Real-Time Systems

Lecture 05: Duration Calculus III

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Contents & Goals

Last Lecture:

- DC Syntax and Semantics: Formulae

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Read (and at best also write) Duration Calculus formulae – including abbreviations.
  - What is Validity/Satisfiability/Realisability for DC formulae?
  - How can we prove a design correct?
- **Content:**
  - Duration Calculus Abbreviations
  - Basic Properties
  - Validity, Satisfiability, Realisability
  - Correctness Proofs: Gas Burner

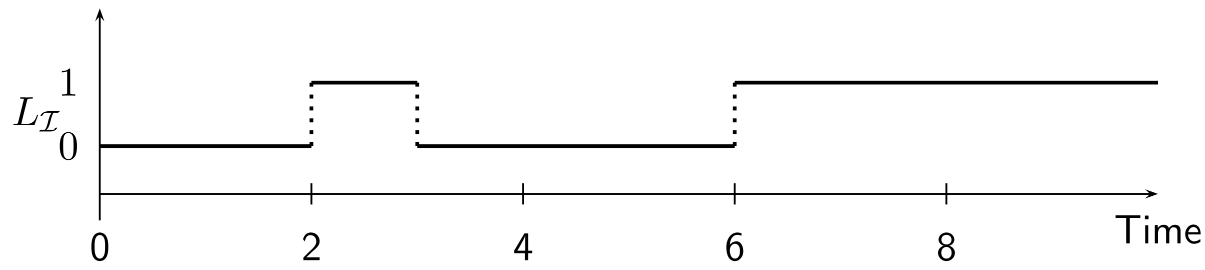
Duration Calculus Abbreviations

Abbreviations

- $\lceil\,\rceil := \ell = 0$ **(point interval)**
- $\lceil P \rceil := \int P = \ell \wedge \ell > 0$ **(almost everywhere)**
- $\lceil P \rceil^t := \lceil P \rceil \wedge \ell = t$ **(for time t)**
- $\lceil P \rceil^{\leq t} := \lceil P \rceil \wedge \ell \leq t$ **(up to time t)**

- $\Diamond F := \mathit{true} \,\mathbf{;}\, F \,\mathbf{;}\, \mathit{true}$ **(for some subinterval)**
- $\Box F := \neg \Diamond \neg F$ **(for all subintervals)**

Abbreviations: Examples



$$
\begin{array}{llll}
\mathcal{I}[\![& \int L = 0 &]\!](\mathcal{V}, & [0,2] \quad) = \\
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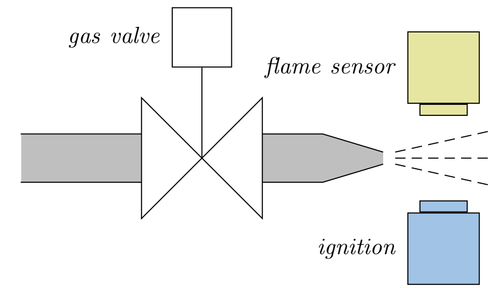
\mathcal{I}[\![& \Diamond \lceil \neg L \rceil^2 \,;\, \lceil \neg L \rceil^1 \,;\, \lceil \neg L \rceil^3 &]\!](\mathcal{V}, & [0,6] \quad) =
\end{array}
$$

Duration Calculus: Looking Back



- $G, F, I, H : \{0, 1\}$
- Define $L : \{0, 1\}$ as $G \wedge \neg F$.

- Duration Calculus is an **interval logic**.
- Formulae are evaluated in an (**implicitly given**) interval.

Strangest operators:

- **almost everywhere** — Example: $\lceil G \rceil$

  (Holds in a given interval $[b, e]$ iff the gas valve is open almost everywhere.)

- **chop** — Example: $(\lceil \neg I \rceil \,;\, \lceil I \rceil \,;\, \lceil \neg I \rceil) \implies \ell \geq 1$

  (Ignition phases last at least one time unit.)

- **integral** — Example: $\ell \geq 60 \implies \int L \leq \frac{\ell}{20}$

  (At most 5% leakage time within intervals of at least 60 time units.)

DC Validity, Satisfiability, Realisability

Validity, Satisfiability, Realisability

Let $\mathcal{I}$ be an interpretation, $\mathcal{V}$ a valuation, $[b, e]$ an interval, and F a DC formula.

- $\mathcal{I}, \mathcal{V}, [b, e] \models F$ ("F **holds** in $\mathcal{I}$, $\mathcal{V}$, $[b, e]$") iff $\mathcal{I}[\![F]\!](\mathcal{V}, [b, e]) = \mathrm{tt}$.

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- $\mathcal{I} \models F$ ("$\mathcal{I}$ **realises** F") iff $\forall\, \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models F$.

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- F is called **realisable** iff some $\mathcal{I}$ and $\mathcal{V}$ realise F.

- $\mathcal{I} \models F$ ("$\mathcal{I}$ **realises** F") iff $\forall\, \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models F$.

- $\models F$ ("F is **valid**") iff $\forall$ interpretation $\mathcal{I} : \mathcal{I} \models F$.

Validity vs. Satisfiability vs. Realisability

Remark 2.13. For all DC formulae F,

- F is satisfiable iff $\neg F$ is not valid,
  F is valid iff $\neg F$ is not satisfiable.
- If F is valid then F is realisable, but not vice versa.
- If F is realisable then F is satisfiable, but not vice versa.

Examples: Valid? Realisable? Satisfiable?

- $\ell \geq 0$
- $\ell = \int 1$
- $\ell = 30 \iff \ell = 10 \,; \ell = 20$
- $((F \,; G) \,; H) \iff (F \,; (G \,; H))$
- $\int L \leq x$
- $\ell = 2$

Initial Values

- $\mathcal{I}, \mathcal{V} \models_0 F$ ("$\mathcal{I}$ and $\mathcal{V}$ **realise** F **from** 0") iff

$$\forall\, t \in \mathsf{Time} : \mathcal{I}, \mathcal{V}, [0, t] \models F.$$

- F is called **realisable from** 0 iff some $\mathcal{I}$ and $\mathcal{V}$ realise F from 0.

- Intervals of the form $[0, t]$ are called **initial intervals**.

- $\mathcal{I} \models_0 F$ ("$\mathcal{I}$ **realises** F **from** 0") iff $\forall\, \mathcal{V} \in \mathsf{Val} : \mathcal{I}, \mathcal{V} \models_0 F$.

- $\models_0 F$ ("F is **valid from** 0") iff $\forall$ interpretation $\mathcal{I} : \mathcal{I} \models_0 F$.

Initial or not Initial...

For all interpretations $\mathcal{I}$, valuations $\mathcal{V}$, and DC formulae F,

(i) $\mathcal{I}, \mathcal{V} \models F$ implies $\mathcal{I}, \mathcal{V} \models_0 F$,

(ii) if F is realisable then F is realisable from 0, but not vice versa,

(iii) F is valid iff F is valid from 0.

Specification and Semantics-based Correctness Proofs of Real-Time Systems with DC

Methodology: Ideal World...

(i) Choose a collection of **observables** ‘Obs’.

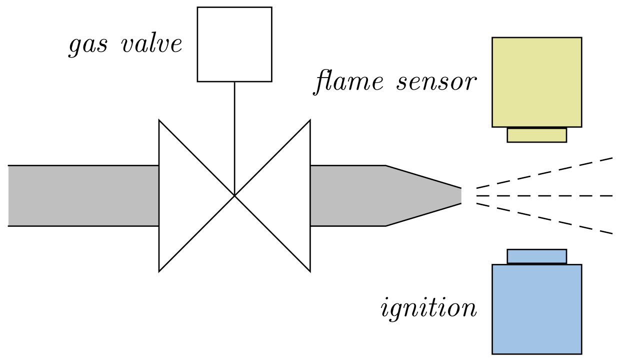
(ii) Provide the **requirement**/**specification** ‘Spec’ as a conjunction of DC formulae (over ‘Obs’).

(iii) Provide a description ‘Ctrl’ of the **controller** in form of a DC formula (over ‘Obs’).

(iv) We say ‘Ctrl’ is **correct** (wrt. ‘Spec’) iff

$$\models_0 \mathsf{Ctrl} \implies \mathsf{Spec}.$$

Gas Burner Revisited



(i) Choose **observables**:

- two boolean observables G and F
  (i.e. Obs $= \{G, F\}$, $\mathcal{D}(G) = \mathcal{D}(F) = \{0, 1\}$)
- $G = 1$: gas valve open (output)
- $F = 1$: have flame (input)
- define $L := G \wedge \neg F$ (leakage)

(ii) Provide the **requirement**:

$$\text{Req} :\iff \square(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)$$

Gas Burner Revisited

(iii) Provide a description 'Ctrl'
of the **controller** in form of a DC formula (over 'Obs').
Here, firstly consider a **design**:

- Des-1 : $\iff \Box(\lceil L \rceil \implies \ell \leq 1)$

- Des-2 : $\iff \Box(\lceil L \rceil \,;\, \lceil \neg L \rceil \,;\, \lceil L \rceil \implies \ell > 30)$

(iv) Prove **correctness**:

- We want (or do we want $\models_0$...?):

$$\models (\text{Des-1} \wedge \text{Des-2} \implies \text{Req}) \qquad \text{(Thm. 2.16)}$$

Gas Burner Revisited

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$$\models (\text{Des-1} \wedge \text{Des-2} \implies \text{Req}) \qquad \text{(Thm. 2.16)}$$

- We do show

$$\models \text{Req-1} \implies \text{Req} \qquad \text{(Lem. 2.17)}$$

with the simplified requirement

$$\text{Req-1} := \Box(\ell \leq 30 \implies \textstyle\int L \leq 1),$$

Gas Burner Revisited: Lemma 2.17

Claim:

$$\models \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}} \implies \underbrace{\Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)}_{\text{Req}}$$

Proof:

Gas Burner Revisited: Lemma 2.17

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Proof:

- Assume ‘Req-1’.

Gas Burner Revisited: Lemma 2.17

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Proof:

- Assume 'Req-1'.
- Let $L_{\mathcal{I}}$ be any interpretation of L, and $[b, e]$ an interval with $e - b \geq 60$.

Gas Burner Revisited: Lemma 2.17

Claim:

$$\models \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}} \implies \underbrace{\Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)}_{\text{Req}}$$

Proof:

- Assume ‘Req-1’.
- Let $L_\mathcal{I}$ be any interpretation of L, and $[b,e]$ an interval with $e - b \geq 60$.
- Show “$20 \cdot \int L \leq \ell$”, i.e.

$$\mathcal{I}[\![20 \cdot \textstyle\int L \leq \ell]\!](\mathcal{V},[b,e]) = \mathrm{tt}$$

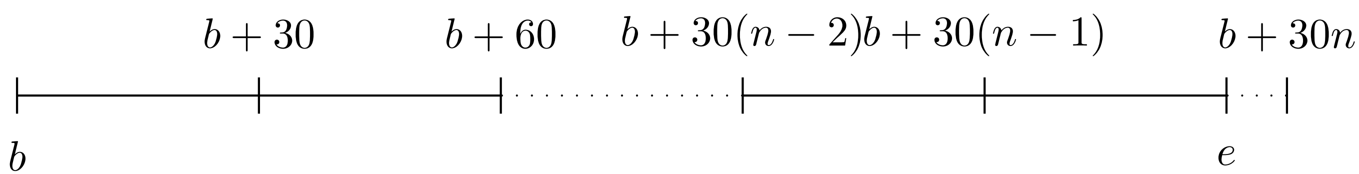
i.e.

$$\hat{20} \mathbin{\hat{\cdot}} \int_b^e L_\mathcal{I}(t)\,dt \mathrel{\hat{\leq}} (e-b)$$

Gas Burner Revisited: Lemma 2.17

$$\models \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}} \implies \Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)$$

- Set $n := \lceil \frac{e-b}{30} \rceil$, i.e. $n \in \mathbb{N}$ with $n - 1 < \frac{e-b}{30} \leq n$, and split the interval



Some Laws of the DC Integral Operator

Theorem 2.18.
For all state assertions P and all real numbers $r_1, r_2 \in \mathbb{R}$,

(i) $\models \int P \leq \ell$,

(ii) $\models (\int P = r_1)\,;(\int P = r_2) \implies \int P = r_1 + r_2$,

(iii) $\models \lceil \neg P \rceil \implies \int P = 0$,

(iv) $\models \lceil\rceil \implies \int P = 0$.

Gas Burner Revisited: Lemma 2.18

Claim:

$$\models (\underbrace{\Box(\lceil L \rceil \implies \ell \leq 1)}_{\text{Des-1}} \land \underbrace{\Box(\lceil L \rceil \,;\, \lceil \neg L \rceil \,;\, \lceil L \rceil \implies \ell > 30)}_{\text{Des-2}}) \implies \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}}$$

Proof:

Gas Burner Revisited: Lemma 2.

(i) $\models \int P \leq \ell$, (iv) $\models \lceil\rceil \implies \int P = 0$
(ii) $\models (\int P = r_1) \,;\, (\int P = r_2) \implies \int P = r_1 + r_2$,
(iii) $\models \lceil \neg P \rceil \implies \int P = 0$,

Claim:

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Proof:

Gas Burner Revisited: Lemma 2.18

Obstacles in Non-Ideal World

Methodology: The World is Not Ideal...

(i) Choose a collection of **observables** 'Obs'.

(ii) Provide **specification** 'Spec' (conjunction of DC formulae (over 'Obs')).

(iii) Provide a description 'Ctrl' of the **controller** (DC formula (over 'Obs')).

(iv) Prove 'Ctrl' is **correct** (wrt. 'Spec').

That looks **too simple to be practical**. Typical **obstacles**:

(i) It may be impossible to realise 'Spec'
if it doesn't consider properties of **the plant**.

(ii) There are typically intermediate **design levels** between 'Spec' and 'Ctrl'.

(iii) 'Spec' and 'Ctrl' may use **different observables**.

(iv) **Proving** validity of the implication is not trivial.

(i) Assumptions As A Form of Plant Model

- Often the controller will (or can) operate correctly only under some **assumptions**.
- For instance, with a level crossing
  - we may assume an upper bound on the speed of approaching trains, (otherwise we'd need to close the gates arbitrarily fast)
  - we may assume that trains are not arbitrarily slow in the crossing, (otherwise we can't make promises to the road traffic)

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- We shall specify such assumptions as a DC formula 'Asm' on the **input observables** and verify correctness of 'Ctrl' wrt. 'Spec' by proving validity (from 0) of

$$\mathsf{Ctrl} \wedge \mathsf{Asm} \implies \mathsf{Spec}$$

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- Shall we **care** whether 'Asm' is satisfiable?

(ii) Intermediate Design Levels

- A top-down development approach may involve
  - Spec — specification/requirements
  - Des — design
  - Ctrl — implementation
- Then correctness is established by proving validity of

$$\text{Ctrl} \implies \text{Des} \tag{1}$$

  and

$$\text{Des} \implies \text{Spec} \tag{2}$$

  (then concluding Ctrl $\implies$ Spec by transitivity)
- Any preference on the order?

(iii): Different Observables

- Assume, 'Spec' uses more abstract observables Obs_A and 'Ctrl' more concrete ones Obs_C.
- For instance:
  - in Obs_A: only consider gas valve open or closed $(\mathcal{D}(G) = \{0,1\})$
  - in Obs_C: may control two valves and care for intermediate positions, for instance, to react to different heating requests $(\mathcal{D}(G_1) = \{0,1,2,3\}, \mathcal{D}(G_2) = \{0,1,2,3\})$

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    ($\mathcal{D}(G_1) = \{0, 1, 2, 3\}, \mathcal{D}(G_2) = \{0, 1, 2, 3\}$)
- To prove correctness, we need information how the observables are related — an **invariant** which **links** the data values of Obs_A and Obs_C.
- **If** we're given the linking invariant as a DC formula, say '$\mathsf{Link}_{C,A}$', **then** proving correctness of 'Ctrl' wrt. 'Spec' amounts to proving validity (from 0) of

$$\mathsf{Ctrl} \wedge \mathsf{Link}_{C,A} \implies \mathsf{Spec}.$$

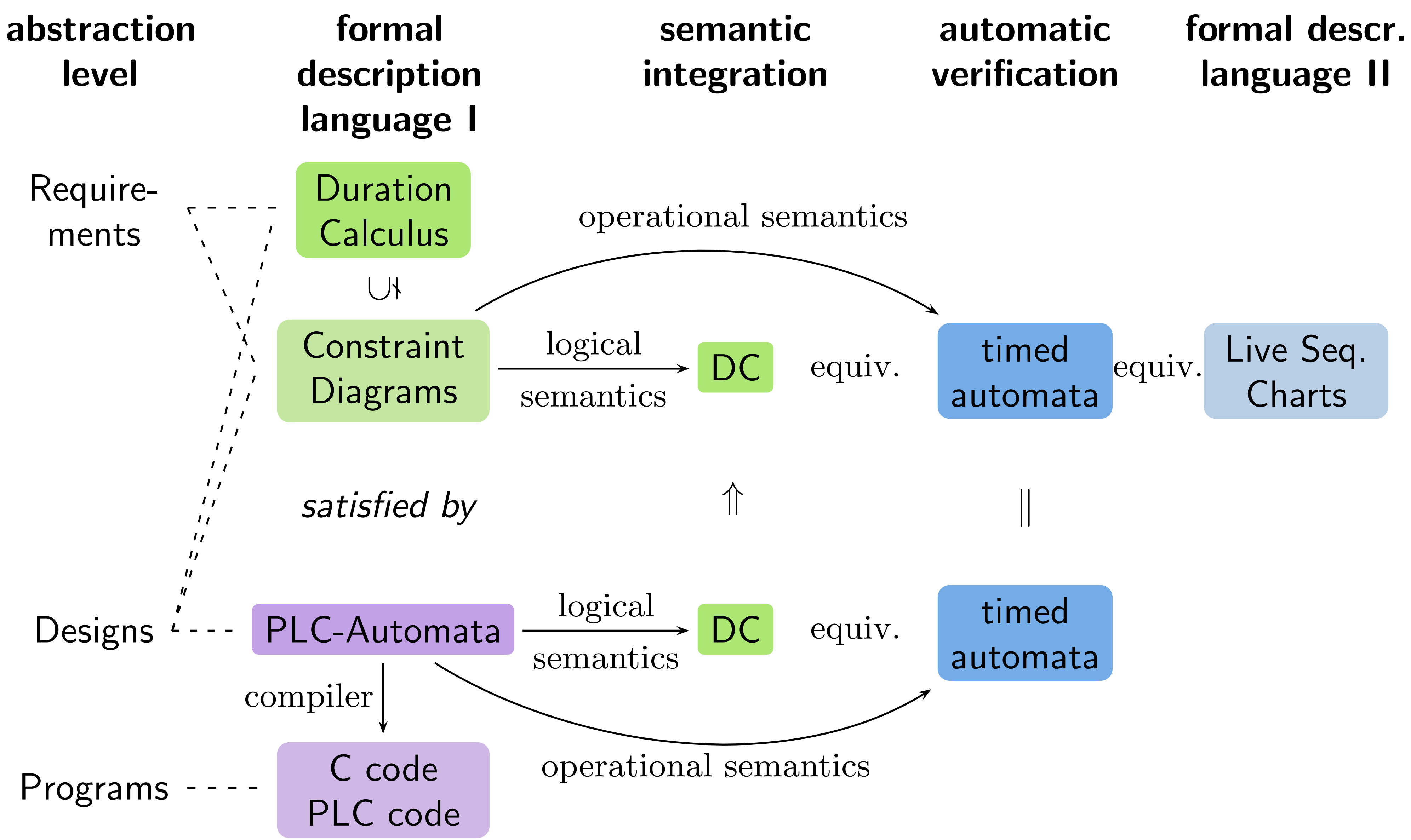
- For instance,

$$\mathsf{Link}_{C,A} = \lceil G \iff (G_1 + G_2 > 0) \rceil$$

Obstacle (iv): How to Prove Correctness?

- by hand on the basis of DC semantics,
- maybe supported by proof rules,
- sometimes a general theorem may fit (e.g. cycle times of PLC automata),
- algorithms as in Uppaal.

Recall: Tying It All Together



References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.