Exercise 1: NP-Completeness I

The directed Hamiltonian path problem asks – like the undirected Hamiltonian path problem (known from the lecture) – whether there exists a path in a graph which visits every vertex exactly once. The only difference is that now the graph is directed, i.e., the edges have a direction and they may only be traversed in the respective direction.

Example: The first graph has a directed Hamiltonian path (going from left to right), while the second graph does not.

Prove that the directed Hamiltonian path problem is NP-complete. For this, use the undirected Hamiltonian path problem.

Hint: The following “guide” always works for showing NP-completeness of a problem $L_1$:

(a) Argue why $L_1 \in \text{NP}$. That is, describe a guess-and-check algorithm. You should always be able to do this part.

(b) Show that $L_1$ is NP-hard. You show this by a polynomial reduction from some problem $L_2$ which is known to be NP-hard: $L_2 \leq_p L_1$.

For this reduction you have to find a way to translate the question of $L_2$ into a question of $L_1$. So you have to be creative.

- Describe the translation $f$ (usually depending on the input).
- Argue why $f$ is total (usually this is automatically the case – you must not make assumptions on the input).
- Argue why $f$ is polynomial (if possible, give a coarse upper bound).
- Show that $w \in L_2 \iff f(w) \in L_1$.

In this exercise: Think about how an undirected graph can be modeled by a directed graph such that every path that was possible before the translation is possible afterwards and vice versa.
Exercise 2: NP-Completeness II

The problem *set cover* is defined as follows.

*Given:* A set \( M \), a set of subsets of \( M \) (i.e., \( S_1, \ldots, S_n \) such that \( S_i \subseteq M \) for \( i = 1, \ldots, n \)) and a natural number \( k \leq n \).

*Question:* Are there \( k \) subsets \( S_{j_1}, \ldots, S_{j_k} \) such that \( M = S_{j_1} \cup \cdots \cup S_{j_k} \)?

Prove that the set cover problem is NP-complete. For this, use the *vertex cover* problem which is NP-complete and defined as follows.

*Given:* An undirected graph \( G = (V, E) \) and a natural number \( k \).

*Question:* Is there a covering set of vertices of size \( k \)? (A covering set of vertices is a subset \( V' \subseteq V \) such that for all edges \( (u, v) \in E \): \( u \in V' \) or \( v \in V' \)).