Antichains in Automata Theory

Jeremias Holub

University of Freiburg

holub@informatik.uni-freiburg.de

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Outline

1 Motivation
2 Definitions
3 Antichains for universality checking
4 Conclusion
Universality Problem

Given an NFA $A$, decide if $\text{Lang}(A) = \Sigma^*$. 
Universality Problem

Given an NFA $A$, decide if $\text{Lang}(A) = \Sigma^*$. 

Common algorithm:

1. Determinization of $A$ with subset construction
2. Find counterexample, i.e. a word that is not accepted
Universality Problem

Given an NFA $A$, decide if $\text{Lang}(A) = \Sigma^*$. 

Common algorithm:

1. Determinization of $A$ with subset construction
2. Find counterexample, i.e. a word that is not accepted

Determinization can lead to exponential blow-up in states.
Universality Problem

Given an NFA $A$, decide if $\text{Lang}(A) = \Sigma^*$. 

Common algorithm:

1. Determinization of $A$ with subset construction
2. Find counterexample, i.e. a word that is not accepted

Determinization can lead to exponential blow-up in states.

$\Rightarrow$ PSPACE complete
Universality Problem: Example

Consider following NFA $A$:

Is $A$ universal?
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA \( A \):

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA A:

A run of the algorithm:

\[s_1, s_2\]

\[s_2, s_3\]

\[s_1, s_2, s_4\]
Universality Problem: Example

NFA $A$:

A run of the algorithm:

$$
\begin{align*}
&\text{start } \rightarrow s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow s_1, s_2, s_3, s_4 \\
&\text{start } \rightarrow s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow a, b
\end{align*}
$$
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$: 

A run of the algorithm:
Universality Problem: Example

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NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$: 

A run of the algorithm:
Universality Problem: Example

NFA $A$: 

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

```
<table>
<thead>
<tr>
<th>State</th>
<th>Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>s1</td>
<td>a, b</td>
</tr>
<tr>
<td>s2</td>
<td>a, b</td>
</tr>
<tr>
<td>s3</td>
<td>b</td>
</tr>
<tr>
<td>s4</td>
<td>a, b</td>
</tr>
</tbody>
</table>
```

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$: 

A run of the algorithm:

\[
\begin{align*}
\text{s1, s2} & \\
\text{s2, s3} & \\
\text{s1, s2, s4} & \\
\text{s1, s3} & \\
\text{s2, s3} & \\
\text{s2, s3} & \\
\text{s1, s2, s3, s4} & \\
\end{align*}
\]
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA A:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA A:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$: 

A run of the algorithm:
Universality Problem: Example

NFA $A$: $s_1$ $s_2$ $s_3$ $s_4$

A run of the algorithm:
Universality Problem: Example

NFA A:

A run of the algorithm:
Universality Problem: Example

NFA A:

A run of the algorithm:
Universality Problem: Example

NFA A:

\[ \begin{array}{c}
  s_1 & b & a & b & b \\
  s_2 & b & a & b & b \\
  s_3 & b & a & b & b \\
  s_4 & b & a & b & b
\end{array} \]

A run of the algorithm:

\[ \begin{array}{c}
  s_1, s_2 \\
  s_2, s_3 \\
  s_1, s_3 \\
  s_1, s_2, s_4 \\
  s_1, s_2 \\
  s_2, s_3 \\
  s_2, s_4 \\
  s_1, s_2, s_3, s_4
\end{array} \]
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:  

A run of the algorithm:
Universality Problem: Example

NFA A:

```
NFA A:

s1 -----> b ------------> b
  ^               ^
  |               |
  v               v
s2 -----> a -----> b
  |               |
  v               v
s3 -----> b
```

A run of the algorithm:

```
A run of the algorithm:

s1, s2
  |       |
  v       v
s2, s3
  |       |
  v       v
s1, s3
  |       |
  v       v
s1, s2, s4
```

```
A run of the algorithm:

s1, s2
  |       |
  v       v
s2, s3
  |       |
  v       v
s1, s3
  |       |
  v       v
s1, s2, s3
  |       |
  v       v
s1, s2, s3, s4
```

```
A run of the algorithm:

s1, s2
  |       |
  v       v
s2, s3
  |       |
  v       v
s1, s3
  |       |
  v       v
s1, s2, s4
```

```
A run of the algorithm:

s1, s2
  |       |
  v       v
s2, s3
  |       |
  v       v
s1, s3
  |       |
  v       v
s1, s2, s3
  |       |
  v       v
s1, s2, s3, s4
```
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Motivation

Universality Problem - Example: Forward Subset Construction

Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$: 

A run of the algorithm:
Universality Problem: Example

NFA $A$: 

A run of the algorithm: 

\[ s_1, s_2 \]

\[ s_1, s_3 \]

\[ s_2, s_3 \]

\[ s_1, s_2, s_4 \]

\[ s_1, s_2, s_3, s_4 \]
Universality Problem: Example

NFA $A$: 

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

NFA A:

A run of the algorithm:
Universality Problem: Example

NFA A:

A run of the algorithm:
Universality Problem: Example

NFA $A$:

A run of the algorithm:
Universality Problem: Example

A run of the algorithm:

NFA A:

\[ A \text{ is universal because all reachable expanded states are accepting!} \]
A partial order \((M, R)\) consists of a set \(M\) and a binary relation \(R \subseteq M \times M\), where for all \(x, y, z \in M\), it holds

- **reflexivity** \(xRx\)
- **antisymmetry** \(xRy \land yRx \Rightarrow x = y\)
- **transitivity** \(xRy \land yRz \Rightarrow xRz\)
Partial Order - Example

Consider $M = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 5\}, \{1, 6\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 3, 5\}\}$ and the $\subseteq$-relation. Then $(M, \subseteq)$ is a partial order.
Consider $M = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 5\}, \{1, 6\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 3, 5\}\}$ and the $\subseteq$-relation. Then $(M, \subseteq)$ is a partial order.

Representation as a Hasse diagram:
Antichain

Consider $M = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 5\}, \{1, 6\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 3, 5\}\}$ and the $\subseteq$-relation. Then $(M, \subseteq)$ is a partial order.

Representation as a Hasse diagram:

\[
\begin{align*}
\{1, 2\} &\rightarrow \{1, 2, 3\} \\
\{1, 3\} &\rightarrow \{1, 3, 4\} \\
\{1\} &\rightarrow \{1, 5\} \rightarrow \{1, 3, 5\} \\
\{1, 6\} &
\end{align*}
\]

**Antichain:** Given $A \subseteq M$, $(A, \subseteq)$. If $\nexists x, y \in A, x \neq y : x \subseteq y \lor y \subseteq x$ then $A$ is called *antichain*. 
Consider $M = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 5\}, \{1, 6\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 3, 5\}\}$ and the $\subseteq$-relation. Then $(M, \subseteq)$ is a partial order.

Representation as a Hasse diagram:

Example: $A = \{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{1, 6\}\}$ with $(A, \subseteq)$ is an antichain.
NFA $A = \langle \text{Loc}, \text{Init}, \text{Fin}, \Sigma, \delta \rangle$ with

- **Loc**: finite set of locations
- **Init $\subseteq$ Loc**: set of initial states
- **Fin $\subseteq$ Loc**: set of final (accepting) states
- **$\Sigma$**: finite alphabet
- **$\delta \subseteq \text{Loc} \times \Sigma \times \text{Loc}$**: nondeterministic transition relation
Definition: Minimal element in set of sets w.r.t \( \subseteq \)-relation

Let \( q \in 2^\text{Loc} \). Then \( s \in q \) minimal \( \iff \forall s' \in q : s' \nsubseteq s \).
Definition: Minimal element in set of sets w.r.t $\subseteq$-relation

Let $q \in 2^{\text{Loc}}$. Then $s \in q$ minimal $\iff \forall s' \in q : s' \notin s$.

Example: $q = \{\{s_2, s_3\}, \{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}\}$
Definition: Minimal element in set of sets w.r.t $\subseteq$-relation

Let $q \in 2^{\text{Loc}}$. Then $s \in q$ minimal $\iff \forall s' \in q: s' \not\subset s$.

Example: $q = \{\{s_2, s_3\}, \{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}\}$

Then: $\{s_2, s_3\}$ is minimal in $q$ because

$$\{s_1, s_2, s_3\} \not\subset \{s_2, s_3\} \text{ and } \{s_2, s_3, s_4\} \not\subset \{s_2, s_3\}$$
Definition: Minimal element in set of sets w.r.t $\subseteq$-relation

Let $q \in 2^{\mathbb{L}_{\text{oc}}}$. Then $s \in q$ minimal $\iff \forall s' \in q: s' \not\subseteq s$.

Example: $q = \{\{s_2, s_3\}, \{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}\}$

Then: $\{s_2, s_3\}$ is minimal in $q$ because

$\{s_1, s_2, s_3\} \not\subseteq \{s_2, s_3\}$ and $\{s_2, s_3, s_4\} \not\subseteq \{s_2, s_3\}$

But: $\{s_1, s_2, s_3\}$ and $\{s_2, s_3, s_4\}$ are not minimal in $q$ because

$\{s_2, s_3\} \subseteq \{s_1, s_2, s_3\}$ and $\{s_2, s_3\} \subseteq \{s_2, s_3, s_4\}$
Definition: Set of minimal elements

Let $q \in 2^{\text{Loc}}$. Then $[q]$ set of minimal elements in $q$. 
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Let $q \in 2^{\text{Loc}}$. Then $[q]$ set of minimal elements in $q$.

Example: $q = \{\{s_1\}, \{s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_3, s_4\}, \{s_3, s_5, s_6\}\}$
Definition: Set of minimal elements

Let \( q \in 2^{\text{Loc}} \). Then \([q]\) set of minimal elements in \( q \).

Example: \( q = \{\{s_1\}, \{s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_3, s_4\}, \{s_3, s_5, s_6\}\} \)

Then: \( [q] = \{\{s_1\}, \{s_2\}, \{s_3, s_4\}, \{s_3, s_5, s_6\}\} \)
**Definition: Set of minimal elements**

Let $q \in 2^{\text{Loc}}$. Then $[q]$ set of minimal elements in $q$.

Example: $q = \{\{s_1\}, \{s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_3, s_4\}, \{s_3, s_5, s_6\}\}$

Then: $[q] = \{\{s_1\}, \{s_2\}, \{s_3, s_4\}, \{s_3, s_5, s_6\}\}$

**Important note:** $[q]$ is always an antichain!
Definition: Partial Order $\sqsubseteq$

Let $q, q'$ be antichains. Then the following holds:

$$q \sqsubseteq q' \iff \forall s' \in q' \cdot \exists s \in q: s \subseteq s'$$
Definition: Partial Order $\subseteq$

Let $q, q'$ be antichains. Then the following holds:

$$q \subseteq q' \iff \forall s' \in q' \cdot \exists s \in q: s \subseteq s'$$

Example:

$$\{\{s_1, s_2\}, \{s_2, s_3\}\} \subseteq \{\{s_1, s_2, s_3\}, \{s_2, s_3, s_5\}\}$$
Definition: Partial Order $\triangleleft$

Let $q, q'$ be antichains. Then the following holds:

$$q \triangleleft q' \iff \forall s' \in q' \cdot \exists s \in q : s \subseteq s'$$

Example:

$$\{\{s_1, s_2\}, \{s_2, s_3\}\} \triangleleft \{\{s_1, s_2, s_3\}, \{s_2, s_3, s_5\}\}$$

Counterexample:

$$\{\{s_1, s_2\}, \{s_2, s_3\}\} \not\triangleleft \{\{s_1, s_2, s_3\}, \{s_2, s_4, s_5\}\}$$
Definition: $\tilde{\sqcap}$-glb (greatest lower bound)

Let $q, q'$ be antichains. Then the $\tilde{\sqcap}$-glb is:

$$q \tilde{\sqcap} q' = \{s | s \in q \lor s \in q'\}$$
Definition: \( \lesssim \)-glb (greatest lower bound)

Let \( q, q' \) be antichains. Then the \( \lesssim \)-glb is:

\[
q \lesssim q' = \left\{ s \mid s \in q \lor s \in q' \right\}
\]

Example:

\[
q = \{\{s_1, s_2\}, \{s_2, s_3\}, \{s_3, s_4, s_5\}\}
\]
\[
q' = \{\{s_2, s_3\}, \{s_3, s_4\}\}
\]
Definition: $\tilde{\sqcap}$-glb (greatest lower bound)

Let $q, q'$ be antichains. Then the $\tilde{\sqcap}$-glb is:

$$q \tilde{\sqcap} q' = \left\{ s \mid s \in q \lor s \in q' \right\}$$

Example:

$$q = \left\{ \{s_1, s_2\}, \{s_2, s_3\}, \{s_3, s_4, s_5\} \right\}$$
$$q' = \left\{ \{s_2, s_3\}, \{s_3, s_4\} \right\}$$

Then:

$$q \tilde{\sqcap} q' = \left\{ \{s_1, s_2\}, \{s_2, s_3\}, \{s_3, s_4\}, \{s_3, s_4, s_5\} \right\}$$
$$= \left\{ \{s_1, s_2\}, \{s_2, s_3\}, \{s_3, s_4\} \right\}$$
Definition: $\gtrless$-glb (greatest lower bound)

Let $q, q'$ be antichains. Then the $\gtrless$-glb is:

$$q \gtrless q' = \left\{ s \mid s \in q \lor s \in q' \right\}$$

Example:

$$q = \left\{ \{s_1, s_2\}, \{s_2, s_3\}, \{s_3, s_4, s_5\} \right\}$$

$$q' = \left\{ \{s_2, s_3\}, \{s_3, s_4\} \right\}$$

Then:

$$q \gtrless q' = \left\{ \{s_1, s_2\}, \{s_2, s_3\}, \{s_3, s_4\}, \{s_3, s_4, s_5\} \right\}$$

$$= \left\{ \{s_1, s_2\}, \{s_2, s_3\}, \{s_3, s_4\} \right\}$$

Important note: $q \gtrless q'$ is always an antichain!
Definition: Successor states

Let $s$ be an set of states and $\sigma$ a letter. Then $\text{post}_\sigma(s)$ is set of all of successor states reachable with $\sigma$:

$$\text{post}_\sigma(s) = \{ \ell' \in \text{Loc} | \exists \ell \in s : \delta(\ell, \sigma, \ell') \}$$
Definition: Successor states

Let $s$ be a set of states and $\sigma$ a letter. Then $\text{post}_\sigma(s)$ is the set of all of successor states reachable with $\sigma$:

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Example:

$$\text{post}_a(\{s_1, s_2\}) =$$
**Definition: Successor states**

Let $s$ be a set of states and $\sigma$ a letter. Then $\text{post}_\sigma(s)$ is set of all of successor states reachable with $\sigma$:

$$
\text{post}_\sigma(s) = \{ \ell' \in \text{Loc} | \exists \ell \in s : \delta(\ell, \sigma, \ell') \}
$$

**Example:**

\[ \text{post}_a(\{s_1, s_2\}) = \]
Definition: Successor states

Let $s$ be an set of states and $\sigma$ a letter. Then $\text{post}_\sigma(s)$ is set of all of successor states reachable with $\sigma$:

$$
\text{post}_\sigma(s) = \{ \ell' \in \text{Loc} | \exists \ell \in s: \delta(\ell, \sigma, \ell') \}
$$

Example:

\[
\text{post}_a(\{s_1, s_2\}) = \]

![Diagram of a state transition graph]

- $s_1$ starts
- Transitions:
  - $a$: $s_1 \rightarrow s_2$
  - $b$: $s_1 \rightarrow s_1$
- $s_2$ starts
- Transitions:
  - $a$: $s_2 \rightarrow s_2$
  - $b$: $s_2 \rightarrow s_1$
- $s_3$ starts
- Transitions:
  - $a$: $s_3 \rightarrow s_3$
  - $b$: $s_3 \rightarrow s_4$
- $s_4$ starts
- Transitions:
  - $b$: $s_4 \rightarrow s_4$

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Definition: Successor states

Let $s$ be an set of states and $\sigma$ a letter. Then $\text{post}_\sigma(s)$ is set of all of successor states reachable with $\sigma$:

$$\text{post}_\sigma(s) = \{l' \in \text{Loc} | \exists l \in s: \delta(l, \sigma, l')\}$$

Example:

$$\text{post}_a(\{s_1, s_2\}) = \{s_2, s_3\}$$
Forward Antichain Algorithm

Let $q$ be an antichain. Then $\text{Post}(q)$ is an antichain of all successor states:

$$\text{Post}(q) = \{s' | \exists s \in q \cdot \sigma \in \Sigma : s' = \text{post}_\sigma(s)\}$$
Forward Antichain Algorithm

Let $q$ be an antichain. Then Post($q$) is an antichain of all successor states:

$$\text{Post}(q) = \{ \{s' | \exists s \in q \cdot \sigma \in \Sigma : s' = \text{post}_\sigma(s) \} \}$$

Theorem: Let $A = \langle \text{Loc}, \text{Init}, \text{Fin}, \Sigma, \delta \rangle$ be an NFA and $\widetilde{F} = \bigcap \{q | q = \text{Post}(q) \cap \{\text{Init}\}\}$. Then $\text{Lang}(A) \neq \Sigma^*$ iff $\widetilde{F} \subseteq \{\text{Fin}\}$. 
Forward Antichain Algorithm

Let $q$ be an antichain. Then $\text{Post}(q)$ is an antichain of all successor states:

$$\text{Post}(q) = \{ s' | \exists s \in q \cdot \sigma \in \Sigma : s' = \text{post}_\sigma(s) \}$$

Theorem: Let $A = \langle \text{Loc}, \text{Init}, \text{Fin}, \Sigma, \delta \rangle$ be an NFA and $\tilde{\mathcal{F}} = \bigcap\{ q | q = \text{Post}(q) \upharpoonright \{ \text{Init} \} \}$. Then $\text{Lang}(A) \neq \Sigma^*$ iff $\tilde{\mathcal{F}} \subseteq \{ \text{Fin} \}$.

Algorithm:

Iterating $\text{Post}(q)$ starting with $q = \{ \text{Init} \}$ until fixed point is reached. Then check for $\tilde{\mathcal{F}} \subseteq \{ \text{Fin} \}$. 

Jeremias Holub
Example I

\[ x_0 = \text{Init} = \{\{s_1, s_2\}\} \]
Example 1

\[ x_0 = \text{Init} = \{\{s_1, s_2\}\} \]
Example 1

\[ x_0 = \text{Init} = \{s_1, s_2\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \emptyset \]
Example 1

\[ x_0 = \text{Init} = \{s_1, s_2\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{\} \]
Example 1

\[ x_0 = \text{Init} = \{s_1, s_2\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{\} \]
Example 1

\[ x_0 = \text{Init} = \{s_1, s_2\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{s_2, s_3\} \]
Example I

\[ x_0 = \text{Init} = \{s_1, s_2\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = [\{s_2, s_3\}, \]

\[ b \]
\[ a \]
\[ b \]
\[ a \]
\[ b \]
\[ b \]

\[ s_1 \]
\[ s_2 \]
\[ s_3 \]
\[ s_4 \]
Example 1

\[ x_0 = \text{Init} = \{ \{s_1, s_2\} \} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{ \{s_2, s_3\} \}, \]
Example 1

\[ x_0 = \text{Init} = \{s_1, s_2\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{\{s_2, s_3\}, \]

Example 1

\[ x_0 = \text{Init} = \{ \{s_1, s_2\} \} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{\{s_2, s_3\}, \{s_1, s_2, s_4\}\} \]
Example 1

\[ x_0 = \text{Init} = \{ s_1, s_2 \} \]

\[ x_1 = \text{Post}(x_0) \cap \{ \text{Init} \} = \left\{ \{ s_2, s_3 \}, \{ s_1, s_2, s_4 \} \right\} \]
Example I

\[ x_0 = \text{Init} = \{\{s_1, s_2\}\} \]

\[ x_1 = \text{Post}(x_0) \cap \text{Init} = [\{\{s_2, s_3\}, \{s_1, s_2, s_4\}\}] \cap \text{Init} \]
Example 1

\[
x_0 = \text{Init} = \{s_1, s_2\}
\]
\[
x_1 = \text{Post}(x_0) \cap \text{Init} = \{s_2, s_3\}, \{s_1, s_2, s_4\}\]

Example 1

\[ x_0 = \text{Init} = \{ \{s_1, s_2\} \} \]
\[ x_1 = \text{Post}(x_0) \cap \{ \text{Init} \} = \{ \{s_2, s_3\}, \{s_1, s_2, s_4\} \} \cap \{ \{s_1, s_2\} \} \]
Example 1

\[
\begin{align*}
\mathcal{x}_0 &= \text{Init} = \{\{s_1, s_2\}\} \\
\mathcal{x}_1 &= \text{Post}(\mathcal{x}_0) \cap \text{Init} = \left[\{\{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_2, s_4\}\}\right]
\end{align*}
\]
Example 1

\[ x_0 = \text{Init} = \{s_1, s_2\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{\{s_1, s_2\}, \{s_2, s_3\}\} \]
Example I

\[ x_0 = \text{Init} = \{\{s_1, s_2\}\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{\{s_1, s_2\}, \{s_2, s_3\}\} \]
**Example 1**

\[ x_0 = \text{Init} = \{s_1, s_2\} \]
\[ x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{s_1, s_2\}, \{s_2, s_3\} \]
\[ x_2 = \text{Post}(x_1) \cap \{\text{Init}\} = \]
Example 1

\[
\begin{align*}
    x_0 &= \text{Init} = \{\{s_1, s_2\}\} \\
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x_2 = \text{Post}(x_1) \cap \{\text{Init}\} = [\{\{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_3\}\}] \cap \{\text{Init}\}
\]
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\[
x_3 = \text{Post}(x_2) \cap \{\text{Init}\} =
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Example 1

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Example I

- $x_0 = \text{Init} = \{\{s_1, s_2\}\}$
- $x_1 = \text{Post}(x_0) \cap \{\text{Init}\} = \{\{s_1, s_2\}, \{s_2, s_3\}\}$
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Example 1

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\[ x_3 = \text{Post}(x_2) \cap \text{Init} = \left\{ \{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_3\} \right\} = x_2 \]
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\[ x_3 = \text{Post}(x_2) \cap \{\text{Init}\} = \{\{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_3\}\} = x_2 \]

⇒ least fixpoint!
Example 1

\[ x_3 = \text{Post}(x_2) \triangleleft \{\text{Init}\} = \{\{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_3\}\} \]

We need to check:

\[ x_3 \triangleleft \{\text{Fin}\} \]
Example 1

\[ x_3 = \text{Post}(x_2) \cap \{\text{Init}\} = \{\{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_3\}\} \]

We need to check:

\[ x_3 \subseteq \{\text{Fin}\} \iff \{s_1, s_2\} \subseteq \{s_4\} \lor \{s_2, s_3\} \subseteq \{s_4\} \lor \{s_1, s_3\} \subseteq \{s_4\} \]
Example 1

\[ x_3 = \text{Post}(x_2) \cap \{\text{Init}\} = \{\{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_3\}\} \]

We need to check:

\[ x_3 \subseteq \{\text{Fin}\} \iff \{s_1, s_2\} \subseteq \{s_4\} \lor \{s_2, s_3\} \subseteq \{s_4\} \lor \{s_1, s_3\} \subseteq \{s_4\} \]

\[ \Rightarrow \text{The automaton is universal!} \]
Example I - Execution tree

Remember the execution tree of the subset construction algorithm for universality checking?

```
s_1, s_2
  /   \  
\   /   \   
s_2, s_3  s_1, s_2, s_4
  /   \  
\   /   \   
s_1, s_3  s_2, s_3
  /   \  
\   /   \   
s_1, s_3  s_1, s_2, s_3, s_4
  /   \  
\   /   \   
s_1, s_2  s_2, s_3
  /   \  
\   /   \   
s_1, s_2  s_2, s_3, s_4
  /   \  
\   /   \   
s_1  s_1, s_2, s_3
  /   \  
\   /   \   
s_1  s_1, s_2, s_3, s_4
```

s_1, s_2, s_3
  /   \  
\   /   \   
s_1, s_3, s_4  s_1, s_2, s_3, s_4
Example I - Execution tree

The forward antichain algorithm only expanded \( \{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_3\} \) which is an antichain!
Conclusion

- problem: checking universality for NFAs
- explicit determinization with subset construction can lead to exponential many states
- new concept: using antichains
- keeps determinization implicit
Literature

