Exercise 1: Formalization in first-order logic

Express the following declarative sentences in first-order logic; in each case state what your respective constant, function, and predicate symbols stand for:

(a) Whatever goes upon four legs, or has wings, is a friend.
(b) No animal shall kill any other animal.
(c) All animals are equal, but some animals are more equal than others.
(d) The array $a$, whose indices and values are integers, is sorted between position 0 and position $l$.

Exercise 2: Quantifiers

(a) Show that the following first-order logic formula is not valid.

$$ \left( (\forall x. P(x)) \to Q \right) \to \left( (\forall x. P(x)) \to Q \right) $$

(b) Is the other direction of the implication (s. below) valid?

$$ \left( (\forall x. P(x) \to Q) \right) \to \left( (\forall x. P(x)) \to Q \right) $$

A short argument is sufficient.

Exercise 3: Minimal unsatisfiable core

Definition (Minimal unsatisfiable core) Let $\Gamma$ be a finite set of formulas such that the conjunction $\bigwedge_{\phi \in \Gamma} \phi$ is unsatisfiable. A subset $\Gamma' \subseteq \Gamma$ is called unsatisfiable core of $\Gamma$ if $\bigwedge_{\phi \in \Gamma'} \phi$ is also unsatisfiable. An unsatisfiable core $\Gamma'$ is called minimal unsatisfiable core if for each proper subset $\Gamma'' \subset \Gamma'$ the conjunction $\bigwedge_{\phi \in \Gamma''} \phi$ is satisfiable.

(a) Give a minimal unsatisfiable core for the following set of formulas.

$$ \{ \neg(X \to \neg Z), \ Y \to \neg U, \ X \to Y, \ X, \ Z \to U \} $$

(b) Is the minimal unsatisfiable core of a set of formulas unique? (Are there sets of formulas $\Gamma, \Gamma_1, \Gamma_2$ such that $\Gamma_1 \neq \Gamma_2$ but both $\Gamma_1$ and $\Gamma_2$ are minimal unsatisfiable cores of $\Gamma$)