Softwaretechnik / Software-Engineering

Lecture 17: Software Verification

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Prof. Dr. Andreas Podelski, **Dr. Bernd Westphal**

Albert-Ludwigs-Universität Freiburg, Germany

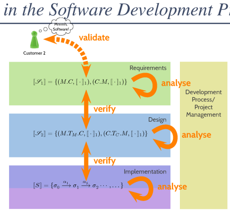
Topic Area Code Quality Assurance: Content

Content

- Software quality assurance in a **larger scope**:
  - vocabulary,
  - fault, error, failure,
  - concepts of software quality assurance (next to testing)
- **Formal Program Verification**
  - **Deterministic Programs**
    - **Syntax**
    - **Semantics**
    - Termination, Divergence
  - **Correctness** of deterministic programs
    - **partial** correctness,
    - **total** correctness.
  - **Proof System PD**
- **The Verifier for Concurrent C**

Software Quality Assurance

Formal Methods in the Software Development Process



validation –
The process of evaluating a system or component during or at the end of the development process to determine whether it satisfies specified requirements.
Contrast with: **verification**.
IEEE 610.12 (1990)

verification –
(1) The process of evaluating a system or component to determine whether the products of a given development phase satisfy the conditions imposed at the start of that phase. Contrast with: **validation**.
(2) Formal proof of program correctness.
IEEE 610.12 (1990)

Vocabulary

software quality assurance – See: quality assurance. **IEEE 610.12 (1990)**

quality assurance –
(1) A planned and systematic pattern of all actions necessary to provide adequate confidence that an item or product conforms to established technical requirements.
(2) A set of activities designed to evaluate the process by which products are developed or manufactured.
IEEE 610.12 (1990)

Note: in order to trust a product, it can be **built well**,
or **proven to be good** (at best: both) – both is QA in the sense of (1).

Fault, Error, Failure

fault – abnormal condition that can cause an element or an item to fail.

Note: Permanent, intermittent and transient **faults** (especially soft-errors) are considered.

Note: An **intermittent fault** occurs time and time again, then disappears. This type of fault can occur when a component is on the verge of breaking down or, for example, due to a glitch in a switch. Some **systematic faults** (e.g. timing marginalities) could lead to intermittent faults.

ISO 26262 (2011)

error – discrepancy between a computed, observed or measured value or condition, and the true, specified, or theoretically correct value or condition.

Note: An error can arise as a result of unforeseen operating conditions or due to a **fault** within the system, subsystem or, component being considered.

Note: A fault can manifest itself as an error within the considered element and the error can ultimately cause a failure.

ISO 26262 (2011)

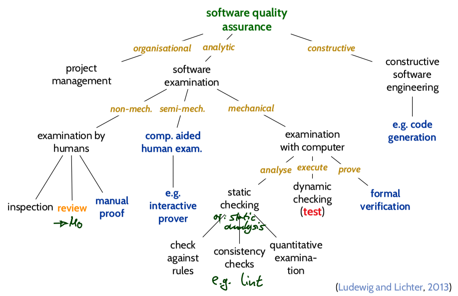
failure – termination of the ability of an element, to perform a function as required.

Note: Incorrect specification is a source of failure.

ISO 26262 (2011)

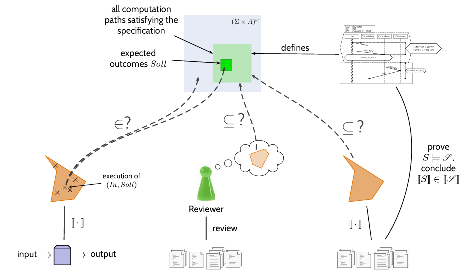
We want to avoid **failures**, thus we try to detect **faults** and **errors**.

Concepts of Software Quality Assurance



(Ludewig and Lichter, 2013)

Three Basic Approaches



Sequential, Deterministic While-Programs

Deterministic Programs

Syntax:

$$S := skip \mid u := t \mid S_1; S_2 \mid \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi} \mid \textbf{while } B \textbf{ do } S_1 \textbf{ od}$$

where $u \in V$ is a **variable**, t is a type-compatible **expression**, B is a Boolean **expression**.

Semantics: (is induced by the following transition relation) – $\sigma : V \to \mathcal{D}(V)$

(i) $\langle skip,\ \sigma\rangle \to \langle E,\ \sigma\rangle$ *empty program*

(ii) $\langle u := t,\ \sigma\rangle \to \langle E,\ \sigma[u := \sigma(t)]\rangle$

(iii) $\dfrac{\langle S_1,\ \sigma\rangle \to \langle S_2,\ \tau\rangle}{\langle S_1; S,\ \sigma\rangle \to \langle S_2; S,\ \tau\rangle}$

(iv) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_1,\ \sigma\rangle$, if $\sigma \models B$,

(v) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_2,\ \sigma\rangle$, if $\sigma \not\models B$,

(vi) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle S; \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle$, if $\sigma \models B$,

(vii) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle E,\ \sigma\rangle$, if $\sigma \not\models B$.

E denotes the **empty program**; define $E; S \equiv S; E \equiv S$.

Note: the first component of $\langle S,\ \sigma\rangle$ is a program (**structural operational semantics** (SOS)).

Example

(i) $\langle skip,\ \sigma\rangle \to \langle E,\ \sigma\rangle$ $\qquad E; S \equiv S; E \equiv S$

(ii) $\langle u := t,\ \sigma\rangle \to \langle E,\ \sigma[u := \sigma(t)]\rangle$

(iii) $\dfrac{\langle S_1,\ \sigma\rangle \to \langle S_2,\ \tau\rangle}{\langle S_1; S,\ \sigma\rangle \to \langle S_2; S,\ \tau\rangle}$

(iv) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_1,\ \sigma\rangle$, if $\sigma \models B$,

(v) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi},\ \sigma\rangle \to \langle S_2,\ \sigma\rangle$, if $\sigma \not\models B$,

(vi) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle S; \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle$, if $\sigma \models B$,

(vii) $\langle \textbf{while } B \textbf{ do } S \textbf{ od},\ \sigma\rangle \to \langle E,\ \sigma\rangle$, if $\sigma \not\models B$.

Consider **program**

$$S \equiv a[0] := 1; a[1] := 0; \textbf{while } a[x] \neq 0 \textbf{ do } x := x + 1 \textbf{ od}$$

and a **state** σ with $\sigma \models x = 0$.

$$\begin{aligned}
\langle S,\ \sigma\rangle &\xrightarrow{(ii),(iii)} \langle a[1] := 0; \textbf{while } a[x] \neq 0 \textbf{ do } x := x+1 \textbf{ od},\ \sigma[a[0] := 1]\rangle \\
&\xrightarrow{(ii),(iii)} \langle \textbf{while } a[x] \neq 0 \textbf{ do } x := x+1 \textbf{ od},\ \sigma'\rangle \\
&\xrightarrow{(vi)} \langle x := x+1; \textbf{while } a[x] \neq 0 \textbf{ do } x := x+1 \textbf{ od},\ \sigma'\rangle \\
&\xrightarrow{(ii),(iii)} \langle \textbf{while } a[x] \neq 0 \textbf{ do } x := x+1 \textbf{ od},\ \sigma'[x := 1]\rangle \\
&\xrightarrow{(vii)} \langle E,\ \sigma'[x := 1]\rangle
\end{aligned}$$

where $\sigma' = \sigma[a[0] := 1][a[1] := 0]$.

Another Example

(i) $\langle skip, \sigma\rangle \to \langle E, \sigma\rangle$ $\qquad E; S \equiv S; E \equiv S$
(ii) $\langle u := t, \sigma\rangle \to \langle E, \sigma[u := \sigma(t)]\rangle$
(iii) $\dfrac{\langle S_1, \sigma\rangle \to \langle S_2, \tau\rangle}{\langle S_1; S, \sigma\rangle \to \langle S_2; S, \tau\rangle}$
(iv) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi}, \sigma\rangle \to \langle S_1, \sigma\rangle$, if $\sigma \models B$,
(v) $\langle \textbf{if } B \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ fi}, \sigma\rangle \to \langle S_2, \sigma\rangle$, if $\sigma \not\models B$,
(vi) $\langle \textbf{while } B \textbf{ do } S \textbf{ od}, \sigma\rangle \to \langle S; \textbf{while } B \textbf{ do } S \textbf{ od}, \sigma\rangle$, if $\sigma \models B$,
(vii) $\langle \textbf{while } B \textbf{ do } S \textbf{ od}, \sigma\rangle \to \langle E, \sigma\rangle$, if $\sigma \not\models B$,

Consider **program**

$$S_1 \equiv y := x; y := (x-1) \cdot x + y$$

and a **state** σ with $\sigma \models x = 3$.

$$\begin{aligned}\langle S_1, \sigma\rangle &\xrightarrow{(ii),(iii)} \langle y := (x-1)\cdot x + y, \{x \mapsto 3, y \mapsto 3\}\rangle \\ &\xrightarrow{(ii)} \langle E, \{x \mapsto 3, y \mapsto 9\}\rangle\end{aligned}$$

Consider **program** $S_3 \equiv y := x;\ y := (x-1)\cdot x + y;\ \textbf{while } 1 \textbf{ do } skip \textbf{ od}$.

$$\begin{aligned}\langle S_3, \sigma\rangle &\xrightarrow{(ii),(iii)} \langle y := (x-1)\cdot x + y; \textbf{while } 1 \textbf{ do } skip \textbf{ od}, \{x \mapsto 3, y \mapsto 3\}\rangle \\ &\xrightarrow{(ii),(iii)} \langle \textbf{while } 1 \textbf{ do } skip \textbf{ od}, \{x \mapsto 3, y \mapsto 9\}\rangle \\ &\xrightarrow{(vi)} \langle skip; \textbf{while } 1 \textbf{ do } skip \textbf{ od}, \{x \mapsto 3, y \mapsto 9\}\rangle \\ &\xrightarrow{(i),(iii)} \langle \textbf{while } 1 \textbf{ do } skip \textbf{ od}, \{x \mapsto 3, y \mapsto 9\}\rangle \\ &\xrightarrow{(vi)} \ldots\end{aligned}$$

Computations of Deterministic Programs

Definition. Let S be a deterministic program.

(i) A **transition sequence** of S (starting in σ) is a finite or infinite sequence

$$\langle S, \sigma\rangle = \langle S_0, \sigma_0\rangle \to \langle S_1, \sigma_1\rangle \to \ldots$$

(that is, $\langle S_i, \sigma_i\rangle$ and $\langle S_{i+1}, \sigma_{i+1}\rangle$ are in transition relation for all i).

(ii) A **computation (path)** of S (starting in σ) is a **maximal** transition sequence of S (starting in σ), i.e. infinite or not extendible.

(iii) A computation of S is said to

a) **terminate** in τ if and only if it is finite and ends with $\langle E, \tau\rangle$,

b) **diverge** if and only if it is infinite.

S **can diverge from** σ if and only if a diverging computation starts in σ.

(iv) We use $\to^*$ to denote the transitive, reflexive closure of $\to$.

Lemma. For each deterministic program S and each state σ, there is exactly one computation of S which starts in σ.

Input/Output Semantics of Deterministic Programs

Definition.
Let S be a deterministic program.

(i) The **semantics of partial correctness** is the function

$$\mathcal{M}[\![S]\!] : \Sigma \to 2^\Sigma$$

with $\mathcal{M}[\![S]\!](\sigma) = \{\tau \mid \langle S, \sigma\rangle \to^* \langle E, \tau\rangle\}$.

(ii) The **semantics of total correctness** is the function

$$\mathcal{M}_{tot}[\![S]\!] : \Sigma \to 2^\Sigma \,\dot\cup\, \{\infty\}$$

with $\mathcal{M}_{tot}[\![S]\!](\sigma) = \mathcal{M}[\![S]\!](\sigma) \cup \{\infty \mid S \text{ can diverge from } \sigma\}$.
∞ is an error state representing **divergence**.

Note: $\mathcal{M}_{tot}[\![S]\!](\sigma)$ has exactly one element, $\quad \mathcal{M}[\![S]\!](\sigma)$ at most one.

Example: $\mathcal{M}[\![S_1]\!](\sigma) = \mathcal{M}_{tot}[\![S_1]\!](\sigma) = \{\tau \mid \tau(x) = \sigma(x) \wedge \tau(y) = \sigma(x)^2\}, \quad \sigma \in \Sigma$.
(Recall: $S_1 \equiv y := x; y := (x-1)\cdot x + y$)

Correctness of While-Programs

Correctness of Deterministic Programs

Definition.
Let S be a program over variables V, and p and q Boolean expressions over V.

(i) The **correctness formula**

$$\{p\}\ S\ \{q\} \qquad \text{("Hoare triple")}$$

holds in the sense of partial correctness, denoted by $\models \{p\}\ S\ \{q\}$, if and only if

$$\mathcal{M}[\![S]\!]([\![p]\!]) \subseteq [\![q]\!].$$

We say S is **partially correct** wrt. p and q.

(ii) A **correctness formula**

$$\{p\}\ S\ \{q\}$$

holds in the sense of total correctness, denoted by $\models_{tot} \{p\}\ S\ \{q\}$, if and only if

$$\mathcal{M}_{tot}[\![S]\!]([\![p]\!]) \subseteq [\![q]\!].$$

We say S is **totally correct** wrt. p and q.

Example: Computing squares (of numbers $0, \ldots, 27$)

- **Pre-condition**: $p \equiv 0 \le x \le 27$,
- **Post-condition**: $q \equiv y = x^2$.

Program S_1:

```
int y = x;
y = (x - 1) * x + y;
```

$\models^? \{p\}\ S_1\ \{q\}$ ✓
$\models^?_{tot} \{p\}\ S_1\ \{q\}$ ✓

Program S_2:

```
int y = x;
int z; // uninitialised
y = ((x - 1) * x + y) + z;
```

$\models^? \{p\}\ S_2\ \{q\}$ ✗
$\models^?_{tot} \{p\}\ S_2\ \{q\}$ ✗

Program S_3:

```
int y = x;
y = (x - 1) * x + y;
while (1);
```

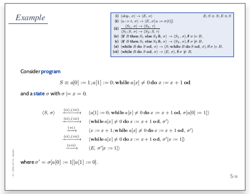
$\models^? \{p\}\ S_3\ \{q\}$ ✓
$\models^?_{tot} \{p\}\ S_3\ \{q\}$ ✗

Program S_4:

```
int x = read_input();
int y = x + (x-1) * x;
```

$\models^? \{p\}\ S_4\ \{q\}$ ✗
$\models^?_{tot} \{p\}\ S_4\ \{q\}$ ✗

Example: Correctness



- By the example, **we have shown**

$$\models \{x = 0\}\ S\ \{x = 1\}$$

and

$$\models_{tot} \{x = 0\}\ S\ \{x = 1\}.$$

(because we only assumed $\sigma \models x = 0$ for the example, which is exactly the precondition.)

- **We have also shown** (= **proved** (!)):

$$\models \{x = 0\}\ S\ \{x = 1 \wedge a[x] = 0\}.$$

- The correctness formula $\{x = 2\}\ S\ \{\mathit{true}\}$ **does not hold** for S.
(For example, if $\sigma \models a[i] \neq 0$ for all $i > 2$.)

- In the sense of **partial correctness**, $\{x = 2 \wedge \forall\, i \geq 2 \bullet a[i] = 1\}\ S\ \{\mathit{false}\}$ also holds.

Proof-System PD

Proof-System PD *(for sequential, deterministic programs)*

Axiom 1: Skip-Statement

$$\{p\}\ \mathit{skip}\ \{p\}$$

Axiom 2: Assignment

$$\{p[u := t]\}\ u := t\ \{p\}$$

Rule 3: Sequential Composition

$$\frac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$$

Rule 4: Conditional Statement

$$\frac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\},}{\{p\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$$

Rule 5: While-Loop

$$\frac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$$

Rule 6: Consequence

$$\frac{p \rightarrow p_1,\ \{p_1\}\ S\ \{q_1\},\ q_1 \rightarrow q}{\{p\}\ S\ \{q\}}$$

Theorem. PD is correct ("sound") and (relative) complete for partial correctness of deterministic programs, i.e. $\vdash_{PD} \{p\}\ S\ \{q\}$ if and only if $\models \{p\}\ S\ \{q\}$.

Example Proof

$$DIV \equiv \overbrace{a := 0;\ b := x;}^{=:S_0^D}\ \textbf{while}\ \overbrace{b \geq y}^{=:B^D}\ \textbf{do}\ \overbrace{b := b - y;\ a := a + 1}^{=:S_1^D}\ \textbf{od}$$

(The first (textually represented) program that has been formally verified (Hoare, 1969).

We can prove $\models \{x \geq 0 \wedge y \geq 0\}\ DIV\ \{a \cdot y + b = x \wedge b < y\}$

by showing $\vdash_{PD} \{\underbrace{x \geq 0 \wedge y \geq 0}_{=:p^D}\}\ DIV\ \{\underbrace{a \cdot y + b = x \wedge b < y}_{=:q^D}\}$, i.e., derivability in PD:

$$\dfrac{\dfrac{(1)}{\{p^D\}\ S_0^D\ \{P\},} \qquad \dfrac{\dfrac{\checkmark}{P \rightarrow P,} \quad \dfrac{\dfrac{(2)}{\{P \wedge (B^D)\}\ S_1^D\ \{P\}}}{\{P\}\ \textbf{while}\ B^D\ \textbf{do}\ S_1^D\ \textbf{od}\ \{P \wedge \neg(B^D)\},}\text{(R5)} \quad \dfrac{(3)}{P \wedge \neg(B^D) \rightarrow q^D}}{\{P\}\ \textbf{while}\ B^D\ \textbf{do}\ S_1^D\ \textbf{od}\ \{q^D\}}\text{(R6)}}{\{p^D\}\ S_0^D;\ \textbf{while}\ B^D\ \textbf{do}\ S_1^D\ \textbf{od}\ \{q^D\}}\text{(R3)}$$

(A1) $\{p\}\ \mathit{skip}\ \{p\}$

(A2) $\{p[u := t]\}\ u := t\ \{p\}$

(R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$

(R4) $\dfrac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\}}{\{p\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$

(R5) $\dfrac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$

(R6) $\dfrac{p \rightarrow p_1,\ \{p_1\}\ S\ \{q_1\},\ q_1 \rightarrow q}{\{p\}\ S\ \{q\}}$

Example Proof

$$DIV \equiv \overbrace{a := 0;\ b := x;}^{=:S_0^D}\ \textbf{while}\ \overbrace{b \geq y}^{=:B^D}\ \textbf{do}\ \overbrace{b := b - y;\ a := a + 1}^{=:S_1^D}\ \textbf{od}$$

(The first (textually represented) program that has been formally verified (Hoare, 1969).

We can prove $\models \{x \geq 0 \wedge y \geq 0\}\ DIV\ \{a \cdot y + b = x \wedge b < y\}$

by showing $\vdash_{PD} \{\underbrace{x \geq 0 \wedge y \geq 0}_{=:p^D}\}\ DIV\ \{\underbrace{a \cdot y + b = x \wedge b < y}_{=:q^D}\}$, i.e., derivability in PD:

$$\dfrac{\dfrac{(1)}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\},} \qquad \dfrac{\dfrac{}{P \rightarrow P,} \quad \dfrac{\dfrac{(2)}{\{P \wedge (b \geq y)\}\ b := b - y;\ a := a + 1\ \{P\}}}{\{P\}\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{P \wedge \neg(b \geq y)\},}\text{(R5)} \quad \dfrac{(3)}{P \wedge \neg(b \geq y) \rightarrow a \cdot y + b = x \wedge b < y}}{\{P\}\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R6)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x;\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R3)}$$

(A1) $\{p\}\ \mathit{skip}\ \{p\}$

(A2) $\{p[u := t]\}\ u := t\ \{p\}$

(R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$

(R4) $\dfrac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\}}{\{p\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$

(R5) $\dfrac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$

(R6) $\dfrac{p \rightarrow p_1,\ \{p_1\}\ S\ \{q_1\},\ q_1 \rightarrow q}{\{p\}\ S\ \{q\}}$

Example Proof Cont'd

$$\dfrac{\dfrac{(1)}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\},} \qquad \dfrac{\dfrac{}{P \rightarrow P,} \quad \dfrac{\dfrac{(2)}{\{P \wedge (b \geq y)\}\ b := b - y;\ a := a + 1\ \{P\}}}{\{P\}\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{P \wedge \neg(b \geq y)\},}\text{(R5)} \quad \dfrac{(3)}{P \wedge \neg(b \geq y) \rightarrow a \cdot y + b = x \wedge b < y}}{\{P\}\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R6)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x;\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R3)}$$

In the following, we show

(1) $\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$,

(2) $\vdash_{PD} \{P \wedge b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$,

(3) $\models P \wedge \neg(b \geq y) \rightarrow a \cdot y + b = x \wedge b < y$.

As **loop invariant**, we choose (**creative act!**):

$$P \equiv a \cdot y + b = x \wedge b \geq 0$$

Proof of (1)

(A1) $\{p\}\ skip\ \{p\}$ (R4) $\dfrac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\}}{\{p\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$

(A2) $\{p[u := t]\}\ u := t\ \{p\}$ (R5) $\dfrac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$

(R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$ (R6) $\dfrac{p \to p_1,\ \{p_1\}\ S\ \{q_1\},\ q_1 \to q}{\{p\}\ S\ \{q\}}$

- (1) claims:
  $\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$
  where $P \equiv a \cdot y + b = x \wedge b \geq 0$.
- $\vdash_{PD} \{0 \cdot y + x = x \wedge x \geq 0\}\ a := 0\ \{a \cdot y + x = x \wedge x \geq 0\}$ by (A2).

p[u:=t]
p[a:=0]

Proof of (1)

- (1) claims:
  $\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$
  where $P \equiv a \cdot y + b = x \wedge b \geq 0$.
- $\vdash_{PD} \{0 \cdot y + x = x \wedge x \geq 0\}\ a := 0\ \{a \cdot y + x = x \wedge x \geq 0\}$ by (A2).
- $\vdash_{PD} \{a \cdot y + x = x \wedge x \geq 0\}\ b := x\ \{\underbrace{a \cdot y + b = x \wedge b \geq 0}_{\equiv P}\}$ by (A2).

p[b:=x]

Proof of (1)

- (1) claims:
  $\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$
  where $P \equiv a \cdot y + b = x \wedge b \geq 0$.
- $\vdash_{PD} \{0 \cdot y + x = x \wedge x \geq 0\}\ a := 0\ \{a \cdot y + x = x \wedge x \geq 0\}$ by (A2).
- $\vdash_{PD} \{a \cdot y + x = x \wedge x \geq 0\}\ b := x\ \{\underbrace{a \cdot y + b = x \wedge b \geq 0}_{\equiv P}\}$ by (A2).
- thus, $\vdash_{PD} \{0 \cdot y + x = x \wedge x \geq 0\}\ a := 0;\ b := x\ \{P\}$ by (R3).
- using $x \geq 0 \wedge y \geq 0 \to 0 \cdot y + x = x \wedge x \geq 0$ and $P \to P$, we obtain

$$\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$$

  by (R6). □

Substitution

The rule '**Assignment**' uses (syntactical) **substitution**: $\{p[u := t]\}\ u := t\ \{p\}$

(In formula p, replace all (free) occurences of (program or logical) variable u by term t.)

Defined as usual, only **indexed** and **bound** variables need to be treated specially:

Expressions:

- plain variable x: $x[u := t] \equiv \begin{cases} t & \text{, if } x = u \\ x & \text{, otherwise} \end{cases}$
- constant c:
  $c[u := t] \equiv c$.
- constant op, terms s_i:
  $op(s_1, \dots, s_n)[u := t]$
  $\equiv op(s_1[u := t], \dots, s_n[u := t])$.
- conditional expression:
  $(B\ ?\ s_1 : s_2)[u := t]$
  $\equiv (B[u := t]\ ?\ s_1[u := t] : s_2[u := t])$

Formulae:

- boolean expression $p \equiv s$:
  $p[u := t] \equiv s[u := t]$
- negation:
  $(\neg q)[u := t] \equiv \neg(q[u := t])$
- conjunction etc.:
  $(q \wedge r)[u := t]$
  $\equiv q[u := t] \wedge r[u := t]$
- **quantifier**:
  $(\forall\, x : q)[u := t] \equiv \forall\, y : q[x := y][u := t]$
  y fresh (not in q, t, u), same type as x.

- **indexed variable**, u plain or $u \equiv b[t_1, \dots, t_m]$ and $a \neq b$:

$$(a[s_1, \dots, s_n])[u := t] \equiv a[s_1[u := t], \dots, s_n[u := t]])$$

- **indexed variable**, $u \equiv a[t_1, \dots, t_m]$:

$$(a[s_1, \dots, s_n])[u := t] \equiv (\textstyle\bigwedge_{i=1}^{n} s_i[u := t] = t_i\ ?\ t\ : a[s_1[u := t], \dots, s_n[u := t]])$$

Proof of (2)

- (2) claims:
  $\vdash_{PD} \{P \wedge b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$
  where $P \equiv a \cdot y + b = x \wedge b \geq 0$.
- $\vdash_{PD} \{(a+1) \cdot y + (b - y) = x \wedge (b - y) \geq 0\}\ b := b - y\ \{(a+1) \cdot y + b = x \wedge b \geq 0\}$
  by (A2).

Proof of (2)

- (2) claims:
  $\vdash_{PD} \{P \wedge b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$
  where $P \equiv a \cdot y + b = x \wedge b \geq 0$.
- $\vdash_{PD} \{(a+1) \cdot y + (b - y) = x \wedge (b - y) \geq 0\}\ b := b - y\ \{(a+1) \cdot y + b = x \wedge b \geq 0\}$
  by (A2).
- $\vdash_{PD} \{(a+1) \cdot y + b = x \wedge b \geq 0\}\ a := a + 1\ \{\underbrace{a \cdot y + b = x \wedge b \geq 0}_{\equiv P}\}$ by (A2).

Proof of (2)

(A1) $\{p\}\ skip\ \{p\}$

(A2) $\{p[u := t]\}\ u := t\ \{p\}$

(R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$

(R4) $\dfrac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\}}{\{p\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$

(R5) $\dfrac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$

(R6) $\dfrac{p \to p_1,\ \{p_1\}\ S\ \{q_1\},\ q_1 \to q}{\{p\}\ S\ \{q\}}$

- (2) claims:
  $\vdash_{PD} \{P \wedge b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$
  where $P \equiv a \cdot y + b = x \wedge b \geq 0$.

- $\vdash_{PD} \{(a+1) \cdot y + (b - y) = x \wedge (b - y) \geq 0\}\ b := b - y\ \{(a+1) \cdot y + b = x \wedge b \geq 0\}$
  by (A2),

- $\vdash_{PD} \{(a+1) \cdot y + b = x \wedge b \geq 0\}\ a := a + 1\ \{\underbrace{a \cdot y + b = x \wedge b \geq 0}_{\equiv P}\}$ by (A2),

- $\vdash_{PD} \{(a+1) \cdot y + (b - y) = x \wedge (b - y) \geq 0\}\ b := b - y;\ a := a + 1\ \{P\}$ by (R3),

- using $(P \wedge b \geq y) \to ((a+1) \cdot y + (b - y) = x \wedge (b - y) \geq 0)$ and $P \to P$ we obtain,
  $$\vdash_{PD} \{P \wedge b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$$
  by (R6). □

Proof of (3)

(3) claims
$$\models (P \wedge \neg(b \geq y)) \to (a \cdot y + b = x \wedge b < y.)$$
where $P \equiv a \cdot y + b = x \wedge b \geq 0$.

Proof: easy.

Back to the Example Proof

We have shown:

(1) $\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}$,

(2) $\vdash_{PD} \{P \wedge b \geq y\}\ b := b - y;\ a := a + 1\ \{P\}$,

(3) $\models P \wedge \neg(b \geq y) \to a \cdot y + b = x \wedge b < y$.

and

$$\dfrac{\dfrac{(1)}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x\ \{P\}},\quad \dfrac{\dfrac{}{P \to P},\ \dfrac{\dfrac{(2)}{\{P \wedge (b \geq y)\}\ b := b - y;\ a := a + 1\ \{P\}}}{\{P\}\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{P \wedge \neg(b \geq y)\}}\text{(R5)},\ \dfrac{(3)}{P \wedge \neg(b \geq y) \to a \cdot y + b = x \wedge b < y}}{\{P\}\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R6)}}{\{x \geq 0 \wedge y \geq 0\}\ a := 0;\ b := x;\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}\ \{a \cdot y + b = x \wedge b < y\}}\text{(R3)}$$

thus

$$\vdash_{PD} \{x \geq 0 \wedge y \geq 0\}\ \underbrace{a := 0;\ b := x;\ \textbf{while}\ b \geq y\ \textbf{do}\ b := b - y;\ a := a + 1\ \textbf{od}}_{\equiv DIV}\ \{a \cdot y + b = x \wedge b < y\}$$

and thus (since PD is sound) DIV is **partially correct** wrt.

- pre-condition: $x \geq 0 \wedge y \geq 0$,
- post-condition: $a \cdot y + b = x \wedge b < y$.

IOW: whenever DIV is called with x and y such that $x \geq 0 \wedge y \geq 0$, then (if DIV terminates) $a \cdot y + b = x \wedge b < y$ will hold.

Once Again

(A1) $\{p\}\ skip\ \{p\}$

(A2) $\{p[u := t]\}\ u := t\ \{p\}$

(R3) $\dfrac{\{p\}\ S_1\ \{r\},\ \{r\}\ S_2\ \{q\}}{\{p\}\ S_1;\ S_2\ \{q\}}$

(R4) $\dfrac{\{p \wedge B\}\ S_1\ \{q\},\ \{p \wedge \neg B\}\ S_2\ \{q\}}{\{p\}\ \textbf{if}\ B\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi}\ \{q\}}$

(R5) $\dfrac{\{p \wedge B\}\ S\ \{p\}}{\{p\}\ \textbf{while}\ B\ \textbf{do}\ S\ \textbf{od}\ \{p \wedge \neg B\}}$

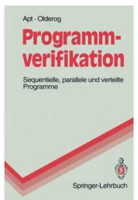
(R6) $\dfrac{p \to p_1,\ \{p_1\}\ S\ \{q_1\},\ q_1 \to q}{\{p\}\ S\ \{q\}}$

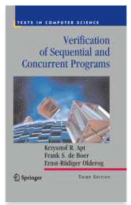
- $P \equiv a \cdot y + b = x \wedge b \geq 0$

$\{x \geq 0 \wedge y \geq 0\}$
$\{0 \cdot y + x = x \wedge x \geq 0\}$
- $a := 0;$
$\{a \cdot y + x = x \wedge x \geq 0\}$
- $b := x;$
$\{a \cdot y + b = x \wedge b \geq 0\}$
$\{P\}$
- $\textbf{while}\ b \geq y\ \textbf{do}$
$\{P \wedge b \geq y\}$
$\{(a+1) \cdot y + (b - y) = x \wedge (b - y) \geq 0\}$
- $b := b - y;$
$\{(a+1) \cdot y + b = x \wedge b \geq 0\}$
- $a := a + 1$
$\{a \cdot y + b = x \wedge b \geq 0\}$
$\{P\}$
- $\textbf{od}$
$\{P \wedge \neg(b \geq y)\}$
$\{a \cdot y + b = x \wedge b < y\}$

A2 A2 R3 R6 A2 A2 R3 R6 R5 R6

Literature Recommendation





Assertions

Assertions

- Extend the **syntax** of **deterministic programs** by

$$S := \cdots \mid \mathbf{assert}(B)$$

- and the **semantics** by rule

$$\langle \mathbf{assert}(B), \sigma \rangle \to \langle E, \sigma \rangle \text{ if } \sigma \models B.$$

(If the asserted boolean expression B does not hold in state σ, the empty program is not reached; otherwise the assertion remains in the first component: **abnormal** program termination).

Extend PD by axiom:

$$\text{(A7)}\ \{p\}\ \mathbf{assert}(p)\ \{p\}$$

- That is, if p holds **before** the assertion, then we can **continue** with the derivation in PD.

  If p does not hold, we **"get stuck"** (and cannot complete the derivation).

- So we **cannot** derive $\{true\}\ x := 0;\ \texttt{assert}(x = 27)\ \{true\}$ in PD.

Modular Reasoning

Modular Reasoning

We can add another rule for calls of functions $f : F$ (simplest case: only global variables):

$$\text{(R7)}\ \frac{\{p\}\ F\ \{q\}}{\{p\}\ f()\ \{q\}}$$

"If we have $\vdash \{p\}\ F\ \{q\}$ for the **implementation** of function f, then if f is **called** in a state satisfying p, the state after return of f will satisfy q."

p is called **pre-condition** and q is called **post-condition** of f.

Example: if we have

- $\{true\}$ `read_number` $\{0 \leq result < 10^8\}$
- $\{0 \leq x \wedge 0 \leq y\}$ `add` $\{(old(x) + old(y) < 10^8 \wedge result = old(x) + old(y)) \vee result < 0\}$
- $\{true\}$ `display` $\{(0 \leq old(x) < 10^8 \implies \text{"}old(x)\text{"}) \wedge (old(x) < 0 \implies \text{"-E-"})\}$

we may be able to prove our pocket calculator correct.


```
int main() {

  while (true) {
    int x = read_number();
    int y = read_number();

    int sum = add( x, y );

    display(sum);
  }
}
```


Return Values and Old Values

- For **modular reasoning**, it's often useful to refer in the post-condition
  - to the **return value** as $result$,
  - the **values** of variable x **at calling time** as $old(x)$.

- Can be defined using **auxiliary variables**:
  - Transform function

$$T\ f()\ \{\ldots; \mathbf{return}\ expr;\}$$

  (over variables $V = \{v_1, \ldots, v_n\}$, $result, v_i^{old} \notin V$) to

$$\begin{array}{l} T\ f()\ \{ \\ \quad v_1^{old} := v_1; \ldots; v_n^{old} := v_n; \\ \quad \ldots; \\ \quad result := expr; \\ \quad \mathbf{return}\ result; \\ \} \end{array}$$

  over $V' = V \cup \{v^{old} \mid v \in V\} \cup \{result\}$.

- Then $old(x)$ is just an abbreviation for x^{old}.

The Verifier for Concurrent C

VCC

- The **Verifier for Concurrent C** (VCC) basically implements Hoare-style reasoning.

- **Special syntax**:
  - `#include <vcc.h>`
  - `_(requires p)` – **pre-condition**, p is (basically) a C expression
  - `_(ensures q)` – **post-condition**, q is (basically) a C expression
  - `_(invariant expr)` – **loop invariant**, $expr$ is (basically) a C expression
  - `_(assert p)` – **intermediate invariant**, p is (basically) a C expression
  - `_(writes &v)` – VCC considers **concurrent** C programs; we need to declare for each procedure which global variables it is allowed to write to (also checked by VCC)
  - **Special expressions**:
    - `\thread_local(&v)` – no other thread writes to variable v (in pre-conditions)
    - `\old(v)` – the value of v when procedure was called (useful for post-conditions)
    - `\result` – return value of procedure (useful for post-conditions)

VCC Syntax Example

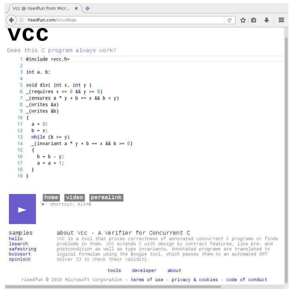
```
#include <vcc.h>

int a, b;

void div( int x, int y )
  _(requires x >= 0 && y >= 0)
  _(ensures a * y + b == x && b < y)
  _(writes &a)
  _(writes &b)
{
  a = 0;
  b = x;
  while (b >= y)
  _(invariant a * y + b == x && b >= 0)
  {
    b = b - y;
    a = a + 1;
  }
}
```

(handwritten annotations: pre-cond. p; post-cond. q; loop invariant p)

$$DIV \equiv a := 0;\ b := x;\ \mathbf{while}\ b \geq y\ \mathbf{do}\ b := b - y;\ a := a + 1\ \mathbf{od}$$

$$\{x \geq 0 \wedge y \geq 0\}\ DIV\ \{x \geq 0 \wedge y \geq 0\}$$

VCC Web-Interface



Example program DIV: `http://rise4fun.com/Vcc/4Kqe`

Interpretation of Results

- VCC says: "**verification succeeded**"

  We can **only** conclude that the tool
  – under its interpretation of the C-standard, under its platform assumptions (32-bit), etc. –
  "thinks" that it can prove $\models \{p\}\ DIV\ \{q\}$.

  Can be due to an error in the tool! (That's a **false negative** then.)

  Yet we can ask **for a printout of the proof** and check it manually
  (hardly possible in practice) or with other tools like interactive theorem provers.

  Note: $\models \{\mathit{false}\}\ f\ \{q\}$ **always** holds.

  That is, a **mistake** in writing down the pre-condition can make errors in the program go undetected.

- VCC says: "**verification failed**"
  - May be a **false positive**.

    The tool **does not provide counter-examples** in the form of a computation path,
    it (only) gives **hints on input values** satisfying p and causing a violation of q.

    → try to construct a (true) counter-example from the hints.

    or: → make pre-condition p or loop-invariant(s) stronger, and try again.

- Other case: "**timeout**" etc. – completely **inconclusive** outcome.

VCC Features

- For the exercises, we use VCC only for **sequential, single-thread programs**.
- VCC checks a number of **implicit assertions**:
  - **no arithmetic overflow** in expressions (according to C-standard),
  - **array-out-of-bounds access**,
  - **NULL-pointer dereference**,
  - and many more.
- VCC also supports:
  - **concurrency**:
    different threads may write to shared global variables; VCC can check whether concurrent access to shared variables is properly managed;
  - **data structure invariants**:
    we may declare invariants that have to hold for, e.g., records (e.g. the length field l is always equal to the length of the string field str); those invariants may **temporarily** be violated when updating the data structure.
  - and much more.
- Verification **does not always succeed**:
  - The backend SMT-solver may not be able to discharge proof-obligations
    (in particular non-linear multiplication and division are challenging);
  - In many cases, we need to provide **loop invariants** manually.

Tell Them What You've Told Them...

- There are **more approaches** to software quality assurance than just **testing**.
- For example, **program verification**.
- **Proof System PD** can be used
  - to **prove**
  - that a given program is
  - **correct** wrt. its specification.

  This approach considers **all inputs** inside the specification!
- Tools like **VCC** implement this approach.

References

References

Hoare, C. A. R. (1969). An axiomatic basis for computer programming. *Commun. ACM*, 12(10):576–580.

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