Propositional Logic
The set of formulas of propositional logic is given by the abstract syntax:

\[
\text{Form} \ni A, B, C \ ::= \ P | \bot | (\neg A) | (A \land B) | (A \lor B) | (A \rightarrow B)
\]

where \( P \) ranges over a countable set \( \text{Prop} \), whose elements are called propositional symbols or propositional variables. (We also let \( Q, R \) range over \( \text{Prop} \).)

- Formulas of the form \( \bot \) or \( P \) are called atomic.
- \( \top \) abbreviates \((\neg \bot)\) and \((A \leftrightarrow B)\) abbreviates \(( (A \rightarrow B) \land (B \rightarrow A) ) \).

Conventions to omit parentheses are:

- outermost parentheses can be dropped;
- the order of precedence (from the highest to the lowest) of connectives is: \( \neg, \land, \lor \) and \( \rightarrow \);
- binary connectives are right-associative.

There are recursion and induction principles (e.g. structural ones) for \( \text{Form} \).

A is a subformula of \( B \) when \( A \) “occurs in” \( B \).
Semantics

Definition

- \( T \) (true) and \( F \) (false) form the set of truth values.
- A valuation is a function \( \rho : \text{Prop} \rightarrow \{F, T\} \) that assigns truth values to propositional symbols.
- Given a valuation \( \rho \), the interpretation function \( [\cdot]_\rho : \text{Form} \rightarrow \{F, T\} \) is defined recursively as follows:

  \[
  \begin{align*}
  [\bot]_\rho &= F \\
  [P]_\rho &= T \text{ iff } \rho(P) = T \\
  [\neg A]_\rho &= T \text{ iff } [A]_\rho = F \\
  [A \land B]_\rho &= T \text{ iff } [A]_\rho = T \text{ and } [B]_\rho = T \\
  [A \lor B]_\rho &= T \text{ iff } [A]_\rho = T \text{ or } [B]_\rho = T \\
  [A \rightarrow B]_\rho &= T \text{ iff } [A]_\rho = F \text{ or } [B]_\rho = T
  \end{align*}
  \]
Semantics

Definition

A propositional model $\mathcal{M}$ is a set of proposition symbols, i.e. $\mathcal{M} \subseteq \text{Prop}$. The validity relation $\models \subseteq \mathcal{P}(\text{Prop}) \times \text{Form}$ is defined inductively by:

- $\mathcal{M} \models P$ iff $P \in \mathcal{M}$
- $\mathcal{M} \models \neg A$ iff $\mathcal{M} \not\models A$
- $\mathcal{M} \models A \land B$ iff $\mathcal{M} \models A$ and $\mathcal{M} \models B$
- $\mathcal{M} \models A \lor B$ iff $\mathcal{M} \models A$ or $\mathcal{M} \models B$
- $\mathcal{M} \models A \rightarrow B$ iff $\mathcal{M} \not\models A$ or $\mathcal{M} \models B$

Remark

The two semantics are equivalent. In fact, valuations are in bijection with propositional models. In particular, each valuation $\rho$ determines a model $\mathcal{M}_\rho = \{P \in \text{Prop} \mid \rho(P) = T\}$ s.t.

$$\mathcal{M}_\rho \models A \iff \llbracket A \rrbracket_\rho = T,$$

which can be proved by induction on $A$. Henceforth, we adopt the latter semantics.

Definition

- A formula $A$ is valid in a model $\mathcal{M}$ (or $\mathcal{M}$ satisfies $A$), iff $\mathcal{M} \models A$. When $\mathcal{M} \not\models A$, $A$ is said refuted by $\mathcal{M}$.
- A formula $A$ is satisfiable iff there exists some model $\mathcal{M}$ such that $\mathcal{M} \models A$. It is refutable iff some model refutes $A$.
- A formula $A$ is valid (also called a tautology) iff every model satisfies $A$. A formula $A$ is a contradiction iff every model refutes $A$. 

Let $\mathcal{M}$ and $\mathcal{M}'$ be two propositional models and let $A$ be a formula. If for any propositional symbol $P$ occurring in $A$, $\mathcal{M} \models P$ iff $\mathcal{M}' \models P$, then $\mathcal{M} \models A$ iff $\mathcal{M}' \models A$.

**Proof.**

By induction on $A$.

**Remark**

The previous proposition justifies that the truth table method suffices for deciding whether or not a formula is valid, which in turn guarantees that the validity problem of PL is decidable.

**Definition**

$A$ is logically equivalent to $B$, (denoted by $A \equiv B$) iff $A$ and $B$ are valid exactly in the same models.

**Some logical equivalences**

- $\neg\neg A \equiv A$ (double negation)
- $\neg(A \land B) \equiv \neg A \lor \neg B$ (De Morgan’s laws)
- $A \rightarrow B \equiv \neg A \lor B$ (interdefinability)
- $A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$ (distributivity)
- $A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$
Remark
- $\equiv$ is an equivalence relation on Form.
- Given $A \equiv B$, the replacement in a formula $C$ of an occurrence of $A$ by $B$ produces a formula equivalent to $C$.
- The two previous results allow for equational reasoning in proving logical equivalence.

Definition
Given a propositional formula $A$, we say that it is in:
- **Conjunctive normal form** (CNF), if it is a conjunction of disjunctions of literals (atomic formulas or negated atomic formulas), i.e. $A = \bigwedge_i \bigvee_j l_{ij}$, for literals $l_{ij}$;
- **Disjunctive normal form** (DNF), if it is a disjunction of conjunctions of literals, i.e. $A = \bigvee_i \bigwedge_j l_{ij}$, for literals $l_{ij}$.

Note that in some treatments, $\bot$ is not allowed in literals.

Proposition
Any formula is equivalent to a CNF and to a DNF.

Proof.
The wanted CNF and DNF can be obtained by rewriting of the given formula, using the logical equivalences listed before.
**Semantics**

**Notation**

We let $\Gamma, \Gamma', \ldots$ range over sets of formulas and use $\Gamma, A$ to abbreviate $\Gamma \cup \{A\}$.

**Definition**

Let $\Gamma$ be a set of formulas.

- $\Gamma$ is **valid in a model** $\mathcal{M}$ (or $\mathcal{M}$ **satisfies** $\Gamma$), iff $\mathcal{M} \models A$ for every formula $A \in \Gamma$. We denote this by $\mathcal{M} \models \Gamma$.
- $\Gamma$ is **satisfiable** iff there exists a model $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$, and it is **refutable** iff there exists a model $\mathcal{M}$ such that $\mathcal{M} \not\models \Gamma$.
- $\Gamma$ is **valid**, denoted by $\models \Gamma$, iff $\mathcal{M} \models \Gamma$ for every model $\mathcal{M}$, and it is **unsatisfiable** iff it is not satisfiable.

**Definition**

Let $A$ be a formula and $\Gamma$ a set of formulas. If every model that validates $\Gamma$ also validates $A$, we say that $\Gamma$ **entails** $A$ (or $A$ is a **logical consequence** of $\Gamma$).

We denote this by $\Gamma \models A$ and call $\models \subseteq \mathcal{P}(\text{Form }) \times \text{Form}$ the **semantic entailment** or **logical consequence** relation.
**Semantics**

**Proposition**

- **A is valid** iff \( \vdash A \), where \( \vdash \) abbreviates \( \emptyset \vdash A \).
- **A is a contradiction** iff \( A \vdash \bot \).
- **A \equiv B** iff \( A \vdash B \) and \( B \vdash A \). (or equivalently, \( A \leftrightarrow B \) is valid).

**Proposition**

The semantic entailment relation satisfies the following properties (of an abstract consequence relation):

- For all \( A \in \Gamma \), \( \Gamma \vdash A \). (inclusion)
- If \( \Gamma \vdash A \), then \( \Gamma, B \vdash A \). (monotonicity)
- If \( \Gamma \vdash A \) and \( \Gamma, A \vdash B \), then \( \Gamma \vdash B \). (cut)

**Proposition**

Further properties of semantic entailment are:

- \( \Gamma \vdash A \land B \) iff \( \Gamma \vdash A \) and \( \Gamma \vdash B \)
- \( \Gamma \vdash A \lor B \) iff \( \Gamma \vdash A \) or \( \Gamma \vdash B \)
- \( \Gamma \vdash A \rightarrow B \) iff \( \Gamma, A \vdash B \)
- \( \Gamma \vdash \neg A \) iff \( \Gamma, A \vdash \bot \)
- \( \Gamma \vdash A \) iff \( \Gamma, \neg A \vdash \bot \)
The natural deduction system $\mathcal{N}_{PL}$

- The proof system we will consider is a "natural deduction in sequent style" (not to confuse with a "sequent calculus"), which we name $\mathcal{N}_{PL}$.
- The "judgments" (or "assertions") of $\mathcal{N}_{PL}$ are sequents $\Gamma \vdash A$, where $\Gamma$ is a set of formulas (a.k.a. context or LHS) and $A$ a formula (a.k.a. conclusion or RHS), informally meaning that "$A$ can be proved from the assumptions in $\Gamma$".
- Natural deduction systems typically have "introduction" and "elimination" rules for each connective. The set of rules of $\mathcal{N}_{PL}$ is below.

**Rules of $\mathcal{N}_{PL}$**

- **Introduction Rules:**
  - $\text{(Ax)}$: \[ \frac{}{\Gamma, A \vdash A} \]
  - $\text{(RAA)}$: \[ \frac{\Gamma, \lnot A \vdash \bot}{\Gamma \vdash A} \]
  - $\text{(I_\land)}$: \[ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \land B} \]
  - $\text{(I_\lor i)}$: \[ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \lor A_2} \quad i \in \{1, 2\} \]
  - $\text{(I_\rightarrow)}$: \[ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \]

- **Elimination Rules:**
  - $\text{(E_\land i)}$: \[ \frac{\Gamma \vdash A_1 \land A_2}{\Gamma \vdash A_i} \quad i \in \{1, 2\} \]
  - $\text{(E_\lor)}$: \[ \frac{\Gamma \vdash A \lor B \quad \Gamma, A \vdash C}{\Gamma, B \vdash C} \]
  - $\text{(E_\rightarrow)}$: \[ \frac{\Gamma \vdash A \quad \Gamma \vdash A \rightarrow B}{\Gamma \vdash B} \]
  - $\text{(E_\lnot)}$: \[ \frac{\Gamma \vdash A}{\Gamma \vdash \lnot \lnot A} \]
Proof system

Definition

- A **derivation** of a sequent $\Gamma \vdash A$ is a tree of sequents, built up from *instances of the inference rules* of $\mathcal{N}_{PL}$, having as root $\Gamma \vdash A$ and as leaves instances of $(Ax)$. (The set of $\mathcal{N}_{PL}$-derivations can formally be given as an inductive definition and has associated recursion and inductive principles.)

- Derivations induce a binary relation $\vdash \in \mathcal{P}(\text{Form}) \times \text{Form}$, called the **derivability/deduction relation**:
  - $(\Gamma, A) \in \vdash$ iff there is a derivation of the sequent $\Gamma \vdash A$ in $\mathcal{N}_{PL}$;
  - typically we overload notation and abbreviate $(\Gamma, A) \in \vdash$ by $\Gamma \vdash A$, reading “$\Gamma \vdash A$ is derivable”, or “$A$ can be derived (or deduced) from $\Gamma$”, or “$\Gamma$ infers $A$”;

- A formula that can be derived from the empty context is called a **theorem**.

Definition

An inference rule is **admissible** in $\mathcal{N}_{PL}$ if every sequent that can be derived making use of that rule can also be derived without it.
**Proof system**

**Proposition**

The following rules are admissible in $\mathcal{N}_{PL}$:

- **Weakening**
  \[ \frac{\Gamma \vdash A}{\Gamma, B \vdash A} \]

- **Cut**
  \[ \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B} \]

- **(⊥)**
  \[ \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \]

**Proof.**

- Admissibility of weakening is proved by induction on the premise’s derivation.
- Cut is actually a derivable rule in $\mathcal{N}_{PL}$, i.e. can be obtained through a combination of $\mathcal{N}_{PL}$ rules.
- Admissibility of (⊥) follows by combining weakening and RAA.

**Definition**

Γ is said inconsistent if $\Gamma \vdash \bot$ and otherwise is said consistent.

**Proposition**

If $\Gamma$ is consistent, then either $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ is consistent (but not both).

**Proof.**

If not, one could build a derivation of $\Gamma \vdash \bot$ (how?), and $\Gamma$ would be inconsistent.
Traditional presentations of natural deduction take formulas as judgements and not sequents. In these presentations:

- derivations are trees of formulas, whose leaves can be either “open” or “closed”;
- open leaves correspond to the assumptions upon which the conclusion formula (the root of the tree) depends;
- some rules allow for the closing of leaves (thus making the conclusion formula not depend on those assumptions).

For example, introduction and elimination rules for implication look like:

\[
\begin{align*}
\text{(E→)} & \quad \frac{A \rightarrow B \quad A}{B} \\
\text{(I→)} & \quad \frac{B}{A \rightarrow B}
\end{align*}
\]

In rule (I→), any number of occurrences of A as a leaf may be closed (signalled by the use of square brackets).
Theorem (Soundness)

*If* $\Gamma \vdash A$, *then* $\Gamma \models A$.

**Proof.**

By induction on the derivation of $\Gamma \vdash A$. Some of the cases are illustrated:

- **If the last step is**

  \[
  (\text{Ax}) \quad \frac{}{\Gamma', A \vdash A}
  \]

  We need to prove $\Gamma', A \models A$, which holds by the inclusion property of semantic entailment.

- **If the last step is**

  \[
  (\text{I}\rightarrow) \quad \frac{\Gamma, B \vdash C}{\Gamma \vdash B \rightarrow C}
  \]

  By IH, we have $\Gamma, B \models C$, which is equivalent to $\Gamma \models B \rightarrow C$, by one of the properties of semantic entailment.

- **If the last step is**

  \[
  (\text{E}\rightarrow) \quad \frac{\Gamma \vdash B \quad \Gamma \vdash B \rightarrow A}{\Gamma \vdash A}
  \]

  By IH, we have both $\Gamma \models B$ and $\Gamma \models B \rightarrow A$. From these, we can easily get $\Gamma \models A$. 

$\square$
Definition

Γ is maximally consistent iff it is consistent and furthermore, given any formula A, either A or ¬A belongs to Γ (but not both can belong).

Proposition

Maximally consistent sets are closed for derivability, i.e. given a maximally consistent set Γ and given a formula A, Γ ⊢ A implies A ∈ Γ.

Lemma

If Γ is consistent, then there exists Γ′ ⊇ Γ s.t. Γ′ is maximally consistent.

Proof.

Let Γ₀ = Γ and consider an enumeration A₁, A₂, . . . of the set of formulas Form. For each of these formulas, define Γᵢ to be Γᵢ₋₁ ∪ {Aᵢ} if this is consistent, or Γᵢ₋₁ ∪ {¬Aᵢ} otherwise. (Note that one of these sets is consistent.) Then, we take Γ′ = ∪ᵢ Γᵢ. Clearly, by construction, Γ′ ⊇ Γ and for each Aᵢ either Aᵢ ∈ Γ′ or ¬Aᵢ ∈ Γ′. Also, Γ′ is consistent (otherwise some Γᵢ would be inconsistent).
Proposition

Γ is consistent iff Γ is satisfiable.

Proof.

The “if statement” follows from the soundness theorem. Let us proof the converse.

Let Γ’ be a maximally consistent extension of Γ (guaranteed to exist by the previous lemma) and define M as the set of proposition symbols that belong to Γ’.

Claim: M |= A iff A ∈ Γ’.

As Γ’ ⊇ Γ, M is a model of Γ, hence Γ is satisfiable.

The claim is proved by induction on A. Two cases are illustrated.

Case A = P. The claim is immediate by construction of M.

Case A = B → C. By IH and the fact that Γ’ is maximally consistent, M |= B → C is equivalent to ¬B ∈ Γ’ or C ∈ Γ’, which in turn is equivalent to B → C ∈ Γ’. The latter equivalence is proved with the help of the fact that Γ’, being maximally consistent, is closed for derivability.