Real-Time Systems

Lecture 06: DC Properties I

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Dr. Bernd Westphal

Albert-Ludwigs-Universität Freiburg, Germany

Contents & Goals

Last Lecture:

- DC Syntax and Semantics: Abbreviations ("almost everywhere")
- Satisfiable/Realisable/Valid (from 0)

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - What are obstacles on proving a design correct in the real-world, and how to overcome them?
  - Facts: decidability properties.
  - What's the idea of the considered (un)decidability proofs?

- **Content:**
  - Semantical Correctness Proof
  - (Un-)Decidable problems of DC variants in discrete and continuous time

Specification and Semantics-based Correctness Proofs of Real-Time Systems with DC

Methodology: Ideal World...

(i) Choose a collection of **observables** 'Obs'.

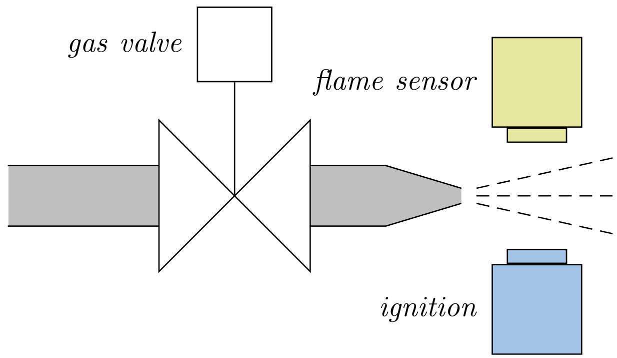
(ii) Provide the **requirement**/**specification** 'Spec' as a conjunction of DC formulae (over 'Obs').

(iii) Provide a description 'Ctrl' of the **controller** in form of a DC formula (over 'Obs').

(iv) We say 'Ctrl' is **correct** (wrt. 'Spec') iff

$$\models_0 \mathsf{Ctrl} \implies \mathsf{Spec}.$$

Gas Burner Revisited



(i) Choose **observables**:

- two boolean observables G and F
  (i.e. Obs $= \{G, F\}$, $\mathcal{D}(G) = \mathcal{D}(F) = \{0, 1\}$)
- $G = 1$: gas valve open (output)
- $F = 1$: have flame (input)
- define $L := G \wedge \neg F$ (leakage)

(ii) Provide the **requirement**:

$$\text{Req} :\iff \square(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)$$

Gas Burner Revisited

(iii) Provide a description 'Ctrl'
of the **controller** in form of a DC formula (over 'Obs').
Here, firstly consider a **design**:

- Des-1 : $\iff \Box(\lceil L \rceil \implies \ell \leq 1)$
- Des-2 : $\iff \Box(\lceil L \rceil \,;\, \lceil \neg L \rceil \,;\, \lceil L \rceil \implies \ell > 30)$

(iv) Prove **correctness**:

- We want (or do we want $\models_0 \ldots$?):

$$\models (\text{Des-1} \wedge \text{Des-2} \implies \text{Req}) \qquad \text{(Thm. 2.16)}$$

- We do show

$$\models \text{Req-1} \implies \text{Req} \qquad \text{(Lem. 2.17)}$$

  with the simplified requirement

$$\text{Req-1} := \Box(\ell \leq 30 \implies \textstyle\int L \leq 1),$$

- and we show

$$\models (\text{Des-1} \wedge \text{Des-2}) \implies \text{Req-1}. \qquad \text{(Lem. 2.19)}$$

Gas Burner Revisited: Lemma 2.17

Claim:

$$\models \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}} \implies \underbrace{\Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)}_{\text{Req}}$$

Proof:

Gas Burner Revisited: Lemma 2.17

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Proof:

- Assume 'Req-1'.

Gas Burner Revisited: Lemma 2.17

Claim:

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Proof:

- Assume 'Req-1'.
- Let $L_\mathcal{I}$ be any interpretation of L, and $[b, e]$ an interval with $e - b \geq 60$.

Gas Burner Revisited: Lemma 2.17

Claim:

$$\models \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}} \implies \underbrace{\Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)}_{\text{Req}}$$

Proof:

- Assume 'Req-1'.
- Let $L_\mathcal{I}$ be any interpretation of L, and $[b, e]$ an interval with $e - b \geq 60$.
- Show "$20 \cdot \int L \leq \ell$", i.e.

$$\mathcal{I}[\![20 \cdot \textstyle\int L \leq \ell]\!](\mathcal{V}, [b, e]) = \mathrm{tt}$$

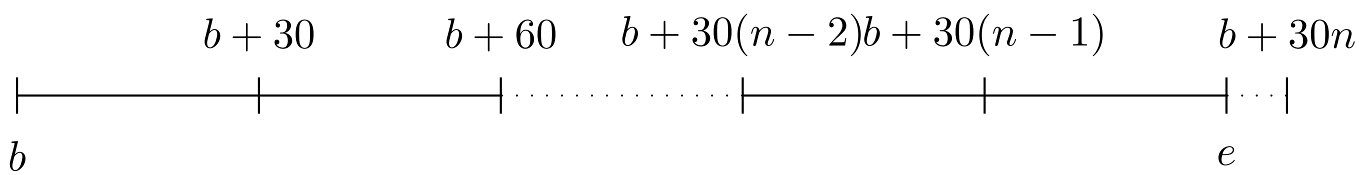
i.e.

$$\hat{20} \;\hat{\cdot} \int_b^e L_\mathcal{I}(t)\, dt \;\hat{\leq}\; (e - b)$$

Gas Burner Revisited: Lemma 2.17

$$\models \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}} \implies \Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)$$

- Set $n := \lceil \frac{e-b}{30} \rceil$, i.e. $n \in \mathbb{N}$ with $n - 1 < \frac{e-b}{30} \leq n$, and split the interval



Some Laws of the DC Integral Operator

Theorem 2.18.
For all state assertions P and all real numbers $r_1, r_2 \in \mathbb{R}$,

(i) $\models \int P \leq \ell$,

(ii) $\models (\int P = r_1) \,;\, (\int P = r_2) \implies \int P = r_1 + r_2$,

(iii) $\models \lceil \neg P \rceil \implies \int P = 0$,

(iv) $\models \lceil\rceil \implies \int P = 0$.

Gas Burner Revisited: Lemma 2.18

Claim:

$$\models (\underbrace{\Box(\lceil L \rceil \implies \ell \leq 1)}_{\text{Des-1}} \wedge \underbrace{\Box(\lceil L \rceil \,;\, \lceil \neg L \rceil \,;\, \lceil L \rceil \implies \ell > 30)}_{\text{Des-2}}) \implies \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}}$$

Proof:

Gas Burner Revisited: Lemma 2.

(i) $\models \int P \leq \ell$, (iv) $\models \lceil\rceil \implies \int P = 0$
(ii) $\models (\int P = r_1) \,;\, (\int P = r_2)$
$\implies \int P = r_1 + r_2$,
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Proof:

Gas Burner Revisited: Lemma 2.18

Obstacles in Non-Ideal World

Methodology: The World is Not Ideal...

(i) Choose a collection of **observables** ‘Obs’.

(ii) Provide **specification** ‘Spec’ (conjunction of DC formulae (over ‘Obs’)).

(iii) Provide a description ‘Ctrl’ of the **controller** (DC formula (over ‘Obs’)).

(iv) Prove ‘Ctrl’ is **correct** (wrt. ‘Spec’).

That looks **too simple to be practical**. Typical **obstacles**:

(i) It may be impossible to realise ‘Spec’
if it doesn’t consider properties of **the plant**.

(ii) There are typically intermediate **design levels** between ‘Spec’ and ‘Ctrl’.

(iii) ‘Spec’ and ‘Ctrl’ may use **different observables**.

(iv) **Proving** validity of the implication is not trivial.

(i) Assumptions As A Form of Plant Model

- Often the controller will (or can) operate correctly only under some **assumptions**.
- For instance, with a level crossing
  - we may assume an upper bound on the speed of approaching trains, (otherwise we'd need to close the gates arbitrarily fast)
  - we may assume that trains are not arbitrarily slow in the crossing, (otherwise we can't make promises to the road traffic)

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- We shall specify such assumptions as a DC formula 'Asm' on the **input observables** and verify correctness of 'Ctrl' wrt. 'Spec' by proving validity (from 0) of

$$\mathsf{Ctrl} \wedge \mathsf{Asm} \implies \mathsf{Spec}$$

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- Shall we **care** whether 'Asm' is satisfiable?

(ii) Intermediate Design Levels

- A top-down development approach may involve
  - Spec — specification/requirements
  - Des — design
  - Ctrl — implementation
- Then correctness is established by proving validity of

$$\text{Ctrl} \implies \text{Des} \tag{1}$$

  and

$$\text{Des} \implies \text{Spec} \tag{2}$$

  (then concluding $\text{Ctrl} \implies \text{Spec}$ by transitivity)
- Any preference on the order?

(iii): Different Observables

- Assume, 'Spec' uses more abstract observables Obs_A and 'Ctrl' more concrete ones Obs_C.
- For instance:
  - in Obs_A: only consider gas valve open or closed $(\mathcal{D}(G) = \{0, 1\})$
  - in Obs_C: may control two valves and care for intermediate positions, for instance, to react to different heating requests $(\mathcal{D}(G_1) = \{0, 1, 2, 3\}, \mathcal{D}(G_2) = \{0, 1, 2, 3\})$

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- To prove correctness, we need information how the observables are related — an **invariant** which **links** the data values of Obs_A and Obs_C.
- **If** we're given the linking invariant as a DC formula, say '$\mathsf{Link}_{C,A}$', **then** proving correctness of 'Ctrl' wrt. 'Spec' amounts to proving validity (from 0) of

$$\mathsf{Ctrl} \wedge \mathsf{Link}_{C,A} \implies \mathsf{Spec}.$$

- For instance,

$$\mathsf{Link}_{C,A} = \lceil G \iff (G_1 + G_2 > 0) \rceil$$

Obstacle (iv): How to Prove Correctness?

- by hand on the basis of DC semantics,
- maybe supported by proof rules,
- sometimes a general theorem may fit (e.g. cycle times of PLC automata),
- algorithms as in Uppaal.

DC Properties

Decidability Results: Motivation

- Recall:

  Given **assumptions** as a DC formula 'Asm' on the input observables, verifying **correctness** of 'Ctrl' wrt. 'Spec' amounts to proving

$$\models_0 \mathsf{Ctrl} \wedge \mathsf{Asm} \implies \mathsf{Spec} \tag{1}$$

- If 'Asm' is **not satisfiable**...

Decidability Results: Motivation

- Recall:

  Given **assumptions** as a DC formula 'Asm' on the input observables, verifying **correctness** of 'Ctrl' wrt. 'Spec' amounts to proving

$$\models_0 \mathsf{Ctrl} \land \mathsf{Asm} \implies \mathsf{Spec} \tag{1}$$

- If 'Asm' is **not satisfiable** then (1) is trivially valid, and thus each 'Ctrl' correct wrt. 'Spec'.
- So: strong interest in assessing the **satisfiability** of DC formulae.
- Question: is there an automatic procedure to help us out? (a.k.a.: is it **decidable** whether a given DC formula is satisfiable?)
- More interesting for 'Spec':

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- Question: is there an automatic procedure to help us out? (a.k.a.: is it **decidable** whether a given DC formula is satisfiable?)
- More interesting for 'Spec': is it **realisable** (from 0)?
- Question: is it **decidable** whether a given DC formula is realisable?

Decidability Results for Realisability: Overview

Fragment	Discrete Time	Continous Time
RDC	decidable	decidable
RDC $+ \ell = r$	decidable for $r \in \mathbb{N}$	undecidable for $r \in \mathbb{R}^+$
RDC $+ \int P_1 = \int P_2$	undecidable	undecidable
RDC $+ \ell = x, \forall\, x$	undecidable	undecidable
DC		

RDC in Discrete Time

Restricted DC (RDC)

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \,;\, F_2$$

where P is a state assertion, but with **boolean** observables **only**.

Note:

- No global variables, thus don't need $\mathcal{V}$.
-

Discrete Time Interpretations

- An interpretation $\mathcal{I}$ is called **discrete time interpretation** if and only if, for each state variable X,

$$X_{\mathcal{I}} : \mathsf{Time} \to \mathcal{D}(X)$$

  with

  - $\mathsf{Time} = \mathbb{R}_0^+$,
  - all discontinuities are in $\mathbb{N}_0$.

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- We say (for a discrete time interpretation $\mathcal{I}$ and a discrete interval $[b, e]$)

$$\mathcal{I}, [b, e] \models F_1 \,;\, F_2$$

if and only if there exists $m \in [b, e] \cap \mathbb{N}_0$ such that

$$\mathcal{I}, [b, m] \models F_1 \qquad \text{and} \qquad \mathcal{I}, [m, e] \models F_2$$

Differences between Continuous and Discrete Time

- Let P be a state assertion.

	Continuous Time	Discrete Time
$\models^? (\lceil P \rceil \,;\, \lceil P \rceil) \implies \lceil P \rceil$		
$\models^? \lceil P \rceil \implies (\lceil P \rceil \,;\, \lceil P \rceil)$		

Differences between Continuous and Discrete Time

- Let P be a state assertion.

	Continuous Time	Discrete Time
$\models^{?} (\lceil P \rceil \,;\, \lceil P \rceil) \implies \lceil P \rceil$	✔	✔
$\models^{?} \lceil P \rceil \implies (\lceil P \rceil \,;\, \lceil P \rceil)$	✔	✘

- In particular: $\ell = 1 \iff (\lceil 1 \rceil \land \neg(\lceil 1 \rceil \,;\, \lceil 1 \rceil))$ (in discrete time).

Expressiveness of RDC

- $\ell = 1 \iff \lceil 1 \rceil \wedge \neg(\lceil 1 \rceil \,;\, \lceil 1 \rceil)$
- $\ell = 0 \iff \neg \lceil 1 \rceil$
- $true \iff \ell = 0 \vee \neg(\ell = 0)$
- $\int P = 0 \iff \lceil \neg P \rceil \vee \ell = 0$
- $\int P = 1 \iff (\int P = 0) \,;\, (\lceil P \rceil \wedge \ell = 1) \,;\, (\int P = 0)$
- $\int P = k + 1 \iff (\int P = k) \,;\, (\int P = 1)$
- $\int P \geq k \iff (\int P = k) \,;\, true$
- $\int P > k \iff \int P \geq k + 1$
- $\int P \leq k \iff \neg(\int P > k)$
- $\int P < k \iff \int P \leq k - 1$

where $k \in \mathbb{N}$.

Decidability of Satisfiability/Realisability from 0

Theorem 3.6.
The satisfiability problem for RDC with discrete time is decidable.

Theorem 3.9.
The realisability problem for RDC with discrete time is decidable.

Sketch: Proof of Theorem 3.6

- give a procedure to construct, given a formula F, a **regular** language $\mathcal{L}(F)$ such that

$$\mathcal{I}, [0, n] \models F \text{ if and only if } w \in \mathcal{L}(F)$$

  where word w describes $\mathcal{I}$ on $[0, n]$
  (suitability of the procedure: **Lemma 3.4**)

- then F is satisfiable in discrete time if and only if $\mathcal{L}(F)$ is not empty (**Lemma 3.5**)

- Theorem 3.6 follows because
  - $\mathcal{L}(F)$ can **effectively** be constructed,
  - the emptyness problem is **decidable** for regular languages.

Construction of $\mathcal{L}(F)$

- **Idea:**
  - alphabet $\Sigma(F)$ consists of basic conjuncts of the state variables in F,
  - a letter corresponds to an interpretation on an interval of length 1,
  - a word of length n describes an interpretation on interval $[0, n]$.

Construction of $\mathcal{L}(F)$

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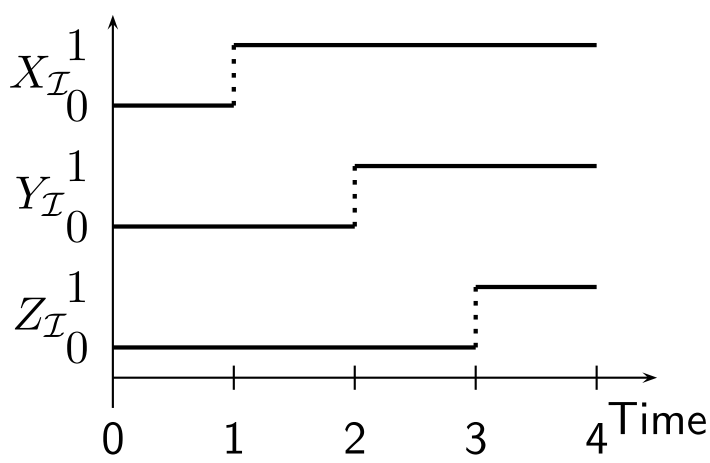
- **Example:** Assume F contains exactly state variables X, Y, Z, then

$$\Sigma(F) = \{X \wedge Y \wedge Z, X \wedge Y \wedge \neg Z, X \wedge \neg Y \wedge Z, X \wedge \neg Y \wedge \neg Z, \\ \neg X \wedge Y \wedge Z, \neg X \wedge Y \wedge \neg Z, \neg X \wedge \neg Y \wedge Z, \neg X \wedge \neg Y \wedge \neg Z\}.$$



$$\begin{aligned} w = {} & (\neg X \wedge \neg Y \wedge \neg Z) \\ & \cdot (X \wedge \neg Y \wedge \neg Z) \\ & \cdot (X \wedge Y \wedge \neg Z) \\ & \cdot (X \wedge Y \wedge Z) \in \Sigma(F)^* \end{aligned}$$

Construction of $\mathcal{L}(F)$ more Formally

Definition 3.2. A word $w = a_1 \dots a_n \in \Sigma(F)^*$ with $n \geq 0$ **describes** a **discrete** interpretation $\mathcal{I}$ on $[0, n]$ if and only if

$$\forall\, j \in \{1, \dots, n\}\ \forall\, t \in\,]j-1, j[\, : \mathcal{I}[\![a_j]\!](t) = 1.$$

For $n = 0$ we put $w = \varepsilon$.

- Each state assertion P can be transformed into an equivalent **disjunctive normal form** $\bigvee_{i=1}^{m} a_i$ with $a_i \in \Sigma(F)$.
- Set $DNF(P) := \{a_1, \dots, a_m\}$ $(\subseteq \Sigma(F))$.
- Define $\mathcal{L}(F)$ inductively:

$$\begin{aligned}
\mathcal{L}(\lceil P \rceil) &= DNF(P)^+, \\
\mathcal{L}(\neg F_1) &= \Sigma(F)^* \setminus \mathcal{L}(F_1), \\
\mathcal{L}(F_1 \vee F_2) &= \mathcal{L}(F_1) \cup \mathcal{L}(F_2), \\
\mathcal{L}(F_1 \,;\, F_2) &= \mathcal{L}(F_1) \cdot \mathcal{L}(F_2).
\end{aligned}$$

Lemma 3.4

Lemma 3.4. For all RDC formulae F, discrete interpretations $\mathcal{I}$, $n \geq 0$, and all words $w \in \Sigma(F)^*$ which **describe** $\mathcal{I}$ on $[0, n]$,

$$\mathcal{I}, [0, n] \models F \text{ if and only if } w \in \mathcal{L}(F).$$

Sketch: Proof of Theorem 3.9

Theorem 3.9.
The realisability problem for RDC with discrete time is decidable.

- $kern(L)$ contains all words of L whose prefixes are again in L.
- If L is regular, then $kern(L)$ is also regular.
- $kern(\mathcal{L}(F))$ can effectively be constructed.
- We have

Lemma 3.8. For all RDC formulae F, F is realisable from 0 in discrete time if and only if $kern(\mathcal{L}(F))$ is infinite.

- Infinity of regular languages is decidable.

References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.