Exercise 1: Regular traces

Consider the program whose set of control flow traces is given by the following regular expression.

\[ \text{assume}(x \text{ is prime}) (x--)^* \text{assume}(x = 0) \]

(a) Consider the pre-/postcondition pair \((\text{true}, \text{true})\).
   
   (i) Is the set of correct control flow traces a regular language?
   
   (ii) Is the set of feasible correct control flow traces a regular language?
   
   (iii) Is the set of infeasible correct control flow traces a regular language?

(b) Consider the pre-/postcondition pair \((\text{true}, \text{false})\). Answer the same questions as above.

Exercise 2: Transitive closure

Let \(R\) be a binary relation over a set \(\Sigma\). Consider the following two relations.

- Let \(R_{tcl1}\) be the smallest set such that the following properties hold.
  
  (a) \(R \subseteq R_{tcl1}\) and
  
  (b) for all \(s, s', s'' \in \Sigma\) if \((s, s') \in R_{tcl1}\) and \((s', s'') \in R_{tcl1}\) then \((s, s'') \in R_{tcl1}\)

- Let \(R_{tcl2}\) be the smallest set such that the following properties hold.
  
  (a) \(R \subseteq R_{tcl2}\) and
  
  (b) for all \(s, s', s'' \in \Sigma\) if \((s, s') \in R_{tcl2}\) and \((s', s'') \in R_{tcl2}\) then \((s, s'') \in R_{tcl2}\)

Prove that the equality \(R_{tcl1} = R_{tcl2}\) holds.

Exercise 3: Transitive closure, abstract version

We consider the same setting as in Exercise 2.

(a) Lift the definitions of \(R_{tcl1}\) and \(R_{tcl2}\) to an abstract domain (call the new relations \(R^\#_{tcl1}\) and \(R^\#_{tcl2}\), respectively).

(b) Does \(R^\#_{tcl1} = R^\#_{tcl2}\) still hold?
Exercise 4: Transition invariants

Consider the program $P = (\Sigma, T, \rho)$, where

- $\Sigma = \mathbb{Z} \times \mathbb{Z}$,
- $T = \{\tau_1, \tau_2, \tau_3, \tau_4\}$,
- $\rho_{\tau_1} \equiv x \geq 0 \land y \geq 0 \land y \leq x \land y' = y - 1$,
- $\rho_{\tau_2} \equiv x \geq 0 \land y \geq 0 \land y \leq x \land x' = y - 1$,
- $\rho_{\tau_3} \equiv x \geq 0 \land y \geq 0 \land x < y \land y' = x - 1$, and
- $\rho_{\tau_4} \equiv x \geq 0 \land y \geq 0 \land x < y \land x' = x - 1$.

(a) Find a ranking function $f$ for the program $P$. Prove that $f$ is a ranking function.

(b) Use transition predicate abstraction to prove that the program is terminating.

- Find a suitable set of transition predicates $Preds$.
- Compute the corresponding set of abstract transitions $P^\# = \{T_1, \ldots, T_n\}$.
- Argue that each abstract transition $T_i$ is well-founded.

(c) Find a disjunctive well-founded transition invariant $T_1 \cup T_2$ such that

- $T_1$ and $T_2$ are well-founded relations and
- both $T_1$ and $T_2$ are formulas that contain only primed and unprimed variables, logical connectives, and (function/relation/constant) symbols from the set $\{>, +, -, 1, 0\}$.