Exercise 1: Loop invariants
Consider the following program $P$.

\[
\begin{align*}
\{ & \text{true} \\
& x := i; \\
& y := j; \\
\text{while } & x \neq 0 \{ \theta \} \{ \\
& \hspace{1cm} x := x - 1 \\
& \hspace{1cm} y := y - 1 \\
\} \\
& \{ i = j \rightarrow y = 0 \}
\end{align*}
\]

(a) Find a suitable loop invariant $\theta$ such that $\text{true} \models wp(P, i = j \rightarrow y = 0)$ holds.

(b) Give two examples for a loop invariant $\theta$ such that $\text{true} \models wp(P, i = j \rightarrow y = 0)$ does not hold.

Exercise 2: Guarded commands
Consider the following modified version of $\text{Fact}$ where we added the variable $u$.

\[
\begin{align*}
\{ & n \geq 0 \\
& u := 1; \\
& f := 1; \\
& i := 1; \\
\text{while } & i \leq n \{ \theta \} \{ \\
& \hspace{1cm} f := f \cdot i; \\
& \hspace{1cm} i := i + 1; \\
& \hspace{1cm} u := u + 1; \\
\} \\
& \{ f = \text{fact}(n) \land u \geq 1 \}
\end{align*}
\]

(a) Transform the program (together with its pre-/postcondition) to a guarded command. Use the old $\theta$ from the previous exercise sheet:

\[ f = \text{fact}(i - 1) \land 1 \leq i \land i \leq n + 1 \]

(b) Why will a correctness proof using $wp$ of your guarded command fail?

(c) Modify $\theta$ above such that it can be used to show correctness of this program.
Exercise 3: Properties of post  
2 Points
We say that \( \text{post} \) distributes over the connective \( \odot \) w.r.t. the first argument if the following equation holds.
\[
\text{post}(\phi_1 \odot \phi_2, \rho) = \text{post}(\phi_1, \rho) \odot \text{post}(\phi_2, \rho)
\]
We say that \( \text{post} \) distributes over the connective \( \odot \) w.r.t. the second argument if the following equation holds.
\[
\text{post}(\phi, \rho_1 \odot \rho_2) = \text{post}(\phi, \rho_1) \odot \text{post}(\phi, \rho_2)
\]

- Determine for \( \odot \in \{\land, \lor, \rightarrow\} \) if \( \text{post} \) distributes over \( \odot \) w.r.t. the first argument or w.r.t. the second argument.
- Determine if the equality \( \text{post}(\neg \phi, \rho) = \neg \text{post}(\phi, \rho) \) holds.
- Determine if the equality \( \text{post}(\phi, \neg \rho) = \neg \text{post}(\phi, \rho) \) holds.

Give a proof for each positive answer, give a counterexample for each negative answer.

Exercise 4: Program representations  
1 Point
Consider again the program from Exercise 1 where we encode the postcondition using an \textbf{assert} statement and omit the precondition and the loop invariant.

\[
\ell_0 : \ x := i; \\
\ell_1 : \ y := j; \\
\ell_2 : \ \textbf{while} \ x \neq 0 \ \textbf{do} \{ \\
\ell_3 : \quad x := x - 1 \\
\ell_4 : \quad y := y - 1 \\
\ell_5 : \} \\
\ell_6 : \ \textbf{assert}(i = j \rightarrow y = 0)
\]

(a) State a formal definition of this program in the notation that was introduced in the lecture on Wednesday, May 30, where a program is given as a tuple
\[
P = (V, pc, \varphi_{init}, R, \varphi_{err}).
\]
(b) Draw the corresponding control flow graph.

Exercise 5: Weakest precondition  
2 Points
Let \( V \) be a tuple of program variables. Let \( \phi \) be a set of states (i.e., \( \phi \) is a formula whose free variables are in \( V \)). Let \( \rho \) be a binary relation over program states (i.e., \( \rho \) is a formula whose free variables are in \( V \cup V' \)).

In the lecture we defined the formula \( \text{post}(\phi, \rho) \) as the image of the set \( \phi \) under the relation \( \rho \).

(a) Define a function \( \text{wp} \) such that the formula \( \text{wp}(\phi, \rho) \) denotes the largest set of states \( \psi \) such that \( \text{post}(\psi, \rho) \) is a subset of \( \phi \).

(b) Compute \( \text{wp}(\phi_i, \rho_i) \) for the following pairs.
\[
\phi_1 \equiv y \geq 7 \quad \rho_1 \equiv x < y \land x' = x \land y' = y \\
\phi_2 \equiv y \geq 7 \quad \rho_2 \equiv x' = x + y + 3 \land y' = y \\
\phi_3 \equiv y \geq 7 \land x = 23 \quad \rho_3 \equiv y' = y
\]