Real-Time Systems

Lecture 13: Location Reachability (or: The Region Automaton)

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Dr. Bernd Westphal

Albert-Ludwigs-Universität Freiburg, Germany

Contents & Goals

Last Lecture:

- Networks of Timed Automata
- Uppaal Demo

This Lecture:

- **Educational Objectives:** Capabilities for following tasks/questions.
  - What are decidable problems of TA?
  - How can we show this? What are the essential premises of decidability?
  - What is a region? What is the region automaton of this TA?
  - What's the time abstract system of a TA? Why did we consider this?
  - What can you say about the complexity of Region-automaton based reachability analysis?
- **Content:**
  - Timed Transition System of network of timed automata
  - Location Reachability Problem
  - Constructive, region-based decidability proof

The Location Reachability Problem

The Location Reachability Problem

Given: A timed automaton $\mathcal{A}$ and one of its control locations ℓ.

Question: Is ℓ **reachable**?

That is, is there a transition sequence of the form

$$\langle \ell_{ini}, \nu_0 \rangle \xrightarrow{\lambda_1} \langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle \xrightarrow{\lambda_3} \dots \xrightarrow{\lambda_n} \langle \ell_n, \nu_n \rangle, \ell_n = \ell$$

in the labelled transition system $\mathcal{T}(\mathcal{A})$?

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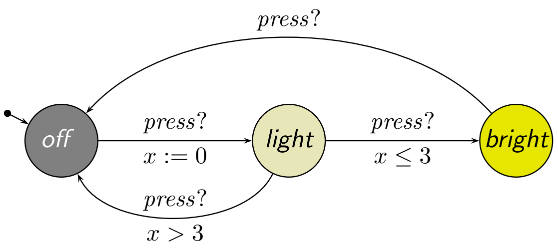
in the labelled transition system $\mathcal{T}(\mathcal{A})$?

- **Note:** Decidability is not **soo** obvious, recall that
  - clocks range over real numbers, thus infinitely many configurations,
  - at each configuration, uncountably many transitions $\xrightarrow{t}$ may originate
- **Consequence:** The timed automata as we consider them here **cannot** encode a 2-counter machine, and they are strictly less expressive than DC.

Decidability of The Location Reachability Problem

Claim: (**Theorem 4.33**)

The location reachability problem is **decidable** for timed automata.



Approach: Constructive proof.

- Observe: clock constraints are **simple** — w.l.o.g. assume constants $c \in \mathbb{N}_0$.
- **Def. 4.19**: **time-abstract transition system** $\mathcal{U}(\mathcal{A})$ — abstracts from uncountably many delay transitions, still infinite-state.
- **Lem. 4.20**: location reachability of $\mathcal{A}$ is **preserved** in $\mathcal{U}(\mathcal{A})$.
- **Def. 4.29**: **region automaton** $\mathcal{R}(\mathcal{A})$ — equivalent configurations collapse into regions
- **Lem. 4.32**: location reachability of $\mathcal{U}(\mathcal{A})$ is **preserved** in $\mathcal{R}(\mathcal{A})$.
- **Lem. 4.28**: $\mathcal{R}(\mathcal{A})$ is **finite**.

Without Loss of Generality: Natural Constants

Recall: Simple clock constraints are $\varphi ::= x \sim c \mid x - y \sim c \mid \varphi \wedge \varphi$
with $x, y \in X$, $c \in \mathbb{Q}_0^+$, and $\sim \in \{<, >, \leq, \geq\}$.

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- Let $C(\mathcal{A}) = \{c \in \mathbb{Q}_0^+ \mid c \text{ appears in } \mathcal{A}\}$ — $C(\mathcal{A})$ is **finite**! (Why?)
- Let $t_\mathcal{A}$ be the **least common multiple of the denominators** in $C(\mathcal{A})$.
- Let $t_\mathcal{A} \cdot \mathcal{A}$ be the TA obtained from $\mathcal{A}$ by **multiplying** all constants by $t_\mathcal{A}$.

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- Let $t_\mathcal{A} \cdot \mathcal{A}$ be the TA obtained from $\mathcal{A}$ by **multiplying** all constants by $t_\mathcal{A}$.

- Then:
  - $C(t_\mathcal{A} \cdot \mathcal{A}) \subset \mathbb{N}_0$.
  - A location ℓ is reachable in $t_\mathcal{A} \cdot \mathcal{A}$ if and only if ℓ is reachable in $\mathcal{A}$.

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- Let $C(\mathcal{A}) = \{c \in \mathbb{Q}_0^+ \mid c \text{ appears in } \mathcal{A}\}$ — $C(\mathcal{A})$ is **finite**! (Why?)
- Let $t_{\mathcal{A}}$ be the **least common multiple of the denominators** in $C(\mathcal{A})$.
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- Then:
  - $C(t_{\mathcal{A}} \cdot \mathcal{A}) \subset \mathbb{N}_0$.
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- That is: we can **without loss of generality** in the following consider only timed automata $\mathcal{A}$ with $C(\mathcal{A}) \subset \mathbb{N}_0$.

Definition. Let x be a clock of timed automaton $\mathcal{A}$ (with $C(\mathcal{A}) \subset \mathbb{N}_0$). We denote by $c_x \in \mathbb{N}_0$ the **largest time constant** c that appears together with x in a constraint of $\mathcal{A}$.

Decidability of The Location Reachability Problem

Claim: (**Theorem 4.33**)

The location reachability problem is **decidable** for timed automata.

Approach: Constructive proof.

- ✔ Observe: clock constraints are **simple** — w.l.o.g. assume constants $c \in \mathbb{N}_0$.
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Helper: Relational Composition

Recall: $\mathcal{T}(\mathcal{A}) = (\mathit{Conf}(\mathcal{A}), \mathsf{Time} \cup B_{?!}, \{\xrightarrow{\lambda} \mid \lambda \in \mathsf{Time} \cup B_{?!}\}, C_{ini})$

- Note: The $\xrightarrow{\lambda}$ are binary relations on configurations.

Definition. Let $\mathcal{A}$ be a TA. For all $\langle \ell_1, \nu_1 \rangle$, $\langle \ell_2, \nu_2 \rangle \in \mathit{Conf}(\mathcal{A})$,

$$\langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_1} \circ \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle$$

if and only if there **exists some** $\langle \ell', \nu' \rangle \in \mathit{Conf}(\mathcal{A})$ such that

$$\langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_1} \langle \ell', \nu' \rangle \text{ and } \langle \ell', \nu' \rangle \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle.$$

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Remark. The following property of **time additivity** holds.

$$\forall\, t_1, t_2 \in \mathsf{Time} : \xrightarrow{t_1} \circ \xrightarrow{t_2} \;=\; \xrightarrow{t_1 + t_2}$$

Time-abstract Transition System

Definition 4.19. [*Time-abstract transition system*]
Let $\mathcal{A}$ be a timed automaton.
The **time-abstract transition system** $\mathcal{U}(\mathcal{A})$
is obtained from $\mathcal{T}(\mathcal{A})$ (Def. 4.4) by taking

$$\mathcal{U}(\mathcal{A}) = (\mathit{Conf}(\mathcal{A}), B_{?!}, \{\overset{\alpha}{\Longrightarrow} \mid \alpha \in B_{?!}\}, C_{ini})$$

where

$$\overset{\alpha}{\Longrightarrow} \subseteq \mathit{Conf}(\mathcal{A}) \times \mathit{Conf}(\mathcal{A})$$

is defined as follows: Let $\langle \ell, \nu \rangle, \langle \ell', \nu' \rangle \in \mathit{Conf}(\mathcal{A})$ be configurations of $\mathcal{A}$ and $\alpha \in B_{?!}$ an action. Then

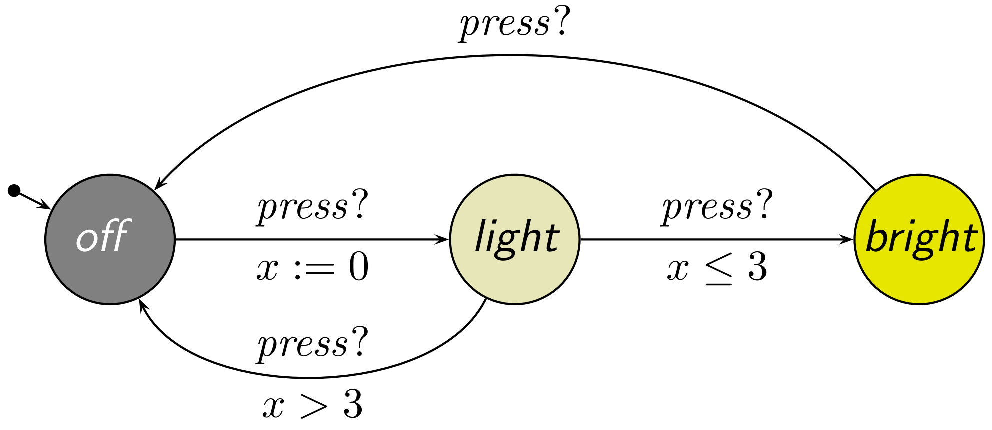
$$\langle \ell, \nu \rangle \overset{\alpha}{\Longrightarrow} \langle \ell', \nu' \rangle$$

if and only if there exists $t \in \mathrm{Time}$ such that

$$\langle \ell, \nu \rangle \xrightarrow{t} \circ \xrightarrow{\alpha} \langle \ell', \nu' \rangle.$$

Example

$$\langle \ell, \nu \rangle \xRightarrow{\alpha} \langle \ell', \nu' \rangle \text{ iff } \exists\, t \in \mathsf{Time} \bullet \langle \ell, \nu \rangle \xrightarrow{t} \circ \xrightarrow{\alpha} \langle \ell', \nu' \rangle$$



Location Reachability is preserved in $\mathcal{U}(\mathcal{A})$

Lemma 4.20. For all locations ℓ of a given timed automaton $\mathcal{A}$ the following holds:

ℓ is reachable in $\mathcal{T}(\mathcal{A})$ if and only if ℓ is reachable in $\mathcal{U}(\mathcal{A})$.

Proof:

Decidability of The Location Reachability Problem

Claim: (**Theorem 4.33**)

The location reachability problem is **decidable** for timed automata.

Approach: Constructive proof.

- ✔ Observe: clock constraints are **simple** — w.l.o.g. assume constants $c \in \mathbb{N}_0$.
- ✔ **Def. 4.19**: **time-abstract transition system** $\mathcal{U}(\mathcal{A})$ — abstracts from uncountably many delay transitions, still infinite-state.
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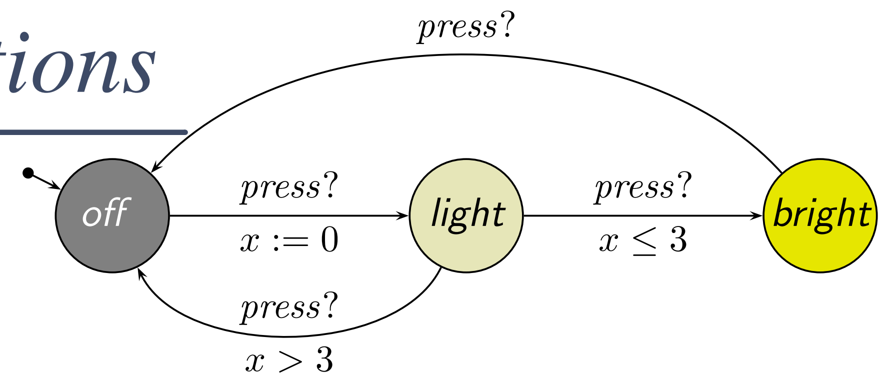
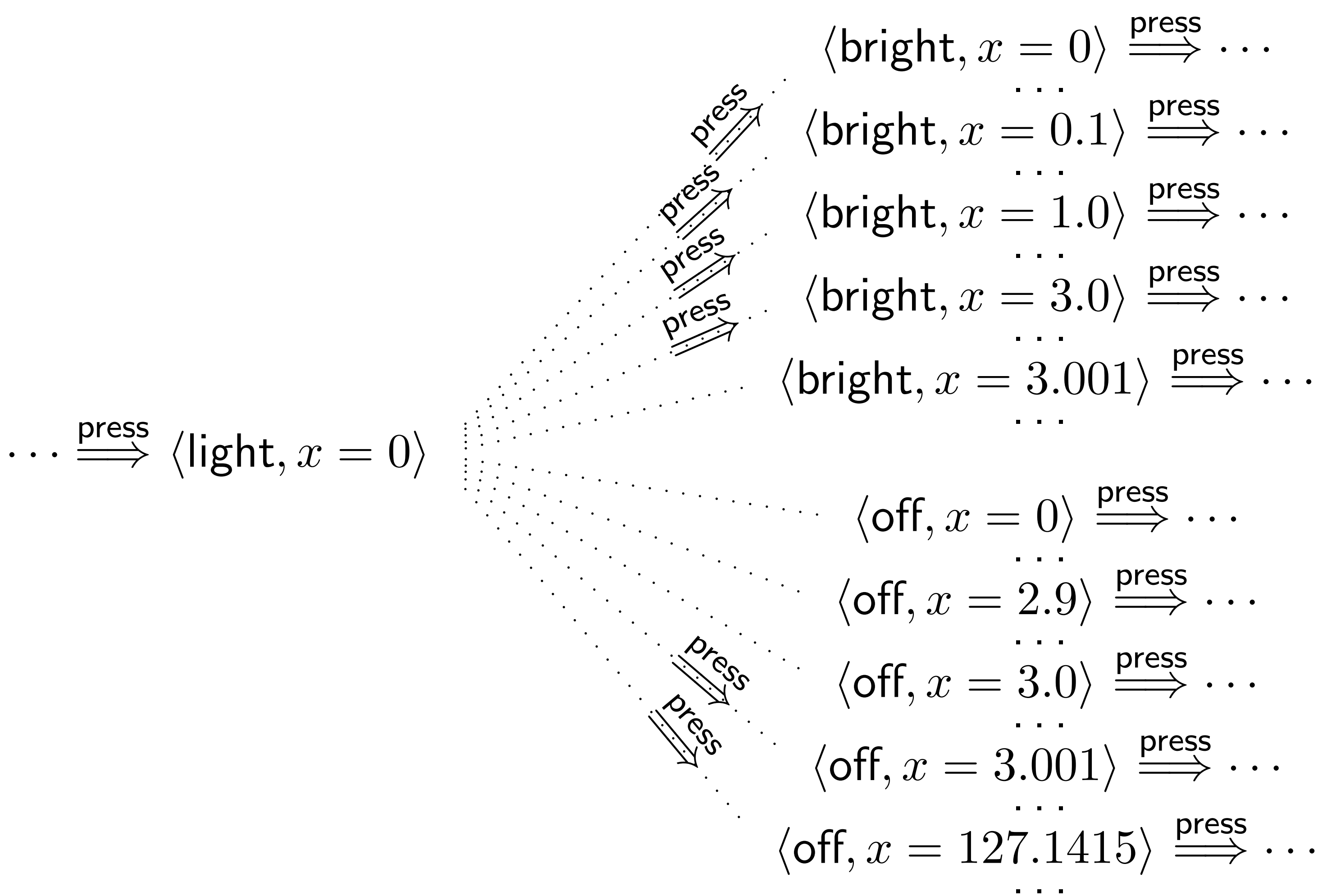
Indistinguishable Configurations



$\mathcal{U}(\mathcal{A})$:



Distinguishing Clock Valuations: One Clock

- Assume $\mathcal{A}$ with only a single clock, i.e. $X = \{x\}$ (**recall**: $C(\mathcal{A}) \subset \mathbb{N}$.)
  - $\mathcal{A}$ **could detect**, for a given ν, whether $\nu(x) \in \{0, \dots, c_x\}$.
  - $\mathcal{A}$ **cannot distinguish** ν_1 and ν_2 if $\nu_i(x) \in (k, k+1)$, $i = 1, 2$, and $k \in \{0, \dots, c_x - 1\}$.
  - $\mathcal{A}$ **cannot distinguish** ν_1 and ν_2 if $\nu_i(x) > c_x$, $i = 1, 2$.

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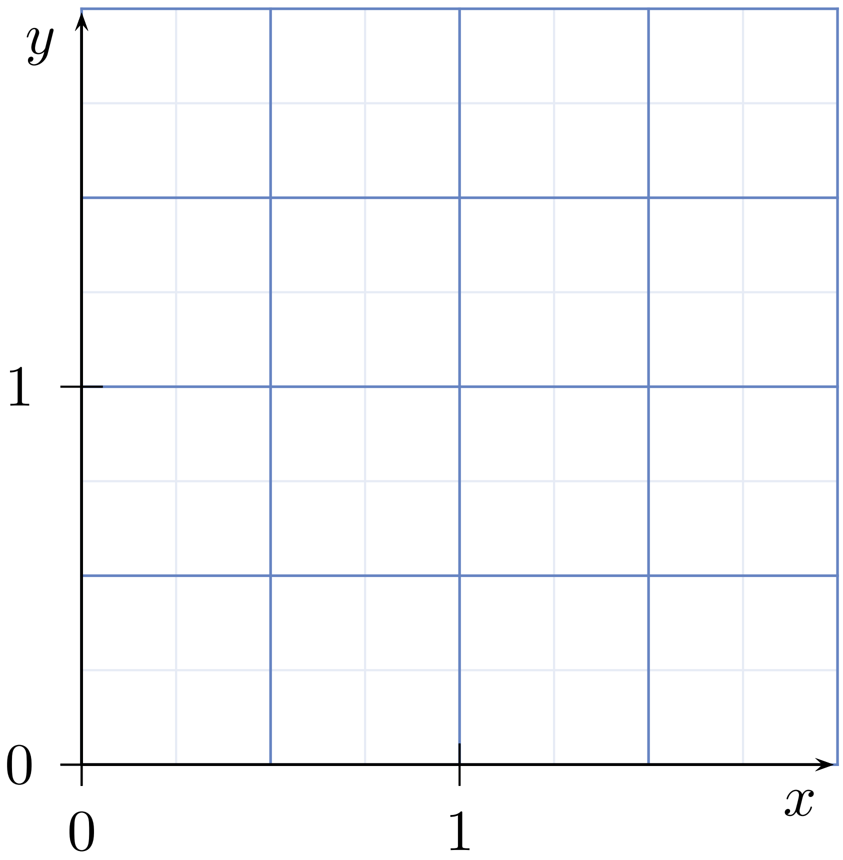
  - $\mathcal{A}$ **cannot distinguish** ν_1 and ν_2 if $\nu_i(x) > c_x$, $i = 1, 2$.
- If $c_x \geq 1$, there are $(2c_x + 2)$ **equivalence classes**:

$$\{\{0\}, (0,1), \{1\}, (1,2), \dots, \{c_x\}, (c_x, \infty)\}$$

If $\nu_1(x)$ and $\nu_2(x)$ are in the **same** equivalence class, then ν_1 and ν_2 are **indistiguishable** by $\mathcal{A}$.

Distinguishing Clock Valuations: Two Clocks

- $X = \{x, y\}$, $c_x = 1$, $c_y = 1$.



Helper: Floor and Fraction

- **Recall**:

  Each $q \in \mathbb{R}_0^+$ can be split into

  - **floor** $\lfloor q \rfloor \in \mathbb{N}_0$ and
  - **fraction** $\mathit{frac}(q) \in [0, 1)$

  such that

$$q = \lfloor q \rfloor + \mathit{frac}(q).$$

An Equivalence-Relation on Valuations

Definition. Let X be a set of clocks, $c_x \in \mathbb{N}_0$ for each clock $x \in X$, and ν_1, ν_2 clock valuations of X.

We set $\nu_1 \cong \nu_2$ iff the following **four** conditions are satisfied.

(1) For all $x \in X$,

$$\lfloor \nu_1(x) \rfloor = \lfloor \nu_2(x) \rfloor \text{ or } \textbf{both } \nu_1(x) > c_x \text{ and } \nu_2(x) > c_x.$$

(2) For all $x \in X$ with $\nu_1(x) \leq c_x$,

$$\mathit{frac}(\nu_1(x)) = 0 \textbf{ if and only if } \mathit{frac}(\nu_2(x)) = 0.$$

(3) For all $x, y \in X$,

$$\lfloor \nu_1(x) - \nu_1(y) \rfloor = \lfloor \nu_2(x) - \nu_2(y) \rfloor$$
$$\text{or } \textbf{both } |\nu_1(x) - \nu_1(y)| > c \text{ and } |\nu_2(x) - \nu_2(y)| > c.$$

(4) For all $x, y \in X$ with $-c \leq \nu_1(x) - \nu_1(y) \leq c$,

$$\mathit{frac}(\nu_1(x) - \nu_1(y)) = 0 \textbf{ if and only if } \mathit{frac}(\nu_2(x) - \nu_2(y)) = 0.$$

Where $c = \max\{c_x, c_y\}$.

Example: Regions

(1) $\forall\, x \in X : \lfloor \nu_1(x) \rfloor = \lfloor \nu_2(x) \rfloor \vee (\nu_1(x) > c_x \wedge \nu_2(x) > c_x)$

(2) $\forall\, x \in X : \nu_1(x) \leq c_x$
$\implies (\mathit{frac}(\nu_1(x)) = 0 \iff \mathit{frac}(\nu_2(x)) = 0)$

(3) $\forall\, x, y \in X : \lfloor \nu_1(x) - \nu_1(y) \rfloor = \lfloor \nu_2(x) - \nu_2(y) \rfloor$
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(4) $\forall\, x, y \in X : -c \leq \nu_1(x) - \nu_1(y) \leq c \implies$
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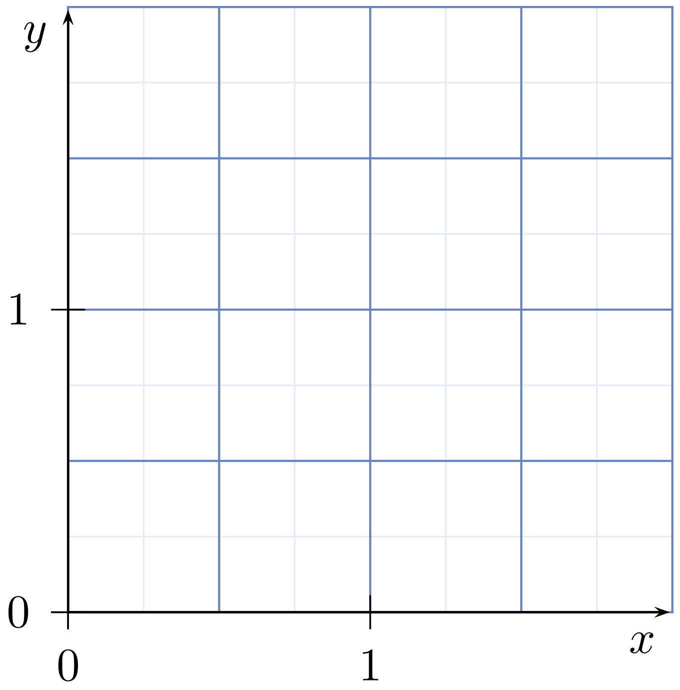


Regions

Proposition. $\cong$ is an **equivalence relation**.

Definition 4.27. For a given valuation ν we denote by $[\nu]$ the equivalence class of ν. We call equivalence classes of $\cong$ **regions**.

The Region Automaton

Definition 4.29. [*Region Automaton*] The **region automaton** $\mathcal{R}(\mathcal{A})$ of the timed automaton $\mathcal{A}$ is the labelled transition system

$$\mathcal{R}(\mathcal{A}) = (\mathit{Conf}(\mathcal{R}(\mathcal{A})), B_{?!}, \{\xrightarrow{\alpha}_{R(\mathcal{A})} \mid \alpha \in B_{?!}\}, C_{ini})$$

where

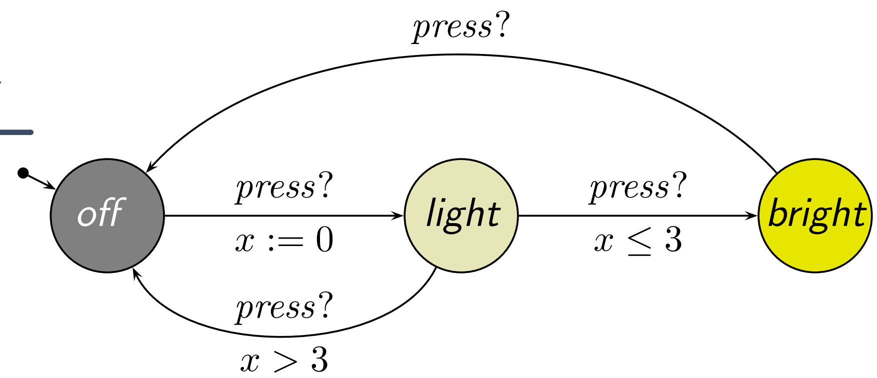
- $\mathit{Conf}(\mathcal{R}(\mathcal{A})) = \{\langle \ell, [\nu] \rangle \mid \ell \in L, \nu : X \to \mathsf{Time}, \nu \models I(\ell)\}$,
- for each $\alpha \in B_{?!}$,

  $$\langle \ell, [\nu] \rangle \xrightarrow{\alpha}_{R(\mathcal{A})} \langle \ell', [\nu'] \rangle \text{ if and only if } \langle \ell, \nu \rangle \overset{\alpha}{\Longrightarrow} \langle \ell', \nu' \rangle$$

  in $\mathcal{U}(\mathcal{A})$, and
- $C_{ini} = \{\langle \ell_{ini}, [\nu_{ini}] \rangle\} \cap \mathit{Conf}(\mathcal{R}(\mathcal{A}))$ with $\nu_{ini}(X) = \{0\}$.

Proposition. The transition relation of $\mathcal{R}(\mathcal{A})$ is **well-defined**, that is, independent of the choice of the representative ν of a region $[\nu]$.

Example: Region Automaton



$\mathcal{U}(\mathcal{A})$:

$\cdots \xRightarrow{\text{press}} \langle \text{light}, [x = 0] \rangle$

$\xRightarrow{\text{press}} \langle \text{bright}, [x = 0] \rangle \xRightarrow{\text{press}} \cdots$

$\xRightarrow{\text{press}} \langle \text{bright}, [x = 0.1] \rangle \xRightarrow{\text{press}} \cdots$

$\xRightarrow{\text{press}} \langle \text{bright}, [x = 1.0] \rangle \xRightarrow{\text{press}} \cdots$

$\cdots$

$\xRightarrow{\text{press}} \langle \text{bright}, [x = 3.0] \rangle \xRightarrow{\text{press}} \cdots$

$\langle \text{bright}, [x = 3.001] \rangle \xRightarrow{\text{press}} \cdots$

$\langle \text{off}, [x = 0] \rangle \xRightarrow{\text{press}} \cdots$

$\cdots$

$\langle \text{off}, [x = 2.9] \rangle \xRightarrow{\text{press}} \cdots$

$\langle \text{off}, [x = 3.0] \rangle \xRightarrow{\text{press}} \cdots$

$\xRightarrow{\text{press}} \langle \text{off}, [x = 3.001] \rangle \xRightarrow{\text{press}} \cdots$

Remark

Remark 4.30. That a configuration $\langle \ell, [\nu] \rangle$ is reachable in $\mathcal{R}(\mathcal{A})$ represents the fact, that all $\langle \ell, \nu \rangle$ are reachable.

IAW: in $\mathcal{A}$, we can observe ν when

location ℓ has **just been entered**.

The clock values reachable by staying/letting time pass in ℓ are **not explicitly** represented by the regions of $\mathcal{R}(\mathcal{A})$.

Decidability of The Location Reachability Problem

Claim: (**Theorem 4.33**)

The location reachability problem is **decidable** for timed automata.

Approach: Constructive proof.

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Region Automaton Properties

Lemma 4.32. [*Correctness*] For all locations ℓ of a given timed automaton $\mathcal{A}$ the following holds:

ℓ is reachable in $\mathcal{U}(\mathcal{A})$ if and only if ℓ is reachable in $\mathcal{R}(\mathcal{A})$.

For the **Proof**:

Definition 4.21. [*Bisimulation*] An equivalence relation $\sim$ on valuations is a **(strong) bisimulation** if and only if, whenever

$$\nu_1 \sim \nu_2 \text{ and } \langle \ell, \nu_1 \rangle \xrightarrow{\alpha} \langle \ell', \nu_1' \rangle$$

then there exists ν_2' with $\nu_1' \sim \nu_2'$ and $\langle \ell, \nu_2 \rangle \xrightarrow{\alpha} \langle \ell', \nu_2' \rangle$.

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The Number of Regions

Lemma 4.28. Let X be a set of clocks, $c_x \in \mathbb{N}_0$ the maximal constant for each $x \in X$, and $c = \max\{c_x \mid x \in X\}$. Then

$$(2c+2)^{|X|} \cdot (4c+3)^{\frac{1}{2}|X|\cdot(|X|-1)}$$

is an **upper bound** on the **number of regions**.

Proof: [Olderog and Dierks, 2008]

Observations Regarding the Number of Regions

- Lemma 4.28 **in particular** tells us that each timed automaton (in our definition) has **finitely** many regions.

- Note: the upper bound is a **worst case**, not an **exact bound**.

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Putting It All Together

Let $\mathcal{A} = (L, B, X, I, E, \ell_{ini})$ be a timed automaton, $\ell \in L$ a location.

- $\mathcal{R}(\mathcal{A})$ can be constructed effectively.
- There are finitely many locations in L (by definition).
- There are finitely many regions by Lemma 4.28.
- So $\mathit{Conf}(\mathcal{R}(\mathcal{A}))$ is finite (by construction).
- It is decidable whether (C_{init} of $\mathcal{R}(\mathcal{A})$ is empty) or whether there exists a sequence

$$\langle \ell_{ini}, [\nu_{ini}] \rangle \xrightarrow{\alpha}_{R(\mathcal{A})} \langle \ell_1, [\nu_1] \rangle \xrightarrow{\alpha}_{R(\mathcal{A})} \dots \xrightarrow{\alpha}_{R(\mathcal{A})} \langle \ell_n, [\nu_n] \rangle$$

  such that $\ell_n = \ell$ (reachability in graphs).

Putting It All Together

Let $\mathcal{A} = (L, B, X, I, E, \ell_{ini})$ be a timed automaton, $\ell \in L$ a location.

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  such that $\ell_n = \ell$ (reachability in graphs).

So we have

> **Theorem 4.33.** [*Decidability*]
> The location reachability problem for timed automata is **decidable**.

The Constraint Reachability Problem

- **Given:** A timed automaton $\mathcal{A}$, one of its control locations ℓ, and a clock constraint φ.
- **Question:** Is a configuration $\langle \ell, \nu \rangle$ **reachable** where $\nu \models \varphi$, i.e. is there a transition sequence of the form

$$\langle \ell_{ini}, \nu_{ini} \rangle \xrightarrow{\lambda_1} \langle \ell_1, \nu_1 \rangle \xrightarrow{\lambda_2} \langle \ell_2, \nu_2 \rangle \xrightarrow{\lambda_3} \ldots \xrightarrow{\lambda_n} \langle \ell_n, \nu_n \rangle = \langle \ell, \nu \rangle$$

  in the labelled transition system $\mathcal{T}(\mathcal{A})$ with $\nu \models \varphi$?

- **Note**: we just observed that $\mathcal{R}(\mathcal{A})$ loses some information about the clock valuations that are possible in/from a region.

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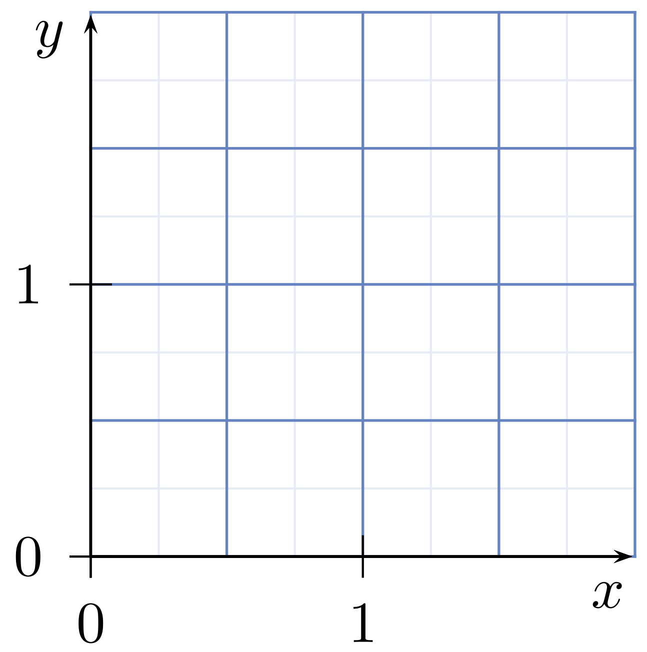
> **Theorem 4.34.** The constraint reachability problem for timed automata is decidable.

The Delay Operation

- Let $[\nu]$ be a clock region.
- We set $delay[\nu] := \{\nu' + t \mid \nu' \cong \nu$ and $t \in \mathsf{Time}\}$.



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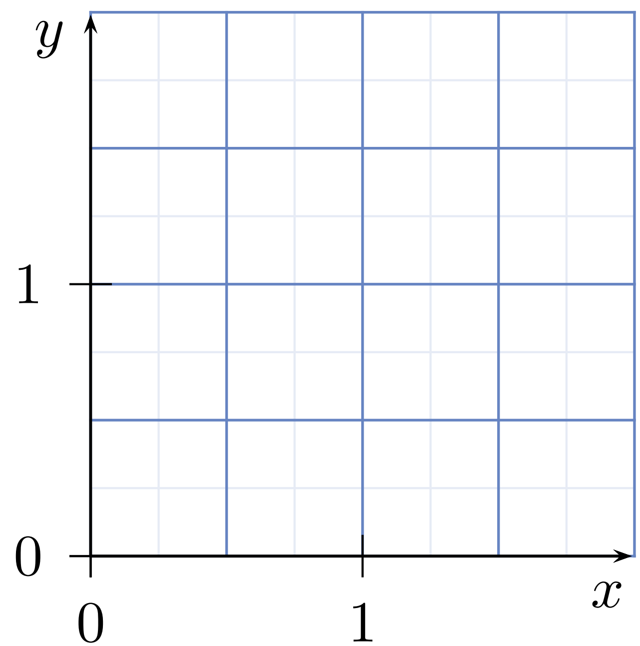

- **Note**: $delay[\nu]$ can be represented as a **finite** union of regions.

  For example, with our two-clock example we have

$$delay[x = y = 0] = [x = y = 0] \cup [0 < x = y < 1] \cup [x = y = 1] \cup [1 < x = y]$$

References

[Olderog and Dierks, 2008] Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.