Contents & Goals

- Last Lecture:
  - DC Implementables

- This Lecture:
  - Educational Objectives:
    - Capabilities for following tasks/questions.
  - Facts: (un)decidability properties of DC in discrete/continuous time.
  - What's the idea of the considered (un)decidability proofs?

- Content:
  - RDC in discrete time cont'd
  - Satisfiability and realisability from 0 is decidable for RDC in discrete time
  - Undecidable problems of DC in continuous time

Restricted DC (RDC)

F ::= ⌈P⌉ | ¬F | F1 | F1 ∨ F2 | F1 ; F2

where P is a state assertion, but with boolean observables only.

Note:
- No global variables, thus don't need V.

Discrete Time Interpretations

- An interpretation I is called discrete time interpretation if and only if,
  for each state variable X, XI:
    Time → D(X)
  with
    - Time = R+0,
    - all discontinuities are in N0.

- An interval [b, e] ⊂ Intv is called discrete if and only if b, e ∈ N0.

- We say (for a discrete time interpretation I and a discrete interval [b, e])
  I, [b, e] |= F1 ; F2 if and only if there exists m ∈ [b, e] ∩ N0 such that
    I, [b, m] |= F1 and I, [m, e] |= F2.

Differences between Continuous and Discrete Time

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- In particular:
  - ℓ = 1 ⇐⇒ (⌈1⌉ ∧ ¬(⌈1⌉; ⌈1⌉)) (in discrete time).
### The satisfiability problem for RDC with discrete time is decidable.

**Theorem 3.6.**

The satisfiability problem for RDC with discrete time is decidable.

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**Sketch**

The satisfiability problem for RDC with discrete time is decidable.

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### Expressiveness of RDC

**Theorem 3.9.**

The satisfiability problem for RDC with discrete time is decidable.
Lemma 3.4.

For all RDC formulae $F$, discrete interpretations $I$, $n \geq 0$, and all words $w \in \Sigma(F)^*$ which describe $I$ on $[0, n]$, $I\mid_{[0, n]} = F$ if and only if $w \in L(F)$.

Sketch: Proof of Theorem 3.9

Theorem 3.9. The realisability problem for RDC with discrete time is decidable.

- $\text{kern}(L)$ contains all words of $L$ whose prefixes are again in $L$.
- If $L$ is regular, then $\text{kern}(L)$ is also regular.
- $\text{kern}(L(F))$ can effectively be constructed.
- We have Lemma 3.8.

For all RDC formulae $F$, $F$ is realisable from 0 in discrete time if and only if $\text{kern}(L(F))$ is infinite.

- Infinity of regular languages is decidable.

(Variants of) RDC in Continuous Time

Recall: Restricted DC (RDC)

$F ::= \lceil P \rceil | \neg F_1 | F_1 \lor F_2 | F_1 ; F_2 | \ell = 1 | \ell = x | \forall x \cdot F_1$

where $P$ is a state assertion, but with boolean observables only.

From now on: "RDC + $\ell = x, \forall x""

$F ::= \lceil P \rceil | \neg F_1 | F_1 \lor F_2 | F_1 ; F_2 | \ell = 1 | \ell = x | \forall x \cdot F_1$

Undecidability of Satisfiability/Realisability from 0

Theorem 3.10. The realisability from 0 problem for DC with continuous time is undecidable, not even semi-decidable.

Theorem 3.11. The satisfiability problem for DC with continuous time is undecidable.

Sketch: Proof of Theorem 3.10

Reduce divergence of two-counter machines to realisability from 0:

- Given a two-counter machine $M$ with final state $q_{\text{fin}}$, construct a DC formula $F(M) := $ encoding $(M)$
- such that $M$ diverges if and only if the DC formula $F(M) \land \neg \diamondsuit \lceil q_{\text{fin}} \rceil$ is realisable from 0.
- If realisability from 0 was (semi-)decidable, divergence of two-counter machines would be (which it isn't).
where 

\[ \mathcal{I}(\mathcal{F})(x) = \{ x \in \mathbb{B}^\omega \mid \exists n, q \in \mathbb{N} : (x, n, q) \in \mathcal{F} \} \] 

is realisable from \( k \).

\[ B \subseteq \mathbb{F} \]

or, using abbreviations,

\[ n \leq 1 \]

Each two subsequent intervals \([n, n+1)\]

by a DC formula.

An interpretation on 'Time' encodes

\[ \text{obs} \{ \text{dec} \} \]

Commands of the form \( \text{dec} : q, n \) is a finite sequence of the form ("where

\[ \text{dec} : q, n \]

\[ \text{dec} : q, n \]

of \( K \) can be realised in any configuration of the form

\[ (x, n, q) \in \mathcal{F} \]
\[Q \equiv \{ \text{both modes} \} \land \{ \text{dec} : q \} \land \{ \text{inc} : q \} \]

\[\begin{align*}
\{ \text{both modes} \} & \equiv \{ \text{dec} : q \} \land \{ \text{inc} : q \} \\
\{ \text{dec} : q \} & \equiv \{ \text{dec} : q \} \\
\{ \text{inc} : q \} & \equiv \{ \text{inc} : q \}
\end{align*}\]

\[Q = \{ \text{both modes} \} \land \{ \text{dec} : q \} \land \{ \text{inc} : q \}\]
Following [Chaochen and Hansen, 2004] we can observe that $M$ halts if and only if the DC formula $F(M) \land \Diamond\lceil q_{\text{fin}} \rceil$ is satisfiable. This yields Theorem 3.11.

The satisfiability problem for DC with continuous time is undecidable. (It is semi-decidable.)

Furthermore, by taking the contraposition, we see $M$ diverges if and only if $M$ does not halt if and only if $F(M) \land \neg \Diamond\lceil q_{\text{fin}} \rceil$ is not satisfiable.

Thus whether a DC formula is not satisfiable is not decidable, not even semi-decidable.

By Remark 2.13, $F$ is valid iff $\neg F$ is not satisfiable, so Corollary 3.12.

The validity problem for DC with continuous time is undecidable, not even semi-decidable.

This provides us with an alternative proof of Theorem 2.23 ("there is no sound and complete proof system for DC"):

• Suppose there were such a calculus $C$.
  • By Lemma 2.22 it is semi-decidable whether a given DC formula $F$ is a theorem in $C$.
  • By the soundness and completeness of $C$, $F$ is a theorem in $C$ if and only if $F$ is valid.
  • Thus it is semi-decidable whether $F$ is valid.
  • Contradiction.

Discussion

Note: the DC fragment defined by the following grammar is sufficient for the reduction $F ::= \lceil P \rceil | \neg F | F_1 \lor F_2 | F_1; F_2 | \ell = 1 | \ell = x | \forall x \cdot F_1$, $P$ a state assertion, $x$ a global variable.

Formulae used in the reduction are abbreviations:

- $\ell = 4 \iff \ell = 1$
- $\ell = 1 \iff \ell = z$
- $\ell \geq 4 \iff \ell = 4$
- $true \iff \ell = x$
- $\ell = x + y + 4 \iff \ell = x$
- $\ell = y$
- $\ell = 4$

Length 1 is not necessary — we can use $\ell = z$ instead, with fresh $z$.

This is RDC augmented by "$\ell = x$" and "$\forall x$", which we denote by $\text{RDC} + \ell = x, \forall x$. 