Real-Time Systems

Lecture 03: Duration Calculus I

2014-05-08

Dr. Bernd Westphal

Albert-Ludwigs-Universität Freiburg, Germany

Contents & Goals

Last Lecture:

- Model of timed behaviour: state variables and their interpretation
- First order predicate-logic for requirements and system properties
- Classes of requirements (safety, liveness, etc.)

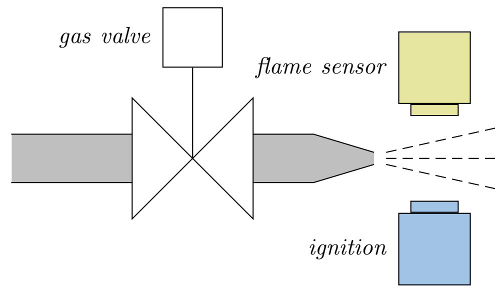
This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Read (and at best also write) Duration Calculus formulae.
- **Content:**
  - Duration Calculus:
    Assertions, Terms, Formulae, Abbreviations, Examples

Duration Calculus

Duration Calculus: Preview

- Duration Calculus is an **interval logic**.
- Formulae are evaluated in an (**implicitly given**) interval.



- $G, F, I, H : \{0, 1\}$
- Define $L : \{0, 1\}$ as $G \wedge \neg F$.

Strangest operators:

- **everywhere** — Example: $\lceil G \rceil$

  (Holds in a given interval $[b, e]$ iff the gas valve is open almost everywhere.)

- **chop** — Example: $(\lceil \neg I \rceil \,;\, \lceil I \rceil \,;\, \lceil \neg I \rceil) \implies \ell \geq 1$

  (Ignition phases last at least one time unit.)

- **integral** — Example: $\ell \geq 60 \implies \int L \leq \frac{\ell}{20}$

  (At most 5% leakage time within intervals of at least 60 time units.)

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$$f, g, \quad true, false, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \int P \mid f(\theta_1, \dots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \, ; F_2$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Symbols: Syntax

- f, g: **function symbols**, each with arity $n \in \mathbb{N}_0$.

  Called **constant** if $n = 0$.

  Assume: constants $0, 1, \dots \in \mathbb{N}_0$; binary '$+$' and '$\cdot$'.

- p, q: **predicate symbols**, also with arity.

  Assume: constants $true, false$; binary $=, <, >, \leq, \geq$.

- $x, y, z \in$ GVar: **global variables**.

- $X, Y, Z \in$ Obs: **state variables** or **observables**, each of a data type $\mathcal{D}$ (or $\mathcal{D}(X), \mathcal{D}(Y), \mathcal{D}(Z)$ to be precise).

  Called **boolean observable** if data type is $\{0, 1\}$.

- d: **elements** taken from data types $\mathcal{D}$ of observables.

Symbols: Semantics

- **Semantical domains** are
  - the **truth values** $\mathbb{B} = \{\mathsf{tt}, \mathsf{ff}\}$,
  - the **real numbers** $\mathbb{R}$,
  - **time** Time,
    (mostly $\mathsf{Time} = \mathbb{R}_0^+$ (continuous), exception $\mathsf{Time} = \mathbb{N}_0$ (discrete time))
  - and **data types** $\mathcal{D}$.

- The semantics of an n-ary **function symbol** f
  is a (mathematical) function from $\mathbb{R}^n$ to $\mathbb{R}$, denoted $\hat{f}$, i.e.

$$\hat{f} : \mathbb{R}^n \to \mathbb{R}.$$

- The semantics of an n-ary **predicate symbol** p
  is a function from $\mathbb{R}^n$ to $\mathbb{B}$, denoted $\hat{p}$, i.e.

$$\hat{p} : \mathbb{R}^n \to \mathbb{B}.$$

Symbols: Examples

- The **semantics** of the function and predicate symbols **assumed above** is fixed throughout the lecture:

  - $\hat{true} = \text{tt}$, $\hat{false} = \text{ff}$
  - $\hat{0} \in \mathbb{R}$ is the (real) number **zero**, etc.
  - $\hat{+} : \mathbb{R}^2 \to \mathbb{R}$ is the **addition** of real numbers, etc.
  - $\hat{=} : \mathbb{R}^2 \to \mathbb{B}$ is the **equality** relation on real numbers,
  - $\hat{<} : \mathbb{R}^2 \to \mathbb{B}$ is the **less-than** relation on real numbers, etc.

- "Since the semantics is the expected one, we shall often simply use the symbols $0, 1, +, \cdot, =, <$ when we mean their semantics $\hat{0}, \hat{1}, \hat{+}, \hat{\cdot}, \hat{=}, \hat{<}$."

Symbols: Semantics

- The semantics of a **global variable** is not fixed (throughout the lecture) but given by a **valuation**, i.e. a mapping

  $$\mathcal{V} : \mathsf{GVar} \to \mathbb{R}$$

  assigning each global variable $x \in \mathsf{GVar}$ a real number $\mathcal{V}(x) \in \mathbb{R}$.

  We use Val to denote the set of all valuations, i.e. $\mathsf{Val} = (\mathsf{GVar} \to \mathbb{R})$.

  Global variables are though **fixed over time** in system evolutions.

Symbols: Semantics

- The semantics of a **global variable** is not fixed (throughout the lecture) but given by a **valuation**, i.e. a mapping

$$\mathcal{V} : \mathsf{GVar} \to \mathbb{R}$$

assigning each global variable $x \in \mathsf{GVar}$ a real number $\mathcal{V}(x) \in \mathbb{R}$.

We use Val to denote the set of all valuations, i.e. $\mathsf{Val} = (\mathsf{GVar} \to \mathbb{R})$.

Global variables are though **fixed over time** in system evolutions.

- The semantics of a **state variable** is **time-dependent**.
It is given by an interpretation $\mathcal{I}$, i.e. a mapping

$$\mathcal{I} : \mathsf{Obs} \to (\mathsf{Time} \to \mathcal{D})$$

assigning each state variable $X \in \mathsf{Obs}$ a function

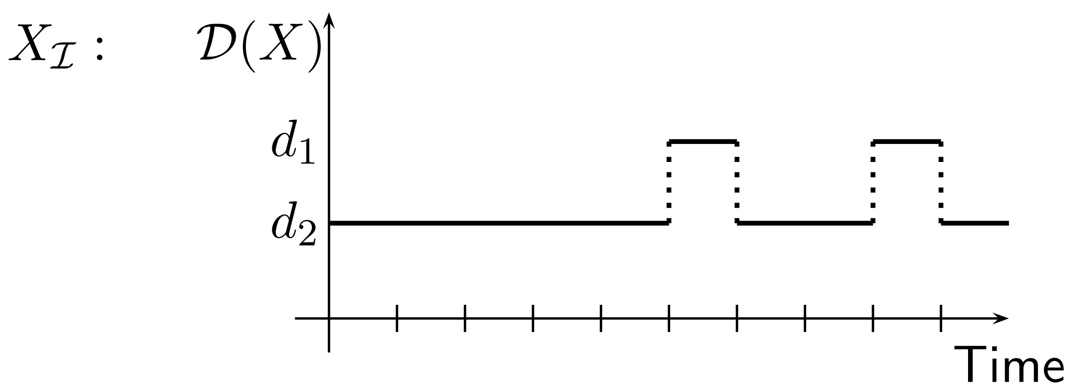
$$\mathcal{I}(X) : \mathsf{Time} \to \mathcal{D}(X)$$

such that $\mathcal{I}(X)(t) \in \mathcal{D}(X)$ denotes the value that X has at time $t \in \mathsf{Time}$.

Symbols: Representing State Variables

- For convenience, we shall abbreviate $\mathcal{I}(X)$ to $X_\mathcal{I}$.
- An **interpretation** (of a state variable) can be displayed in form of a **timing diagram**.

  For instance,



with $\mathcal{D}(X) = \{d_1, d_2\}$.

Duration Calculus: Overview

We will introduce three (or five) syntactical “levels”:

(i) **Symbols:**

$$f, g, \quad \mathit{true}, \mathit{false}, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \dots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

State Assertions: Syntax

- The set of **state assertions** is defined by the following grammar:

  $$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

  with $d \in \mathcal{D}(X)$.

  We shall use P, Q, R to denote state assertions.

- **Abbreviations**:
  - We shall write X instead of $X = 1$ if $\mathcal{D}(X) = \mathbb{B}$.
  - Define $\vee$, $\implies$, $\iff$ as usual.

State Assertions: Semantics

- The **semantics** of **state assertion** P is a function

$$\mathcal{I}[\![P]\!] : \mathsf{Time} \to \{0, 1\}$$

i.e. $\mathcal{I}[\![P]\!](t)$ denotes the truth value of P at time $t \in \mathsf{Time}$.

- The value is defined **inductively** on the structure of P:

$$\mathcal{I}[\![0]\!](t) = 0,$$

$$\mathcal{I}[\![1]\!](t) = 1,$$

$$\mathcal{I}[\![X = d]\!](t) = \begin{cases} 1 & \text{, if } X_{\mathcal{I}} = d \\ 0 & \text{, otherwise,} \end{cases}$$

$$\mathcal{I}[\![\neg P_1]\!](t) = 1 - \mathcal{I}[\![P_1]\!](t)$$

$$\mathcal{I}[\![P_1 \wedge P_2]\!](t) = \begin{cases} 1 & \text{, if } \mathcal{I}[\![P_1]\!](t) = \mathcal{I}[\![P_2]\!](t) = 1 \\ 0 & \text{, otherwise,} \end{cases}$$

State Assertions: Notes

- $\mathcal{I}[\![X]\!](t) = \mathcal{I}[\![X = 1]\!](t) = \mathcal{I}(X)(t) = X_\mathcal{I}(t)$, if X boolean.

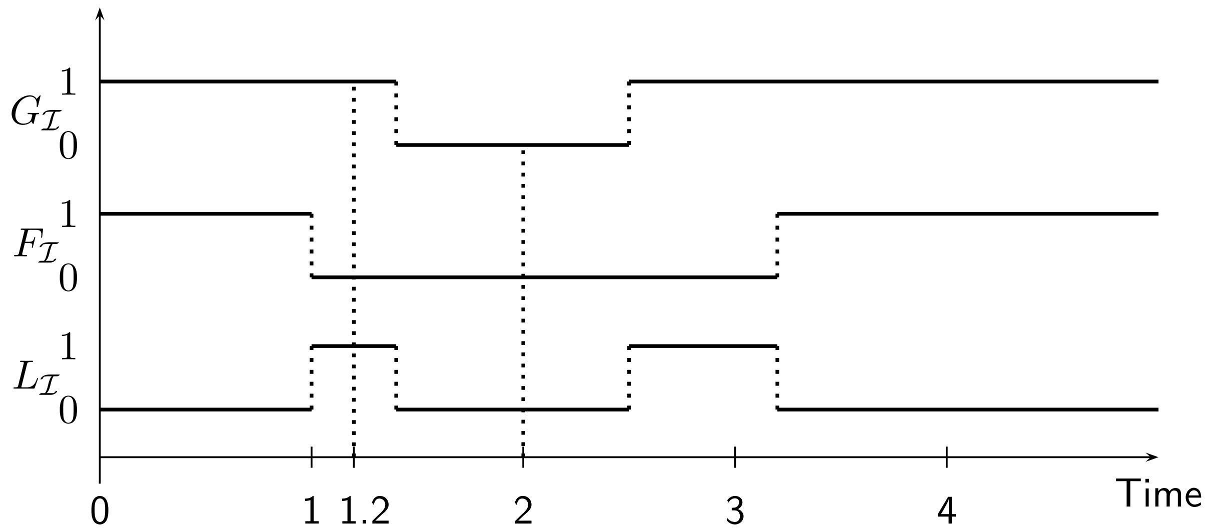
- $\mathcal{I}[\![P]\!]$ is also called **interpretation** of P.

  We shall write $P_\mathcal{I}$ for it.

- Here we prefer 0 and 1 as boolean values (instead of tt and ff) — for reasons that will become clear immediately.

State Assertions: Example

- Boolean observables G and F.
- State assertion $L := G \wedge \neg F$.



- $L_{\mathcal{I}}(1.2) = 1$, because

- $L_{\mathcal{I}}(2) = 0$, because

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$$f, g, \quad true, false, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \dots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall\, x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Terms: Syntax

- **Duration terms** (DC terms or just terms) are defined by the following grammar:

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \dots, \theta_n)$$

  where x is a global variable, ℓ and $\int$ are special symbols, P is a state assertion, and f a function symbol (of arity n).

- ℓ is called **length operator**, $\int$ is called **integral operator**

- Notation: we may write function symbols in **infix notation** as usual, i.e. write $\theta_1 + \theta_2$ instead of $+(\theta_1, \theta_2)$.

Terms: Syntax

- **Duration terms** (DC terms or just terms) are defined by the following grammar:

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \ldots, \theta_n)$$

  where x is a global variable, ℓ and $\int$ are special symbols, P is a state assertion, and f a function symbol (of arity n).

- ℓ is called **length operator**, $\int$ is called **integral operator**

- Notation: we may write function symbols in **infix notation** as usual, i.e. write $\theta_1 + \theta_2$ instead of $+(\theta_1, \theta_2)$.

Definition 1. [*Rigid*]
A term **without** length and integral symbols is called rigid.

Terms: Semantics

- Closed **intervals** in the time domain

$$\mathsf{Intv} := \{[b,e] \mid b,e \in \mathsf{Time} \text{ and } b \le e\}$$

Point intervals: $[b,b]$

Terms: Semantics

- The **semantics** of a **term** is a function

$$\mathcal{I}[\![\theta]\!] : \mathsf{Val} \times \mathsf{Intv} \to \mathbb{R}$$

i.e. $\mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$ is the real number that θ denotes under interpretation $\mathcal{I}$ and valuation $\mathcal{V}$ in the interval $[b, e]$.

- The value is defined **inductively** on the structure of θ:

$$\mathcal{I}[\![x]\!](\mathcal{V}, [b, e]) = \mathcal{V}(x),$$

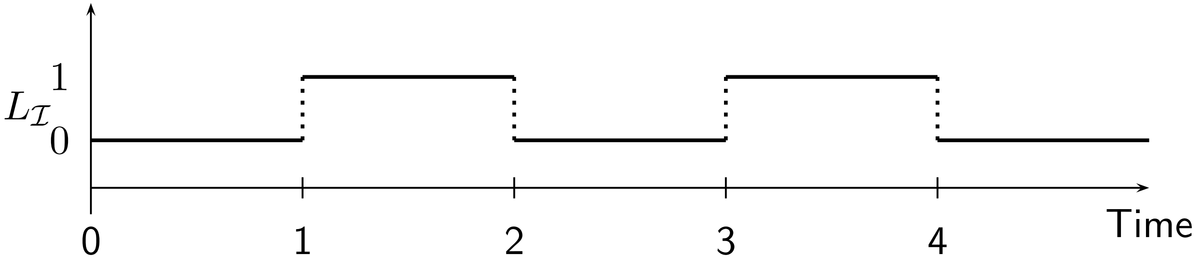
$$\mathcal{I}[\![\ell]\!](\mathcal{V}, [b, e]) = e - b,$$

$$\mathcal{I}[\![\textstyle\int P]\!](\mathcal{V}, [b, e]) = \int_b^e P_{\mathcal{I}}(t)\, dt,$$

$$\mathcal{I}[\![f(\theta_1, \dots, \theta_n)]\!](\mathcal{V}, [b, e]) = \hat{f}(\mathcal{I}[\![\theta_1]\!](\mathcal{V}, [b, e]), \dots, \mathcal{I}[\![\theta_n]\!](\mathcal{V}, [b, e])),$$

Terms: Example

$$\theta = x \cdot \int L$$



$$\mathcal{V}(x) = 20.$$

Terms: Semantics Well-defined?

- So, $\mathcal{I}[\![\int P]\!](\mathcal{V},[b,e])$ is $\displaystyle\int_b^e P_\mathcal{I}(t)\,dt$ — but does the integral always exist?

Terms: Semantics Well-defined?

- So, $\mathcal{I}[\![\int P]\!](\mathcal{V}, [b, e])$ is $\displaystyle\int_b^e P_\mathcal{I}(t)\,dt$ — but does the integral always exist?
- IOW: is there a $P_\mathcal{I}$ which is not (Riemann-)integrable?

Terms: Semantics Well-defined?

- So, $\mathcal{I}[\![\int P]\!](\mathcal{V}, [b, e])$ is $\displaystyle\int_b^e P_\mathcal{I}(t)\,dt$ — but does the integral always exist?
- IOW: is there a $P_\mathcal{I}$ which is not (Riemann-)integrable? Yes. For instance

$$P_\mathcal{I}(t) = \begin{cases} 1 & \text{, if } t \in \mathbb{Q} \\ 0 & \text{, if } t \notin \mathbb{Q} \end{cases}$$

Terms: Semantics Well-defined?

- So, $\mathcal{I}[\![\int P]\!](\mathcal{V}, [b,e])$ is $\int_b^e P_\mathcal{I}(t)\,dt$ — but does the integral always exist?
- IOW: is there a $P_\mathcal{I}$ which is not (Riemann-)integrable? Yes. For instance

$$P_\mathcal{I}(t) = \begin{cases} 1 & \text{, if } t \in \mathbb{Q} \\ 0 & \text{, if } t \notin \mathbb{Q} \end{cases}$$

- To exclude such functions, DC considers only interpretations $\mathcal{I}$ satisfying the following condition of **finite variability**:

  For each state variable X and each interval $[b,e]$ there is a **finite partition** of $[b,e]$ such that the interpretation $X_\mathcal{I}$ is **constant on each part**.

  Thus on each interval $[b,e]$ the function $X_\mathcal{I}$ has only **finitely many points of discontinuity**.

Terms: Remarks

Remark 2.5. The semantics $\mathcal{I}[\![\theta]\!]$ of a term is insensitive against changes of the interpretation $\mathcal{I}$ at individual time points.

Remark 2.6. The semantics $\mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$ of a **rigid** term does not depend on the interval $[b, e]$.

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$$f, g, \quad \mathit{true}, \mathit{false}, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \dots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall\, x \bullet F_1 \mid F_1 \,;\, F_2$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

Formulae: Syntax

- The set of **DC formulae** is defined by the following grammar:

$$F ::= p(\theta_1, \dots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall\, x \bullet F_1 \mid F_1 \,;\, F_2$$

  where p is a predicate symbol, θ_i a term, x a global variable.

- **chop operator**: '**;**'
- **atomic formula**: $p(\theta_1, \dots, \theta_n)$
- **rigid formula**: all terms are rigid
- **chop free**: '**;**' doesn't occur
- usual notion of **free** and **bound** (global) variables

- Note: quantification only over (**first-order**) global variables, not over (**second-order**) state variables.

Formulae: Priority Groups

- To avoid parentheses, we define the following five priority groups from highest to lowest priority:
  - $\neg$ **(negation)**
  - **;** **(chop)**
  - $\wedge$, $\vee$ **(and/or)**
  - $\implies$, $\iff$ **(implication/equivalence)**
  - $\exists$, $\forall$ **(quantifiers)**

Examples:

- $\neg F \,\mathbf{;}\, F \vee H$
- $\forall x \bullet F \wedge G$

Syntactic Substitution...

...of a term θ for a variable x in a formula F.

- We use

$$F[x := \theta]$$

to denote the formula that results from performing the following steps:

(i) transform F into $\tilde{F}$ by (consistently) renaming bound variables such that no free occurrence of x in $\tilde{F}$ appears within a quantified subformula $\exists\, z \bullet G$ or $\forall\, z \bullet G$ for some z occurring in θ,

(ii) textually replace all free occurrences of x in $\tilde{F}$ by θ.

Examples: $F := (x \geq y \implies \exists\, z \bullet z \geq 0 \wedge x = y + z)$, $\quad \theta_1 := \ell$, $\theta_2 := \ell + z$,

- $F[x := \theta_1] = (x \geq y \implies \exists\, z \bullet z \geq 0 \wedge x = y + z)$

- $F[x := \theta_2] = (x \quad \geq y \implies \exists\, z \bullet z \geq 0 \wedge x \quad = y + z)$

Formulae: Semantics

- The **semantics** of a **formula** is a function

$$\mathcal{I}[\![F]\!] : \mathsf{Val} \times \mathsf{Intv} \to \{\mathsf{tt}, \mathsf{ff}\}$$

i.e. $\mathcal{I}[\![F]\!](\mathcal{V}, [b, e])$ is the truth value of F under interpretation $\mathcal{I}$ and valuation $\mathcal{V}$ in the interval $[b, e]$.

- This value is defined **inductively** on the structure of F:

$$\mathcal{I}[\![p(\theta_1, \dots, \theta_n)]\!](\mathcal{V}, [b, e]) = \hat{p}(\mathcal{I}[\![\theta_1]\!](\mathcal{V}, [b, e]), \dots, \mathcal{I}[\![\theta_n]\!](\mathcal{V}, [b, e])),$$

$$\mathcal{I}[\![\neg F_1]\!](\mathcal{V}, [b, e]) = \mathsf{tt} \text{ iff } \mathcal{I}[\![F_1]\!](\mathcal{V}, [b, e]) = \mathsf{ff},$$

$$\mathcal{I}[\![F_1 \wedge F_2]\!](\mathcal{V}, [b, e]) = \mathsf{tt} \text{ iff } \mathcal{I}[\![F_1]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F_2]\!](\mathcal{V}, [b, e]) = \mathsf{tt},$$

$$\mathcal{I}[\![\forall\, x \bullet F_1]\!](\mathcal{V}, [b, e]) = \mathsf{tt} \text{ iff for all } a \in \mathbb{R},$$

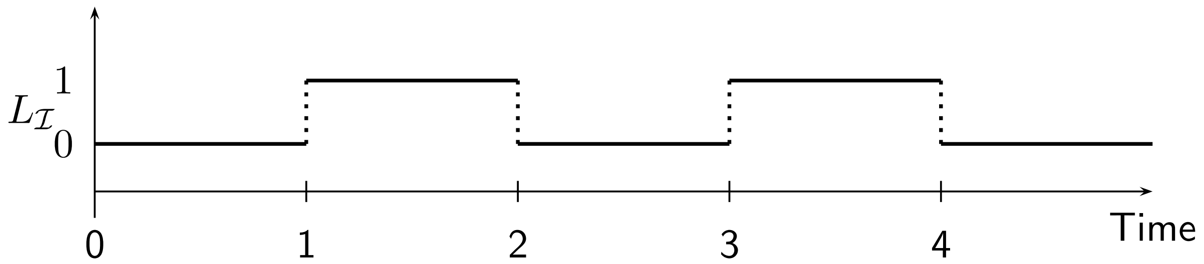
$$\mathcal{I}[\![F_1[x := a]]\!](\mathcal{V}, [b, e]) = \mathsf{tt}$$

$$\mathcal{I}[\![F_1 \,;\, F_2]\!](\mathcal{V}, [b, e]) = \quad \text{iff there is an } m \in [b, e] \text{ such that}$$

$$\mathcal{I}[\![F_1]\!](\mathcal{V}, [b, m]) = \mathcal{I}[\![F_2]\!](\mathcal{V}, [m, e]) = \mathsf{tt}.$$

Formulae: Example

$$F := \int L = 0 \,;\, \int L = 1$$



- $\mathcal{I}[\![F]\!](\mathcal{V}, [0, 2]) =$

Formulae: Remarks

Remark 2.10. [*Rigid and chop-free*] Let F be a duration formula, $\mathcal{I}$ an interpretation, $\mathcal{V}$ a valuation, and $[b, e] \in \mathsf{Intv}$.

- If F is **rigid**, then

$$\forall\,[b', e'] \in \mathsf{Intv} : \mathcal{I}[\![F]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F]\!](\mathcal{V}, [b', e']).$$

- If F is **chop-free** or θ is **rigid**,
  then in the calculation of the semantics of F,
  every occurrence of θ denotes the same value.

Substitution Lemma

Lemma 2.11. [*Substitution*]
Consider a formula F, a global variable x, and a term θ such that F is **chop-free** or θ is **rigid**.

Then for all interpretations $\mathcal{I}$, valuations $\mathcal{V}$, and intervals $[b, e]$,

$$\mathcal{I}[\![F[x := \theta]]\!](\mathcal{V}, [b, e]) = \mathcal{I}[\![F]\!](\mathcal{V}[x := a], [b, e])$$

where $a = \mathcal{I}[\![\theta]\!](\mathcal{V}, [b, e])$.

- $F := \ell = x \,\mathbf{;}\, \ell = x \implies \ell = 2 \cdot x, \quad \theta := \ell$

Duration Calculus: Overview

We will introduce three (or five) syntactical "levels":

(i) **Symbols:**

$$f, g, \quad \mathit{true}, \mathit{false}, =, <, >, \leq, \geq, \quad x, y, z, \quad X, Y, Z, \quad d$$

(ii) **State Assertions:**

$$P ::= 0 \mid 1 \mid X = d \mid \neg P_1 \mid P_1 \wedge P_2$$

(iii) **Terms:**

$$\theta ::= x \mid \ell \mid \textstyle\int P \mid f(\theta_1, \ldots, \theta_n)$$

(iv) **Formulae:**

$$F ::= p(\theta_1, \ldots, \theta_n) \mid \neg F_1 \mid F_1 \wedge F_2 \mid \forall x \bullet F_1 \mid F_1 \,\mathbf{;}\, F_2$$

(v) **Abbreviations:**

$$\lceil\,\rceil, \quad \lceil P \rceil, \quad \lceil P \rceil^t, \quad \lceil P \rceil^{\leq t}, \quad \Diamond F, \quad \Box F$$

References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.