ALBERT-LUDWIGS-
UNIVERSITÄT FREIBURG

Prof. Dr. Andreas Podelski
Matthias Heizmann
Christian Schilling

8. Exercise Sheet for the Tutorial

Computer Science Theory

Exercise 1: NP-Completeness I

The *directed Hamiltonian path problem* asks – like the undirected Hamiltonian path problem (known from the lecture) – whether there exists a path in a graph which visits every vertex exactly once. The only difference is that now the graph is directed, i.e., the edges have a direction and they may only be traversed in the respective direction.

Example: The first graph has a directed Hamiltonian path (going from left to right), while the second graph does not.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \qquad\qquad \bullet \longrightarrow \bullet \longleftarrow \bullet$$

Prove that the directed Hamiltonian path problem is NP-complete. For this, use the undirected Hamiltonian path problem.

Hint: The following "guide" always works for showing NP-completeness of a problem L_1:

(a) Argue why $L_1 \in \mathsf{NP}$. That is, describe a guess-and-check algorithm. You should always be able to do this part.

(b) Show that L_1 is NP-hard. You show this by a polynomial reduction from some problem L_2 which is known to be NP-hard: $L_2 \leq_p L_1$.

For this reduction you have to find a way to translate the question of L_2 into a question of L_1. So you have to be creative.

- Describe the translation f (usually depending on the input).
- Argue why f is total (usually this is automatically the case – you must not make assumptions on the input).
- Argue why f is polynomial (if possible, give a coarse upper bound).
- Show that $w \in L_2 \Leftrightarrow f(w) \in L_1$.

In this exercise: Think about how an undirected graph can be modeled by a directed graph such that every path that was possible before the translation is possible afterwards and vice versa.

Exercise 2: NP-Completeness II
The problem *set cover* is defined as follows.

> *Given:* A set M, a set of subsets of M (i.e., $S_1, \dots, S_n$ such that $S_i \subseteq M$ for $i = 1, \dots, n$) and a natural number $k \leq n$.
>
> *Question:* Are there k subsets $S_{j_1}, \dots, S_{j_k}$ such that $M = S_{j_1} \cup \dots \cup S_{j_k}$?

Prove that the set cover problem is NP-complete. For this, use the *vertex cover* problem which is NP-complete and defined as follows.

> *Given:* An undirected graph $G = (V, E)$ and a natural number k.
>
> *Question:* Is there a covering set of vertices of size k? (A covering set of vertices is a subset $V' \subseteq V$ such that for all edges $(u, v) \in E$: $u \in V'$ or $v \in V'$.)