# Real-Time Systems

## Lecture 8: DC Properties II

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## Contents & Goals

**Last Lecture:**

- DC Implementables

**This Lecture:**

- **Educational Objectives:** Capabilities for following tasks/questions.
  - Facts: (un)decidability properties of DC in discrete/continuous time.
  - What's the idea of the considered (un)decidability proofs?
- **Content:**
  - DC Implementables Cont'd
  - RDC in discrete time
  - Satisfiability and realisability from 0 is decidable for RDC in discrete time
  - Undecidable problems of DC in continuous time



## DC Implementables Cont'd



## Recall: DC Implementables

- DC Implementables are special patterns of DC Standard Forms (due to A.P. Ravn).
- Within one pattern,
  - $\pi, \pi_1, \dots, \pi_n$, $n \geq 0$, denote **phases** of **the same** state variable $X_i$,
  - $\varphi$ denotes a state assertion not depending on $X_i$.
- $\theta$ denotes a **rigid** term.
- **Initialisation**:
  $$\lceil\rceil \vee \lceil \pi \rceil \,;\, true$$
- **Sequencing**:
  $$\lceil \pi \rceil \longrightarrow \lceil \pi \vee \pi_1 \vee \dots \vee \pi_n \rceil$$
- **Progress**:
  $$\lceil \pi \rceil \xrightarrow{\theta} \lceil \neg\pi \rceil$$
- **Synchronisation**:
  $$\lceil \pi \wedge \varphi \rceil \xrightarrow{\theta} \lceil \neg\pi \rceil$$



## Recall: DC Implementables Cont'd

- **Bounded Stability**:
  $$\lceil \neg\pi \rceil \,;\, \lceil \pi \wedge \varphi \rceil \xrightarrow{\leq\theta} \lceil \pi \vee \pi_1 \vee \dots \vee \pi_n \rceil$$
- **Unbounded Stability**:
  $$\lceil \neg\pi \rceil \,;\, \lceil \pi \wedge \varphi \rceil \longrightarrow \lceil \pi \vee \pi_1 \vee \dots \vee \pi_n \rceil$$
- **Bounded initial stability**:
  $$\lceil \pi \wedge \varphi \rceil \xrightarrow{\leq\theta}_0 \lceil \pi \vee \pi_1 \vee \dots \vee \pi_n \rceil$$
- **Unbounded initial stability**:
  $$\lceil \pi \wedge \varphi \rceil \longrightarrow_0 \lceil \pi \vee \pi_1 \vee \dots \vee \pi_n \rceil$$

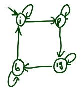


## Recall: Control Automata

Model of Gas Burner controller as a system of four control automata:

- $H$ : Boolean,
  representing **heat request**, (input)
- $F$ : Boolean,
  representing **flame**, (input)
- $C$ with $\mathcal{D}(C) = \{\text{idle}, \text{purge}, \text{ignite}, \text{burn}\}$,
  representing the **controller**, (local)
- $G$ : Boolean,
  representing **gas valve**. (output)

## *Gas Burner Controller Specification*




| | |
|---|---|
| $\lceil\rceil \vee \lceil \text{idle} \rceil \,;\, true, \quad \lceil\rceil \vee \lceil \neg H \rceil \,;\, true, \quad \lceil\rceil \vee \lceil \neg F \rceil \,;\, true, \quad \lceil\rceil \vee \lceil \neg G \rceil \,;\, true$ | (Init-1 - 4) |
| $\lceil \text{idle} \rceil \longrightarrow \lceil \text{idle} \vee \text{purge} \rceil$ | (Seq-1) |
| $\lceil \text{purge} \rceil \longrightarrow \lceil \text{purge} \vee \text{ignite} \rceil$ | (Seq-2) |
| $\lceil \text{ignite} \rceil \longrightarrow \lceil \text{ignite} \vee \text{burn} \rceil$ | (Seq-3) |
| $\lceil \text{burn} \rceil \longrightarrow \lceil \text{burn} \vee \text{idle} \rceil$ | (Seq-4) |
| $\lceil \text{purge} \rceil \xrightarrow{30+\varepsilon} \lceil \neg\text{purge} \rceil$ | (Prog-1) |
| $\lceil \text{ignite} \rceil \xrightarrow{0.5+\varepsilon} \lceil \neg\text{ignite} \rceil$ | (Prog-2) |
| $\lceil \text{idle} \wedge H \rceil \xrightarrow{\varepsilon} \lceil \neg\text{idle} \rceil$ ! | (Syn-1) |
| $\lceil \text{burn} \wedge (\neg H \vee \neg F) \rceil \xrightarrow{\varepsilon} \lceil \neg\text{burn} \rceil$ | (Syn-2) |
| $\lceil G \wedge (\text{idle} \vee \text{purge}) \rceil \xrightarrow{\varepsilon} \lceil \neg G \rceil$ | (Syn-3) |
| $\lceil \neg G \wedge (\text{ignite} \vee \text{burn}) \rceil \xrightarrow{\varepsilon} \lceil G \rceil$ | (Syn-4) |
| $\lceil \neg\text{idle} \rceil \,;\, \lceil \text{idle} \wedge \neg H \rceil \longrightarrow \lceil \text{idle} \rceil$ ! | (Stab-1) |
| $\lceil \text{idle} \wedge \neg H \rceil \longrightarrow_0 \lceil \text{idle} \rceil$ | (Stab-1-init) |
| $\lceil \neg\text{purge} \rceil \,;\, \lceil \text{purge} \rceil \xrightarrow{\leq 30} \lceil \text{purge} \rceil$ | (Stab-2) |
| $\lceil \neg\text{ignite} \rceil \,;\, \lceil \text{ignite} \rceil \xrightarrow{\leq 0.5} \lceil \text{ignite} \rceil$ | (Stab-3) |
| $\lceil \neg\text{burn} \rceil \,;\, \lceil \text{burn} \wedge H \wedge F \rceil \longrightarrow \lceil \text{burn} \rceil$ | (Stab-4) |
| $\lceil F \rceil \,;\, \lceil \neg F \wedge \neg\text{ignite} \rceil \longrightarrow \lceil \neg F \rceil$ | (Stab-5) |
| $\lceil \neg F \wedge \neg\text{ignite} \rceil \longrightarrow_0 \lceil \neg F \rceil$ | (Stab-5-init) |
| $\lceil G \rceil \,;\, \lceil \neg G \wedge (\text{idle} \vee \text{purge}) \rceil \longrightarrow \lceil \neg G \rceil$ | (Stab-6) |
| $\lceil \neg G \wedge (\text{idle} \vee \text{purge}) \rceil \longrightarrow_0 \lceil \neg G \rceil$ | (Stab-6-init) |
| $\lceil \neg G \rceil \,;\, \lceil G \wedge (\text{ignite} \vee \text{burn}) \rceil \longrightarrow \lceil G \rceil$ | (Stab-7) |



## *Gas Burner Controller Correctness Proof*

$$\text{GB-Ctrl} := \text{Init-1} \wedge \cdots \wedge \text{Stab-7} \wedge \varepsilon > 0$$

**Recall:**

$$\text{Req} :\iff \Box(\ell \geq 60 \implies 20 \cdot \textstyle\int L \leq \ell)$$

and (cf. [Olderog and Dierks, 2008])

$$\models \text{Req-1} \implies \text{Req}$$

for the **simplified**

$$\text{Req-1} := \Box(\ell \leq 30 \implies \textstyle\int L \leq 1).$$

Here we show

$$\models \text{GB-Ctrl} \wedge A(\varepsilon) \implies \text{Req-1}.$$



## *Lemma 3.15*

$$\models \text{GB-Ctrl} \implies \Box \begin{pmatrix} & (\lceil \text{idle} \rceil \implies & \int G \leq \varepsilon) \\ \wedge & (\lceil \text{purge} \rceil \implies & \int G \leq \varepsilon) \\ \wedge & (\lceil \text{ignite} \rceil \implies & \ell \leq 0.5 + \varepsilon) \\ \wedge & (\lceil \text{burn} \rceil \implies & \int \neg F \leq 2\varepsilon) \end{pmatrix} \ (*)$$

**Proof**: Let $\mathcal{I}$ be an interpretation, $\mathcal{V}$ a valuation, and $[c,d]$ an interval with $\mathcal{I}, \mathcal{V}, [c,d] \models \text{GB-Ctrl}$. Let $\overbrace{[b,e] \subseteq [c,d]}^{\text{e.g. } \Box}$.

- Case 1: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil \text{idle} \rceil$

| | |
|---|---|
| $\lceil G \wedge (\text{idle} \vee \text{purge}) \rceil \xrightarrow{\varepsilon} \lceil \neg G \rceil$ | (Syn-3) |
| $\lceil G \rceil \,;\, \lceil \neg G \wedge (\text{idle} \vee \text{purge}) \rceil \longrightarrow \lceil \neg G \rceil$ | (Stab-6) |

conclude

$$\mathcal{I}, \mathcal{V}, [b,e] \models \Box(\lceil G \rceil \implies \ell \leq \varepsilon) \wedge \underbrace{\neg\Diamond(\lceil G \rceil \,;\, \lceil \neg G \rceil \,;\, \lceil G \rceil)}_{\text{gas valve doesn't open up again in idle phase}}$$

(*)

- Case 2: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil \text{purge} \rceil$ Analogously to case 1.



## *Lemma 3.15 Cont'd*

$$\begin{array}{r} (\lceil \text{idle} \rceil \implies \quad \int G \leq \varepsilon) \\ (\lceil \text{purge} \rceil \implies \quad \int G \leq \varepsilon) \\ (\lceil \text{ignite} \rceil \implies \ell \leq 0.5 + \varepsilon) \\ (\lceil \text{burn} \rceil \implies \int \neg F \leq 2\varepsilon) \end{array}$$

- Case 3: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil \text{ignite} \rceil$

$$\lceil \text{ignite} \rceil \xrightarrow{0.5+\varepsilon} \lceil \neg\text{ignite} \rceil \qquad \text{(Prog-2)}$$

$$\mathcal{I}, \mathcal{V}, [b,e] \models \ell \leq 0.5 + \varepsilon$$

- Case 4: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil \text{burn} \rceil$

$\lceil\rceil$
$\lceil \neg F \rceil$
$\lceil F \rceil$
$\lceil F \rceil ; \lceil \neg F \rceil$
$\lceil \neg F \rceil ; \lceil F \rceil$
$\lceil \neg F \rceil ; \lceil F \rceil ; \lceil \neg F \rceil$

| | |
|---|---|
| $\lceil \text{burn} \wedge (\neg H \vee \neg F) \rceil \xrightarrow{\varepsilon} \lceil \neg\text{burn} \rceil$ | (Syn-2) |
| $\lceil F \rceil \,;\, \lceil \neg F \wedge \neg\text{ignite} \rceil \longrightarrow \lceil \neg F \rceil$ | (Stab-5) |

$$\mathcal{I}, \mathcal{V}, [b,e] \models \Box(\lceil \neg F \rceil \implies \ell \leq \varepsilon) \wedge \neg\Diamond(\lceil F \rceil \,;\, \lceil \neg F \rceil \,;\, \lceil F \rceil)$$

(*)



## *Lemma 3.16*

$$\models \exists\, \varepsilon \bullet \text{GB-Ctrl} \implies \underbrace{\Box(\ell \leq 30 \implies \textstyle\int L \leq 1)}_{\text{Req-1}}$$

Proof Sketch

Choose $\mathcal{I}, \mathcal{V}, [b,e]$ s.t. $\mathcal{I}, \mathcal{V}, [b,e] \models \text{GB-Ctrl} \wedge \ell \leq 30$.

Distinguish 5 cases:

$$\begin{array}{lll} \mathcal{I}, \mathcal{V}, [b,e] \models & \lceil\rceil & (0) \\ & \vee\ (\lceil \text{idle} \rceil ; true \wedge \ell \leq 30) & (1) \\ & \vee\ (\lceil \text{purge} \rceil ; true \wedge \ell \leq 30) & (2) \\ & \vee\ (\lceil \text{ignite} \rceil ; true \wedge \ell \leq 30) & (3) \\ & \vee\ (\lceil \text{burn} \rceil ; true \wedge \ell \leq 30) & (4) \end{array}$$



## *Lemma 3.16 Cont'd*

- Case 0: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil\rceil$ ✓

- Case 1: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil \text{idle} \rceil \,;\, true \wedge \ell \leq 30$

| | |
|---|---|
| $\lceil \text{idle} \rceil \longrightarrow \lceil \text{idle} \vee \text{purge} \rceil$ | (Seq-1) |
| $\lceil \neg\text{purge} \rceil \,;\, \lceil \text{purge} \rceil \xrightarrow{\leq 30} \lceil \text{purge} \rceil$ | (Stab-2) |

$\leadsto \mathcal{I}, \mathcal{V}, [b,e] \models \lceil \text{idle} \rceil \vee \lceil \text{idle} \rceil ; \lceil \text{purge} \rceil$

3.15 $\leadsto \mathcal{I}, \mathcal{V}, [b,e] \models \int L \leq \varepsilon \vee \int L \leq \varepsilon ; \int L \leq \varepsilon$

$\leadsto \mathcal{I}, \mathcal{V}, [b,e] \models \int L \leq 2\varepsilon$

Thus $\boxed{\varepsilon \leq 0.5}$ is sufficient for Req-1 <u>in this case</u>.

## Lemma 3.16 Cont'd

- Case 2: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil \mathsf{burn} \rceil \,;\, true \wedge \ell \leq 30$

$$\lceil \mathsf{burn} \rceil \longrightarrow \lceil \mathsf{burn} \vee \mathsf{idle} \rceil \qquad \text{(Seq-4)}$$

$\Rightarrow \mathcal{I}, \mathcal{V}, [b,e] \models (\lceil burn \rceil \vee \lceil burn \rceil ; \underbrace{\lceil idle \rceil ; true}_{(1)}) \wedge \ell \leq 30$

3.15, (1) $\Rightarrow \mathcal{I}, \mathcal{V}, [b,e] \models (\int L \leq 2\varepsilon \vee \int L \leq 2\varepsilon ; \int L \leq 2\varepsilon) \wedge \ell \leq 30$

$\Rightarrow \mathcal{I}, \mathcal{V}, [b,e] \models \int L \leq 4\varepsilon$

Thus $\boxed{\varepsilon \leq 0.25}$ sufficient for Req-1 in this case.



## Lemma 3.16 Cont'd

- Case 3: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil \mathsf{ignite} \rceil \,;\, true \wedge \ell \leq 30$

$$\lceil \mathsf{ignite} \rceil \longrightarrow \lceil \mathsf{ignite} \vee \mathsf{burn} \rceil \qquad \text{(Seq-3)}$$

$\Rightarrow \mathcal{I}, \mathcal{V}, [b,e] \models (\lceil ignite \rceil \vee (\lceil ignite \rceil ; \underbrace{\lceil burn \rceil ; true}_{(2)})) \wedge \ell \leq 30$

3.15 (2) $\Rightarrow \mathcal{I}, \mathcal{V}, [b,e] \models (\int L \leq 0.5 + \varepsilon \vee \int L \leq 0.5 + \varepsilon ; \int L \leq 4\varepsilon) \wedge \ell \leq 30$

$\Rightarrow \mathcal{I}, \mathcal{V}, [b,e] \models \int L \leq 0.5 + 5\varepsilon$

So $\boxed{\varepsilon \leq 0.1}$ sufficient in this case.



## Lemma 3.16 Cont'd

- Case 4: $\mathcal{I}, \mathcal{V}, [b,e] \models \lceil \mathsf{purge} \rceil \,;\, true \wedge \ell \leq 30$

$$\lceil \mathsf{purge} \rceil \longrightarrow \lceil \mathsf{purge} \vee \mathsf{ignite} \rceil \qquad \text{(Seq-2)}$$

3.15 (3) $\Rightarrow \mathcal{I}, \mathcal{V}, [b,e] \models \int L \leq 0.5 + 6\varepsilon$

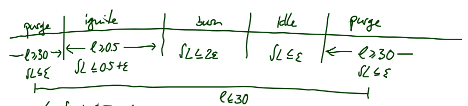
Thus $\boxed{\varepsilon \leq \frac{1}{12}}$ is sufficient for Req-1 in this case.



## Correctness Result

**Theorem 3.17.**

$$\models \left( \text{GB-Ctrl} \wedge \varepsilon \leq \frac{1}{12} \right) \implies \text{Req}$$




$\Rightarrow \int L \leq 0.5 + 6\varepsilon$

$\Rightarrow A(\varepsilon) := \varepsilon \leq \frac{1}{12}$



## Discussion

- We used only

'Seq-1', 'Seq-2', 'Seq-3', 'Seq-4',
'Prog-2', 'Syn-2', 'Syn-3',
'Stab-2', 'Stab-5', 'Stab-6'.

What about

$$\text{Prog-1} = \lceil \mathsf{purge} \rceil \xrightarrow{30+\varepsilon} \lceil \neg \mathsf{purge} \rceil$$

for instance?

Nope, there is the requirement (not noted down) that the system does something finally, e.g. get the heating going on request.



## RDC in Discrete Time Cont'd

## Restricted DC (RDC)

$$F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \,;\, F_2$$

where $P$ is a state assertion, but with **boolean** observables **only**.

Note:

- No global variables, thus don't need $\mathcal{V}$.
- chop is there
- no $\int$, no $\ell$ (in general)
- no predicates, no function symbols (in general)
- $\Diamond F$ ...?
- $\lceil 1 \rceil$ ...?



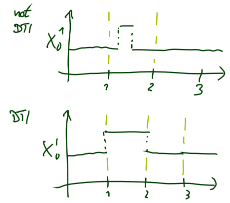
## Discrete Time Interpretations

- An interpretation $\mathcal{I}$ is called **discrete time interpretation** if and only if, for each state variable $X$,

$$X_\mathcal{I} : \text{Time} \to \mathcal{D}(X)$$

with

- $\text{Time} = \mathbb{R}_0^+$,
- all discontinuities are in $\mathbb{N}_0$.



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- An interval $[b, e] \subset \text{Intv}$ is called **discrete** if and only if $b, e \in \mathbb{N}_0$.



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with

- $\text{Time} = \mathbb{R}_0^+$,
- all discontinuities are in $\mathbb{N}_0$.

- We say $\mathcal{I}, [b,e] \models \lceil P \rceil$ if $\int_b^e P_\mathcal{I}(t)\,dt = (e-b) \wedge (e-b) > 0$

- An interval $[b, e] \subset \text{Intv}$ is called **discrete** if and only if $b, e \in \mathbb{N}_0$.
- We say (for a discrete time interpretation $\mathcal{I}$ and a discrete interval $[b, e]$)

$$\mathcal{I}, [b,e] \models F_1 \,;\, F_2$$

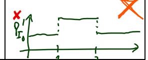
if and only if there exists $m \in [b,e] \cap \mathbb{N}_0$ such that

$$\mathcal{I}, [b,m] \models F_1 \quad \text{and} \quad \mathcal{I}, [m,e] \models F_2$$



## Differences between Continuous and Discrete Time

- Let $P$ be a state assertion.

| | Continuous Time | Discrete Time |
|---|---|---|
| $\models^? (\lceil P \rceil \,;\, \lceil P \rceil) \implies \lceil P \rceil$ | ✔ ✓ | ✔ ✓ |
| $\models^? \lceil P \rceil \implies (\lceil P \rceil \,;\, \lceil P \rceil)$ | ✔ ✓ | ✗  ✗ |

$\lceil P \rceil$
only chop-point candidates are $m=1$ and $m=2$ but then $m-b=0$ or $e-m=0$

- In particular: $\ell = 1 :\iff (\lceil 1 \rceil \wedge \neg(\lceil 1 \rceil \,;\, \lceil 1 \rceil))$ (in discrete time).



## Expressiveness of RDC

- $\ell = 1 \iff \lceil 1 \rceil \wedge \neg(\lceil 1 \rceil \,;\, \lceil 1 \rceil)$
- $\ell = 0 \iff \neg\lceil 1 \rceil$
- $true \iff \ell = 0 \vee \neg(\ell = 0)$
- $\int P = 0 \iff \lceil \neg P \rceil \vee \ell = 0$
- $\int P = 1 \iff (\int P = 0) \,;\, (\lceil P \rceil \wedge \ell = 1) \,;\, (\int P = 0)$
- $\int P = k+1 \iff (\int P = k) \,;\, (\int P = 1)$
- $\int P \geq k \iff (\int P = k) \,;\, true$
- $\int P > k \iff \int P \geq k+1$
- $\int P \leq k \iff \neg(\int P > k)$
- $\int P < k \iff \int P \leq k-1$

where $k \in \mathbb{N}^+$.

so still $\Diamond F := true \,;\, F \,;\, true$ in RDC

## Decidability of Satisfiability/Realisability from 0

**Theorem 3.6.**
The satisfiability problem for RDC with discrete time is decidable.

**Theorem 3.9.**
The realisability problem for RDC with discrete time is decidable.





RDC formula $\mathcal{F}$.

$$\begin{tikzcd}
\mathcal{F} \arrow[d, squiggly, "\text{construct}", "1"'] & \Dashv \arrow[d, Leftrightarrow, "\text{prove } 5"] & \mathcal{I} \text{ DTI} \\
\Sigma^* \supseteq \mathcal{L}(\mathcal{F}) & \ni & w \in \Sigma^* \arrow[u, squiggly, "\text{construct } 4"']
\end{tikzcd}$$

regular language    3    word 2

- $\mathcal{L}(\mathcal{F}) = \emptyset \Rightarrow \mathcal{F}$ not SAT
- $\mathcal{L}(\mathcal{F}) = \emptyset$ is decidable

## References





[Chaochen and Hansen, 2004] Chaochen, Z. and Hansen, M. R. (2004). *Duration Calculus: A Formal Approach to Real-Time Systems*. Monographs in Theoretical Computer Science. Springer-Verlag. An EATCS Series.

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.