Antichain algorithms.
Using Antichains to solve reachability problems on non-deterministic finite automata.

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  Partial orders
  Antichains
  Downward closure, Maximum

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Partial orders

- $V$ be a finite set
- $\leq$ a binary relation $\leq \subseteq V \times V$
- $\leq$ reflexive, transitive and anti-symmetric then it is called a partial order
- $(\leq, V)$ is called a partially ordered set.
Partial orders

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Example

- $(\leq, \{1, 2, 3, 42\})$, the $\leq$ order of the natural numbers.
Partial orders

- \( V \) be a finite set
- \( \leq \) a binary relation \( \leq \subseteq V \times V \)
- \( \leq \) reflexive, transitive and anti-symmetric then it is called a partial order
- \( (\leq, V) \) is called a partially ordered set.

Example

- \( (\leq, \{1, 2, 3, 42\}) \), the \( \leq \) order of the natural numbers.
- \( (\subseteq, 2^V) \), subset-inclusion in a powerset.
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Antichain

- subsets of $V$ pairwise *incompatible* with regard to $\preceq$.
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Example

- Powerset of \{x, y, z\} with $\subseteq$ as partial order.
Antichain

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Example

- Powerset of $\{x, y, z\}$ with $\subseteq$ as partial order.
- $\{\{x, y\}, \{x, z\}\}$ is a antichain.
Antichain

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Example

- Powerset of $\{x, y, z\}$ with $\subseteq$ as partial order.
- $\{\{x, y\}, \{x, z\}\}$ is a antichain.
- $\{\{x\}, \{x, z\}\}$ is not a antichain.
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Conclusion
Downward closure, Maximum of $S \subseteq V$

**Downward closure**

$$\text{Down}(\preceq, S) := \{ v' \in V \mid \exists v \in Sv' \preceq v \}$$

**Maximum**

$$\text{Max}(\preceq, S) := \{ v \in S \mid \forall v' \in S: v \preceq v' \Rightarrow v' \preceq v \}$$

Examples

$$\text{Max}(\subseteq, \{\{x\}, \{x, y\}, \{x, z\}, \{x, y, z\}\}) = \{\{x, y\}, \{x, z\}\}$$

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Downward closure, Maximum of $S \subseteq V$

**Downward closure**

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\text{Down}(\subseteq, S) := \{v' \in V \mid \exists v \in Sv' \subseteq v\}
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**Examples**

- \(\text{Down}(\subseteq, \{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}\)
Downward closure, Maximum of $S \subseteq V$

### Downward closure

$$\text{Down} (\preceq, S) := \{ v' \in V \mid \exists v \in S \ v' \preceq v \}$$

### Maximum

$$\text{Max} (\preceq, S) := \{ v \in S \mid \forall v' \in S : v \preceq v' \Rightarrow v' \preceq v \}$$

#### Examples

- $\text{Max} (\subseteq, \{\{x\}, \{x, y\}, \{x, z\}, \{y, z\}\}) = \{\{x, y, z\}\}$
- $\text{Max} (\subseteq, \{\{x\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}) = \{\{x, y, z\}\}$
Downward closure, Maximum of $S \subseteq V$

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**Examples**

- \(Max(\subseteq, \{\{x\}, \{x, y\}, \{x, z\}\}) = \{\{x, y\}, \{x, z\}\}\)
Downward closure, Maximum of $S \subseteq V$

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$\text{Down}(\subseteq, S) := \{v' \in V \mid \exists v \in Sv' \subseteq v\}$

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**Examples**

- $\text{Max}(\subseteq, \{\{x\}, \{x, y\}, \{x, z\}\}) = \{\{x, y\}, \{x, z\}\}$
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Conclusion
Antichain as a representation for a downward closed set $S \subseteq V$.

- Use $S' := \text{Max}(\preceq, S)$ to represent $S$. 

Example $S_1 := \{0\}, \{x\}, \{y\}$ so $S_1' = \{\{x\}, \{y\}\}$

$S_2 := \{0\}, \{x\}, \{y\}, \{x, y\}$ so $S_2' = \{\{x, y\}\}$
Antichain as a representation for a downward closed set $S \subseteq V$.

- Use $S' := Max(\preceq, S)$ to represent $S$.
- The question $v \in S$ becomes $\exists v' \in S' : v \preceq v'$
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**Example**

$S_1 := \{\emptyset, \{x\}, \{y\}\}$ so $S'_1 = \{\{x\}, \{y\}\}$

$S_2 := \{\emptyset, \{x\}, \{y\}, \{x,y\}\}$ so $S'_2 = \{\{x,y\}\}$
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Powerset determinization of Non-deterministic finite automata

- Let $A := (Loc, Init, Fin, \delta, \Sigma)$ be a finite automaton.
- $G(A) := (V, E, In, Fin)$ is the corresponding powerset automaton.
Powerset determinization of Non-deterministic finite automata

- Let $A := (\text{Loc}, \text{Init}, \text{Fin}, \delta, \Sigma)$ be a finite automaton.
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- $V := 2^{\text{Loc}}$
Powerset determinization of Non-deterministic finite automata

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- $In := \{ v \in 2^{Loc} | Init \in v \}$
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- $(v_1, v_2) \in E$ iff there exists a $\sigma \in \Sigma$ such that $\bigcup_{q \in v_1} \delta(q, \sigma) = v_2$. 
Example $A := (\text{Loc}, \text{Init}, \text{Fin}, \delta, \Sigma)$ to $G(A) := (V, E, \text{In}, \text{Fin})$

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![Diagram](image-url)
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Conclusion
Reachability problem in $G(A) = (V, E; In, Fin)$

- Asks if a subset $S \subseteq V$ is reachable from $In$.
- Where *Reachable* here means there is a *path* from $In$ to $S$. This is $v_1, \cdots v_n$ such that $(v_i, v_{i+1}) \in E$ for all $0 < i < n$ and $v_1 \in In$ and $v_n \in S$.
Reachability problem in $G(A) = (V, E; \text{In}, \overline{\text{Fin}})$

- Asks if a subset $S \subseteq V$ is reachable from $\text{In}$.
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**Example**

$S := \{\{1\}, \{1,2,3\}\}$ is reachable from $\text{In}$.
The predecessors of $S \subseteq V$ in $G(A) = (V, E; \text{In}, \text{Fin})$

The predecessors of $S \subseteq V$ are

$$\text{pre}(S) := \{v_1 \in V \mid \exists v_2 \in S : (v_1, v_2) \in E\}$$
The predecessors of \( S \subseteq V \) in \( G(A) = (V, E; In, Fin) \)

- The predecessors of \( S \subseteq V \) are

\[
pre(S) := \{ v_1 \in V \mid \exists v_2 \in S : (v_1, v_2) \in E \}
\]

Example

\( S := \{\{1\}, \{1, 2, 3\}\} \) then \( pre(S) = \{\{1\}, \{2\}, \{1, 2, 4\}, \{1, 4\}\} \).
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Backward reachability fixpoint algorithm in
\( G(A) = (V, E; \text{In}, \text{Fin}) \)

- Solves the reachability problem for \( S \subseteq V \) by computing the monotone growing sequence of sets

\[
B_0 = S; B_i = B_{i-1} \cup \text{pre}(B_{i-1})
\]
Backward reachability fixpoint algorithm in $G(A) = (V, E; \text{In}, \text{Fin})$

Example starting with $\text{Fin}$

$B_0 = S; B_i = B_{i-1} \cup \text{pre}(B_{i-1})$
Backward reachability fixpoint algorithm in
\( G(A) = (V, E; \text{In}, \overline{\text{Fin}}) \)

Example starting with \( \overline{\text{Fin}} \)

\[ B_0 = S; B_i = B_{i-1} \cup \text{pre}(B_{i-1}) \]

\[ B_0 = \{\{\}\}, \{1\}\} \]
Backward reachability fixpoint algorithm in
\( G(A) = (V, E; \text{In, Fin}) \)

**Example starting with \( \text{Fin} \)**

\[
B_0 = S; B_i = B_{i-1} \cup \text{pre}(B_{i-1}) \\
B_0 = \{\}, \{1\} \} \\
B_1 = B_0 \cup \text{pre}(B_0)
\]
Backward reachability fixpoint algorithm in $G(A) = (V, E; \text{In}, \text{Fin})$

**Example starting with $\text{Fin}$**

\[ B_0 = S; B_i = B_{i-1} \cup \text{pre}(B_{i-1}) \]

\[ B_0 = \{\}, \{1\} \]

\[ B_1 = B_0 \cup \text{pre}(B_0) \]

\[ = \{\}, \{1\} \cup \text{pre}(\{\}, \{1\}) \]
Backward reachability fixpoint algorithm in $G(A) = (V, E; \text{In}, \text{Fin})$

**Example starting with $\text{Fin}$**

$B_0 = S; B_i = B_{i-1} \cup \text{pre}(B_{i-1})$

$B_0 = \{\{\}, \{1\}\}$

$B_1 = B_0 \cup \text{pre}(B_0)$

$= \{\{\}, \{1\}\} \cup \text{pre}(\{\{\}, \{1\}\})$

$= \{\{\}, \{1\}\} \cup \{\{1\}, \{2\}\} = \{\{\}, \{1\}, \{2\}\}$

![Graph](image)
Backward reachability fixpoint algorithm in
\[ G(A) = (V, E; In, Fin) \]

Example starting with \( \overline{Fin} \)

\[ B_0 = S; B_i = B_{i-1} \cup \text{pre}(B_{i-1}) \]

\[ B_0 = \emptyset, \{1\} \]

\[ B_1 = B_0 \cup \text{pre}(B_0) \]
\[ = \emptyset, \{1\} \cup \text{pre}(\emptyset, \{1\}) \]
\[ = \emptyset, \{1\} \cup \{\{1\}, \{2\}\} = \emptyset, \{1\}, \{2\} \]

\[ B_2 = B_1 \cup \text{pre}(B_1) \]
Backward reachability fixpoint algorithm in
\[ G(A) = (V, E; In, Fin) \]

Example starting with \( Fin \)

\[ B_0 = S; B_i = B_{i-1} \cup pre(B_{i-1}) \]

\[ B_0 = \{\}, \{1\} \]
\[ B_1 = B_0 \cup pre(B_0) \]
\[ = \{\}, \{1\} \cup pre(\{\}, \{1\}) \]
\[ = \{\}, \{1\} \cup \{\{1\}, \{2\}\} = \{\}, \{1\}, \{2\} \]
\[ B_2 = B_1 \cup pre(B_1) \]
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Backward reachability fixpoint algorithm in $G(A) = (V, E; In, Fin)$

Example starting with $\overline{Fin}$

$B_0 = S; B_i = B_{i-1} \cup pre(B_{i-1})$

$B_0 = \{\}, \{1\}$

$B_1 = B_0 \cup pre(B_0)$

$= \{\}, \{1\} \cup pre(\{\}, \{1\})$

$= \{\}, \{1\} \cup \{\{1\}, \{2\}\} = \{\}, \{1\}, \{2\}$

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Antichain Backward reachability algorithm

- Antichains can be used as representations for closed sets.
- Where can we introduce antichains in our algorithm?
Antichain Backward reachability algorithm

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Lemma 1

Given $G = (V, E; \text{In}, \text{Fin})$ then $\text{pre}(S)$ is downward closed for all downward closed sets $S \subseteq V$
Antichain Backward reachability algorithm

- Antichains can be used as representations for closed sets.
- Where can we introduce antichains in our algorithm?

**Lemma 1**

Given $G = (V, E; In, Fin)$ then $pre(S)$ is downward closed for all downward closed sets $S \subseteq V$.

**Lemma 2**

$Fin$ is downward closed.
Proof "pre(S) is downward closed for all downward closed S."

Let $S \subseteq V$ be downward closed. We need to show that $v_1 \in \text{pre}(S)$ and $v_2 \subseteq v_1 \Rightarrow v_2 \in \text{pre}(S)$.

Let $v_1 \in \text{pre}(S)$, this means there exists $v_3 \in S$ where $(v_1, v_3) \in E$ and a $\sigma \in \Sigma$ s.t. $\bigcup q \in v_1 \delta(q, \sigma) = v_3$.

Looking at $v_2 \subseteq v_1$ we get $\bigcup q \in v_2 \delta(q, \sigma) = v_4 \subseteq v_3$.

Hence $v_2 \in \text{pre}(S)$.
Proof "\( \text{pre}(S) \) is downward closed for all downward closed \( S \)."

Let \( S \subseteq V \) be downward closed. We need to show that

\[
v_1 \in \text{pre}(S) \text{ and } v_2 \subseteq v_1 \Rightarrow v_2 \in \text{pre}(S)
\]
Proof "\(\text{pre}(S)\) is downward closed for all downward closed \(S\)."

Let \(S \subseteq V\) be downward closed. We need to show that

\[
\forall v_1 \in \text{pre}(S) \text{ and } v_2 \subseteq v_1 \Rightarrow v_2 \in \text{pre}(S)
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Let \(v_1 \in \text{pre}(S)\) this means there exists \(v_3 \in S\) where \((v_1, v_3) \in E\) and a \(\sigma \in \Sigma\) s.t. \(\bigcup_{q \in v_1} \delta(q, \sigma) = v_3\)
Proof "pre(S) is downward closed for all downward closed S."

Let $S \subseteq V$ be downward closed. We need to show that

$$v_1 \in \text{pre}(S) \text{ and } v_2 \subseteq v_1 \Rightarrow v_2 \in \text{pre}(S)$$

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Looking at $v_2 \subseteq v_1$ we get $\bigcup_{q \in v_2} \delta(q, \sigma) = v_4 \subseteq v_3$
Proof "\( \text{pre}(S) \) is downward closed for all downward closed \( S \)."

Let \( S \subseteq V \) be downward closed. We need to show that

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\forall v_1 \in \text{pre}(S) \text{ and } v_2 \subseteq v_1 \Rightarrow v_2 \in \text{pre}(S)
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Let \( v_1 \in \text{pre}(S) \) this means there exists \( v_3 \in S \) where \( (v_1, v_3) \in E \) and a \( \sigma \in \Sigma \) s.t.

\[
\bigcup_{q \in v_1} \delta(q, \sigma) = v_3
\]

Looking at \( v_2 \subseteq v_1 \) we get \( \bigcup_{q \in v_2} \delta(q, \sigma) = v_4 \subseteq v_3 \)

Hence \( v_2 \in \text{pre}(S) \)
Recap: Definition of $\overline{\text{Fin}}$

$\overline{\text{Fin}} := \{ v \in 2^{\text{Loc}} | v \subseteq \text{Loc} \setminus \text{Fin} \}$

Proof "$\overline{\text{Fin}}$ is downward closed."

If $v_1 \in \overline{\text{Fin}}$ and $v_2 \subseteq v_1$ then $v_2 \subseteq v_1 \subseteq \text{Loc} \setminus \text{Fin}$ hence $v_2 \in \overline{\text{Fin}}$
Antichain Backward reachability algorithm

- We extend the fixpoint algorithm

\[ B_0 = S; B_i = B_{i-1} \cup \text{pre}(B_{i-1}) \]

- to the antichain fixpoint algorithm

\[ \tilde{B}_0 = \text{Max}(\subseteq, S); \tilde{B}_i = \text{Max}(\subseteq, \tilde{B}_{i-1} \cup \text{pre}(\text{Down}(\subseteq, \tilde{B}_{i-1}))) \]
Antichain Backward reachability algorithm. Starting with $S = \overline{Fin} = \{\{\}, \{1\}\}$;

Example starting with $\overline{Fin}$

$\overline{B}_0 = \text{Max}(\subseteq, S)$;

$\overline{B}_i = \text{Max}(\subseteq, \overline{B}_{i-1} \cup \text{pre}(\text{Down}(\subseteq, \overline{B}_{i-1})))$
Antichain Backward reachability algorithm. Starting with $S = \overline{\text{Fin}} = \{\{\}\}, \{1\}$;

Example starting with $\overline{\text{Fin}}$

$\widetilde{B}_0 = \text{Max}(\subseteq, S)$;
$\widetilde{B}_i = \text{Max}(\subseteq, \widetilde{B}_{i-1} \cup \text{pre}(\text{Down}(\subseteq, \widetilde{B}_{i-1})))$

$\widetilde{B}_0 = \text{Max}(\subseteq, \{\{\\}, \{1\}\}) = \{\{1\}\}$
Antichain Backward reachability algorithm. Starting with $S = \overline{Fin} = \{\{\}, \{1\}\}$;

Example starting with $\overline{Fin}$

$\overline{B}_0 = \text{Max}(\subseteq, S);$  
$\overline{B}_i = \text{Max}(\subseteq, \overline{B}_{i-1} \cup \text{pre}(\text{Down}(\subseteq, \overline{B}_{i-1})))$

$\overline{B}_0 = \text{Max}(\subseteq, \{\{\}, \{1\}\}) = \{\{1\}\}$  
$\overline{B}_1 = \text{Max}(\subseteq, \{1\} \cup \text{pre}(\text{Down}(\subseteq, \{1\})))$
Antichain Backward reachability algorithm. Starting with 
\( S = \overline{\text{Fin}} = \{\{\}, \{1\}\}; \)

Example starting with \( \overline{\text{Fin}} \)

\[
\begin{align*}
\tilde{B}_0 &= \text{Max}(\subseteq, S); \\
\tilde{B}_i &= \text{Max}(\subseteq, \tilde{B}_{i-1} \cup \text{pre}(\text{Down}(\subseteq, \tilde{B}_{i-1}))) \\
\tilde{B}_0 &= \text{Max}(\subseteq, \{\{\}, \{1\}\}) = \{\{1\}\} \\
\tilde{B}_1 &= \text{Max}(\subseteq, \{1\} \cup \text{pre}(\text{Down}(\subseteq, \{1\}))) \\
&= \text{Max}(\subseteq, \{1\} \cup \text{pre}(\{\}, \{1\})) \\
&= \{\{1\}, \{2\}\}
\end{align*}
\]
Antichain Backward reachability algorithm. Starting with 
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Example starting with \( \overline{\text{Fin}} \)

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\( \overline{B}_1 = \text{Max}(\subseteq, \{ 1 \} \cup \text{pre}(\text{Down}(\subseteq, \{ 1 \}))) \)
\( \quad = \text{Max}(\subseteq, \{ 1 \} \cup \text{pre}(\{ \}, \{ 1 \})) \)
\( \quad = \text{Max}(\subseteq, \{ 1 \} \cup \{ \{ 1 \}, \{ 2 \} \}) = \{ \{ 1 \}, \{ 2 \} \} \)
\( \overline{B}_2 = \text{Max}(\subseteq, \{ \{ 1 \}, \{ 2 \} \} \cup \text{pre}(\text{Down}(\subseteq, \{ \{ 1 \}, \{ 2 \} \}))) \)
Antichain Backward reachability algorithm. Starting with $S = \overline{\text{Fin}} = \{\{\}, \{1\}\}$;

Example starting with $\overline{\text{Fin}}$

$\widetilde{B}_0 = \text{Max}(\subseteq, S)$;
$\widetilde{B}_i = \text{Max}(\subseteq, \widetilde{B}_{i-1} \cup \text{pre}(\text{Down}(\subseteq, \widetilde{B}_{i-1})))$

$\widetilde{B}_0 = \text{Max}(\subseteq, \{\{\}, \{1\}\}) = \{\{1\}\}$
$\widetilde{B}_1 = \text{Max}(\subseteq, \{\{\}, \{1\}\} \cup \text{pre}(\text{Down}(\subseteq, \{\{1\}\})))$

$\widetilde{B}_2 = \text{Max}(\subseteq, \{\{\}, \{1\}\} \cup \text{pre}(\text{Down}(\subseteq, \{\{1\}, \{2\}\})))$

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$\widetilde{B}_2 = \text{Max}\left(\subseteq, \{\{\}, \{1\}\} \cup \{\{1\}\} \cup \{\{1\}, \{2\}\}\right) = \{\{1\}\}$
Antichain Backward reachability algorithm. Starting with
$S = \overline{\text{Fin}} = \{\{\}, \{1\}\}$;

Example starting with $\overline{\text{Fin}}$

$\tilde{B}_0 = \text{Max}(\subseteq, S)$;
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$\tilde{B}_0 = \text{Max}(\subseteq, \{\{\}, \{1\}\}) = \{\{1\}\}$
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  = $\text{Max}(\subseteq, \{1\} \cup \text{pre}(\{\}, \{1\})))$
  = $\text{Max}(\subseteq, \{1\} \cup \{\{1\}, \{2\}\}) = \{\{1\}, \{2\}\}$
$\tilde{B}_2 = \text{Max}(\subseteq, \{\{1\}, \{2\}\} \cup \text{pre}(\text{Down}(\subseteq, \{\{1\}, \{2\}\})))$
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Content

Preliminaries
  Partial orders
  Antichains
  Downward closure, Maximum

Antichains as representations of closed sets

Powerset determinization of Non-deterministic finite automata

Reachability problem
  Backward reachability fixpoint algorithm
  Antichain Backward reachability algorithm

Conclusion
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- Antichains can be used as representations of closed sets.
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- In the powerset construction $\text{Fin}$ is a downward closed set.
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- On the powerset automaton $\text{pre}(S)$ for closed $S$ is downward closed. Thus we can use antichains in the classic backward reachability algorithm.
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On the powerset automaton $\text{pre}(S)$ for closed $S$ is downward closed. Thus we can use antichains in the classic backward reachability algorithm.

To be efficient, further improvements are possible and will be shown in antichain talk II.
Doyen, Laurent and Raskin, Jean-François
Antichain algorithms for finite automata

De Wulf, Martin and Doyen, Laurent and Henzinger, Thomas A and Raskin, J-F.