# Antichain algorithms.

Using Antichains to solve reachabillity problems
on non-deterministic finite automata.

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Proseminar on Automata Theory at the chair of Software Engineering.
Supervised by Alexander Nutz.

# Content

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# Partial orders

- $V$ be a finite set
- $\preceq$ a binary relation $\preceq \subseteq V \times V$
- $\preceq$ reflexive, transitive and anti-symmetric then it is called a partial order
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**Example**

- $(\leq, \{1,2,3,42\})$, the $\leq$ order of the natural numbers.
- $(\subseteq, 2^V))$, subset-inclusion in a powerset.

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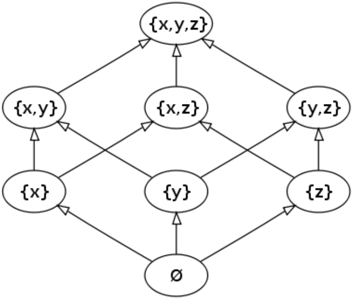


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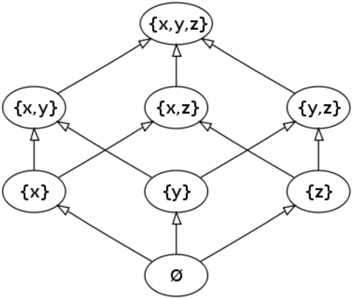


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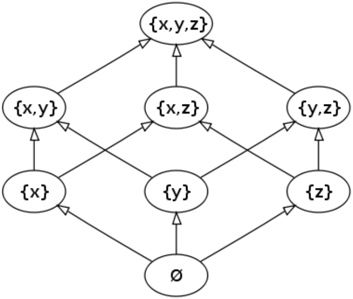


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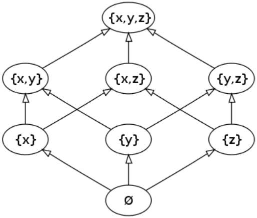


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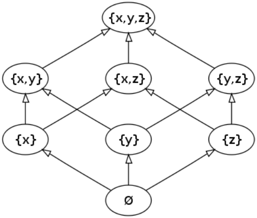


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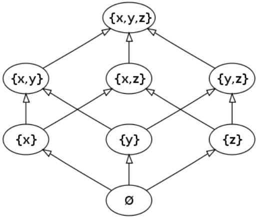


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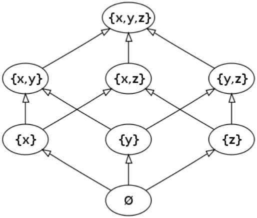


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**Example**

$S_1 := \{\emptyset, \{x\}, \{y\}\}$ so $S'_1 = \{\{x\}, \{y\}\}$
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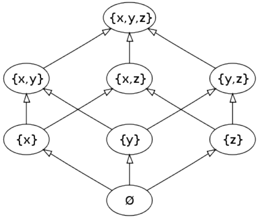

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# Powerset determinization of Non-deterministic finite automata

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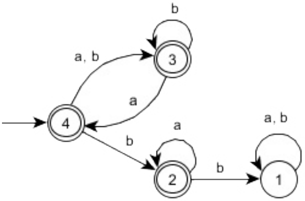

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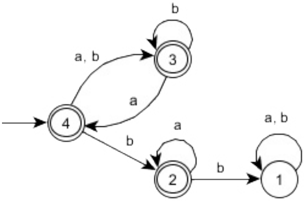


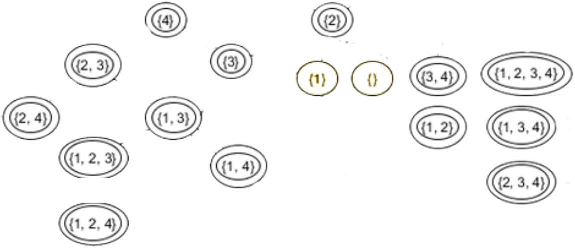

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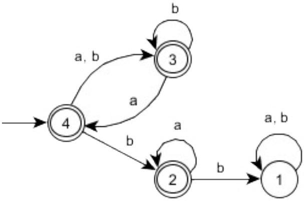


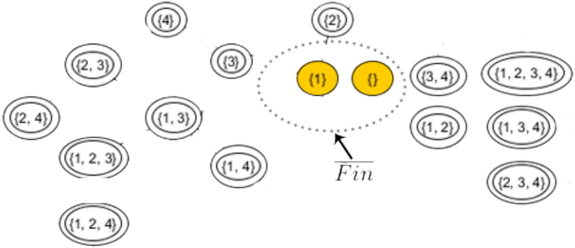

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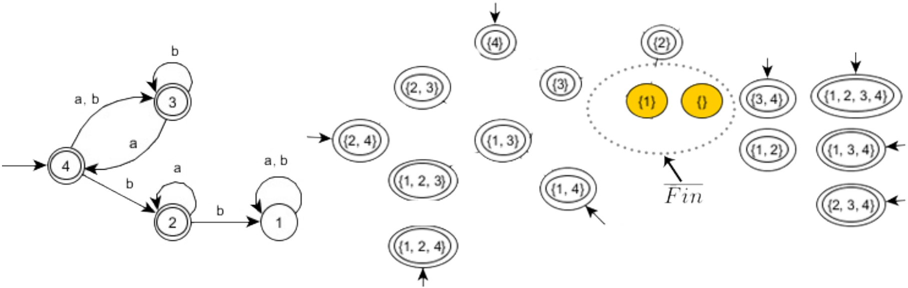

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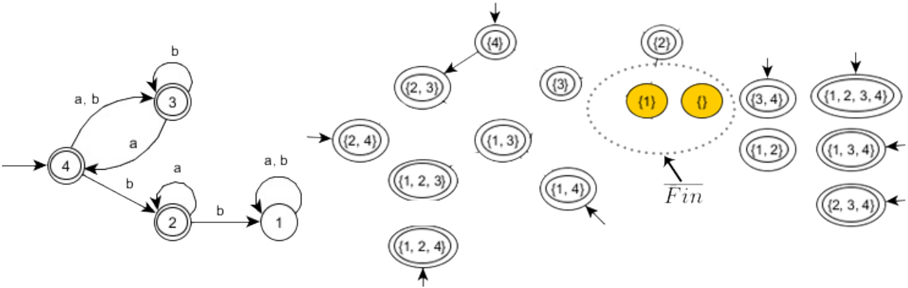

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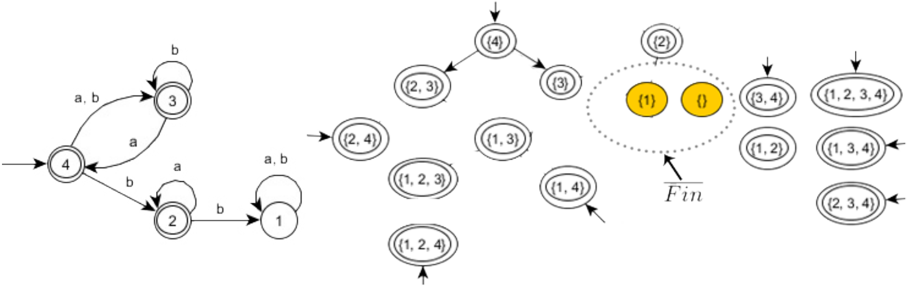

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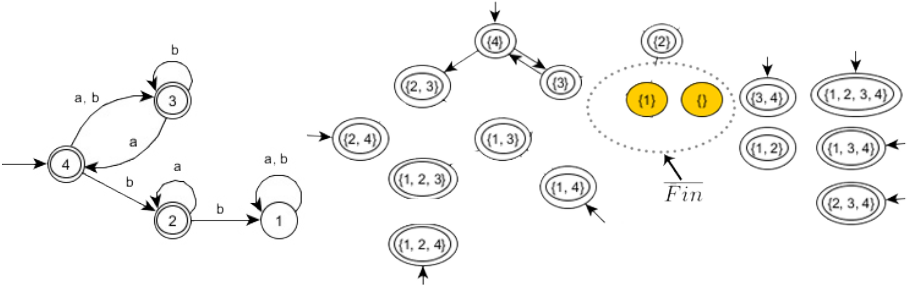

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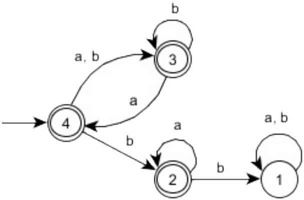


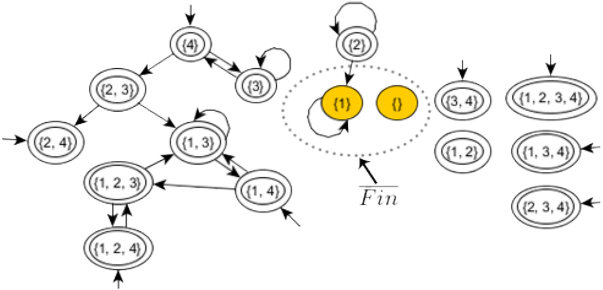

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# Reachability problem in $G(A) = (V, E; In, \overline{Fin})$

- Asks if a subset $S \subseteq V$ is reachable from $In$.
- Where *Reachable* here means there is a *path* from $In$ to $S$. This is $v_1, \cdots v_n$ such that $(v_i, v_{i+1}) \in E$ for all $0 < i < n$ and $v_1 \in In$ and $v_n \in S$.

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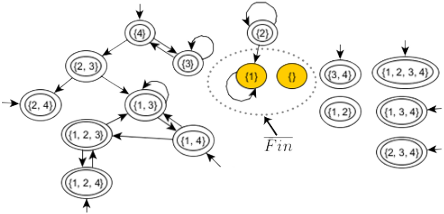


**Example**

$S := \{\{1\}, \{1,2,3\}\}$ is reachable from *In*.

# The predecessors of $S \subseteq V$ in $G(A) = (V, E; In, \overline{Fin})$

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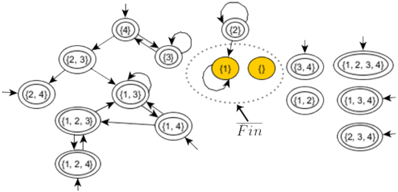


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# Backward reachability fixpoint algorithm in $G(A) = (V, E; In, \overline{Fin})$

- Solves the reachability problem for $S \subseteq V$ by computing the monotone growing sequence of sets

$$B_0 = S; B_i = B_{i-1} \cup pre(B_{i-1})$$

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## Example starting with $\overline{Fin}$

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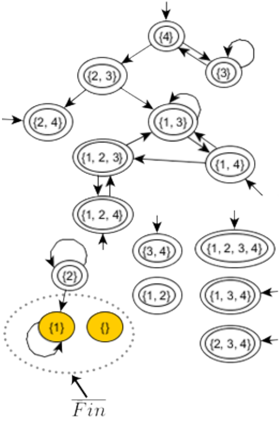

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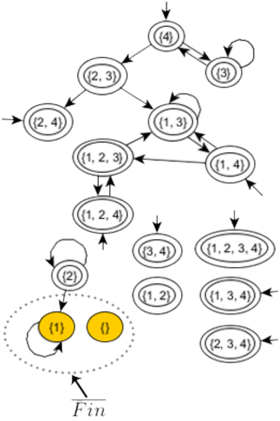

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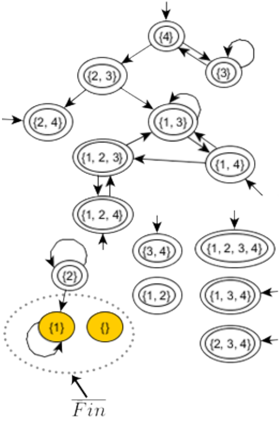

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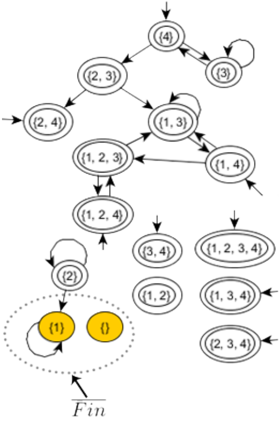

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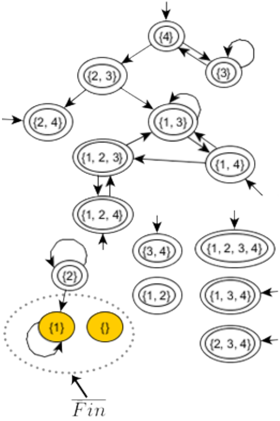

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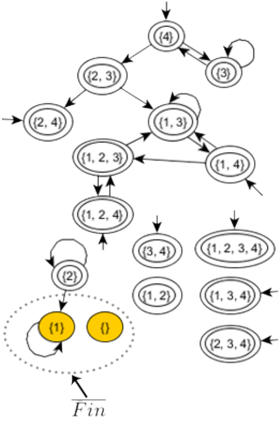

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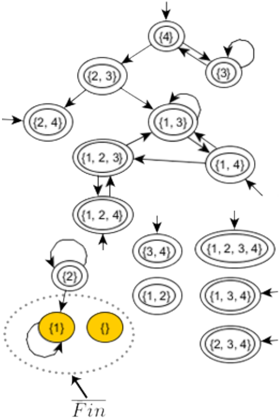

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$$
\begin{aligned}
B_0 &= \{\{\}, \{1\}\} \\
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&= \{\{\}, \{1\}\} \cup pre(\{\{\}, \{1\}\}) \\
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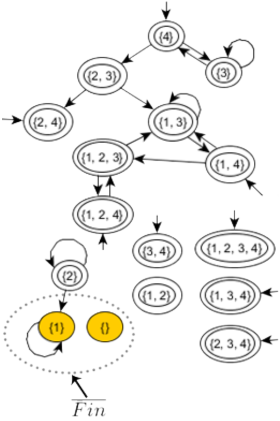

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**Lemma 1**

Given $G = (V, E; In, \overline{Fin})$ then $pre(S)$ is downward closed for all downward closed sets $S \subseteq V$

**Lemma 2**

$\overline{Fin}$ is downward closed.

# Antichain Backward reachability algorithm
# Proof for Lemma 1

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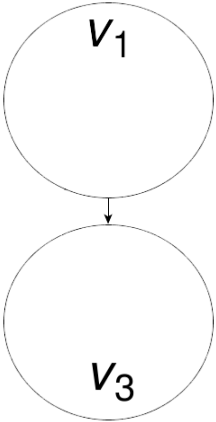

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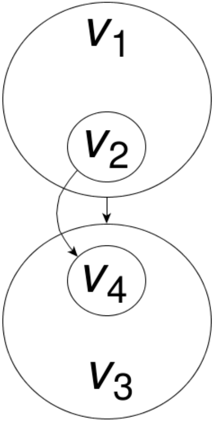

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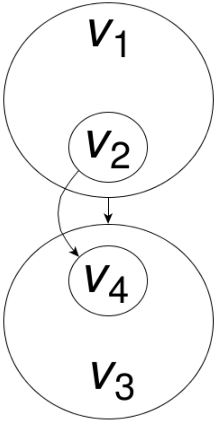
Hence $v_2 \in pre(S)$ □



# Antichain Backward reachability algorithm
# Proof for Lemma 2

**Recap: Definition of $\overline{Fin}$**

$\overline{Fin} := \{v \in 2^{Loc} | v \subseteq Loc \setminus Fin\}$

**Proof "$\overline{Fin}$ is downward closed."**

If $v_1 \in \overline{Fin}$ and $v_2 \subseteq v_1$ then $v_2 \subseteq v_1 \subseteq Loc \setminus Fin$ hence $v_2 \in \overline{Fin}$ $\square$

# Antichain Backward reachability algorithm

- We extend the fixpoint algorithm

$$B_0 = S; B_i = B_{i-1} \cup pre(B_{i-1})$$

to the antichain fixpoint algorithm

$$\widetilde{B}_0 = Max(\subseteq, S); \widetilde{B}_i = Max(\subseteq, \widetilde{B}_{i-1} \cup pre(Down(\subseteq, \widetilde{B}_{i-1})))$$

# Antichain Backward reachability algorithm. Starting with $S = \overline{Fin} = \{\{\},\{1\}\}$;

## Example starting with $\overline{Fin}$

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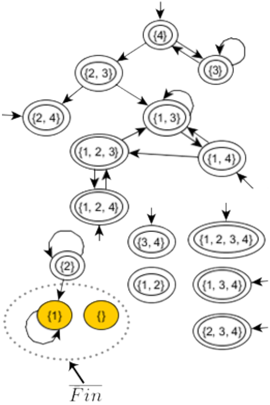

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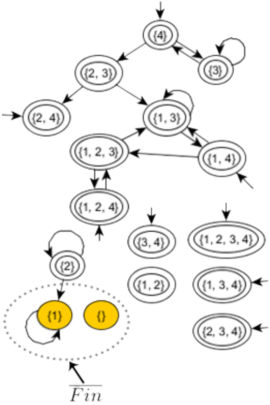

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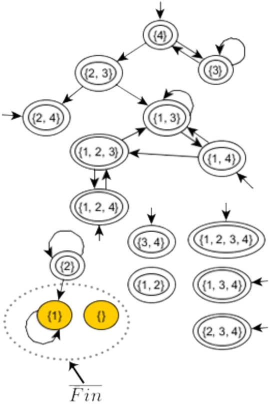

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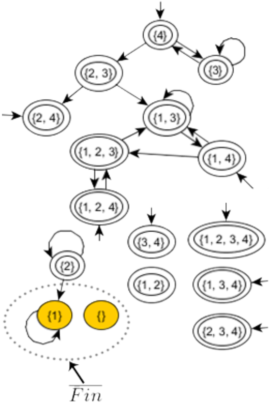

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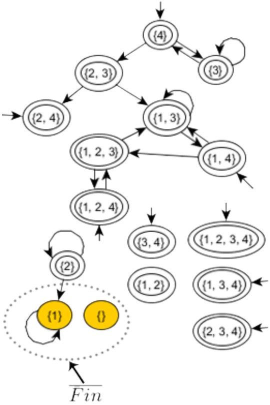

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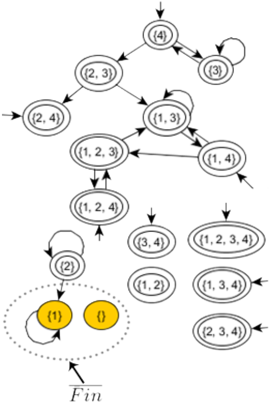

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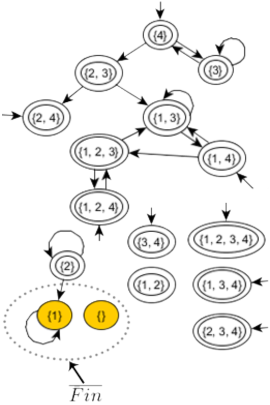

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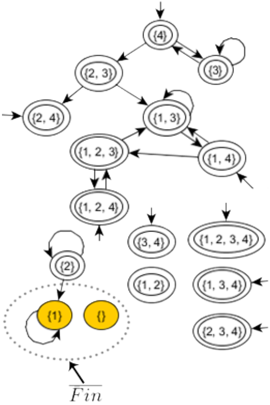

# Content

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- In the powerset construction $\overline{Fin}$ is a downward closed set.
- On the powerset automaton $pre(S)$ for closed $S$ Is downward closed. Thus we can use antichains in the classic backward reachability algorithm.
- To be efficient, further improvements are possible and will be shown in antichain talk II.

# References I

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