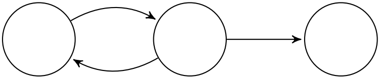

Figure : A graph

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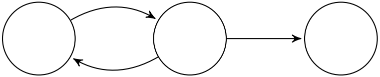

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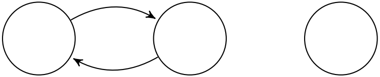

Figure : Its strongly connected components

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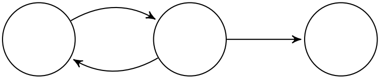

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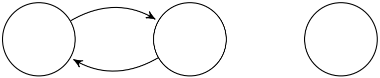

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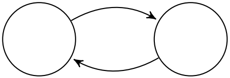

Figure : Its ball

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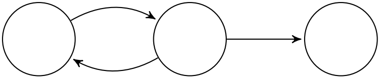

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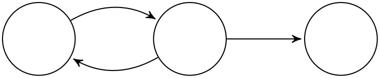

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We have $\mathrm{lc}(G) = 1$.

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In general, given an automaton $\mathcal{A}$ and two total orders on its states ω_1 and ω_2, we have

$$\mathsf{h}(\mathsf{BMC}(\mathcal{A}, \omega_1)) \neq \mathsf{h}(\mathsf{BMC}(\mathcal{A}, \omega_2))$$

as the following example shows:

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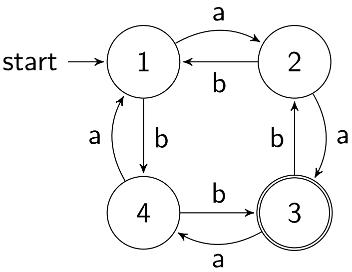

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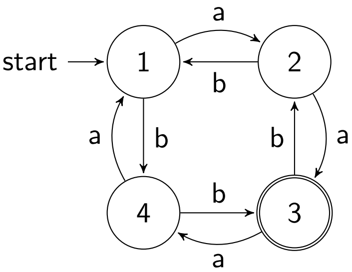
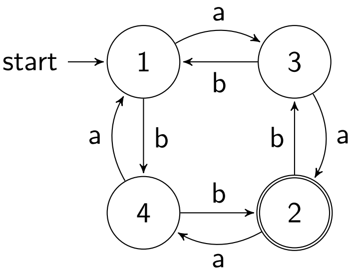


Star height of result of BMC algorithm: 3

Example continued



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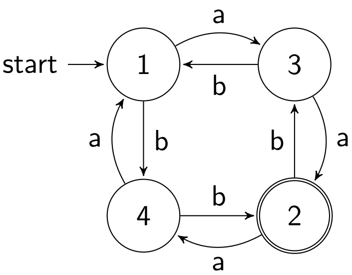

Star height of result of BMC algorithm: 2

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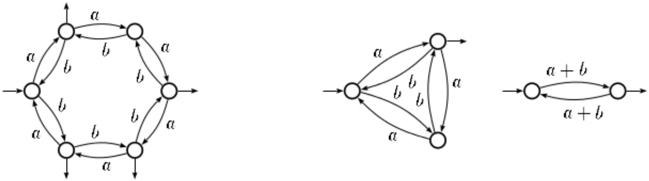

Figure : Minimal automata $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ for their respective languages[1]

[1] LoSa00, On the star height of regular languages

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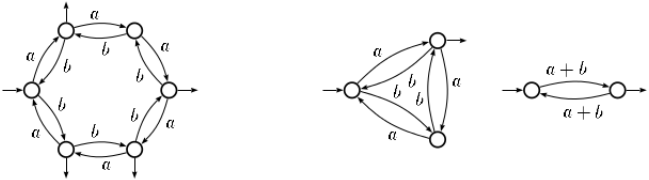

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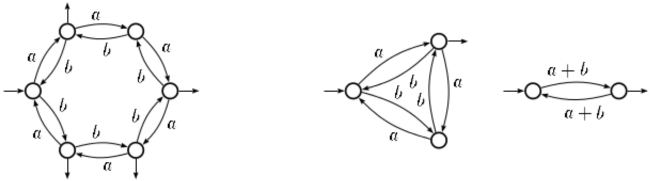

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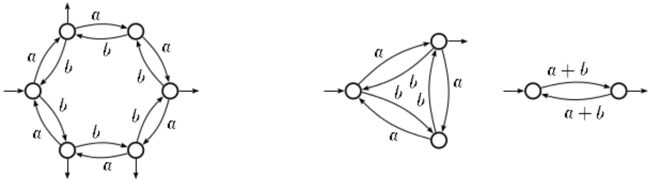

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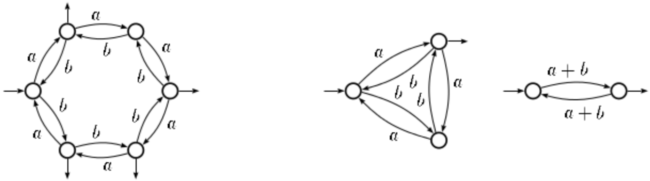

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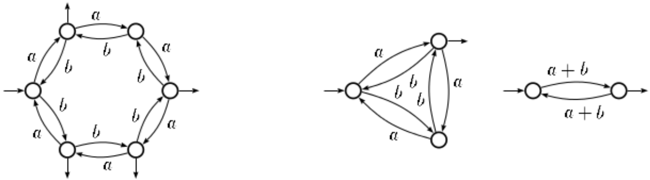

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- But: $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{B}) \cup \mathcal{L}(\mathcal{C})$
- Therefore: $\mathrm{h}(\mathcal{L}(\mathcal{A})) \leq 2 < 3 = \mathrm{lc}(\mathcal{A})$

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Computability

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Computability

- Is the star height of a regular language computable?
- Not in general
- But: For a subset of the regular languages (*pure-group languages*) it is computable

Overview

References

[LoSa00] Lombardy, S.; Sakarovitch, J.: On the star height of rational languages. In: *Proceedings of the 3rd International Conference on Words, Languages and Combinatorics, Kyoto (Japan), March 2000*