Live Sequence Charts — Semantics
Plan:

(i) Given an LSC $\mathcal{L}$ with body

$((L, \preceq, \sim), \mathcal{I}, \text{Msg}, \text{Cond}, \text{LocInv}, \Theta)$,

(ii) construct a TBA $B_{\mathcal{L}}$, and

(iii) define language $\mathcal{L}(\mathcal{L})$ of $\mathcal{L}$ in terms of $\mathcal{L}(B_{\mathcal{L}})$,

in particular taking activation condition and activation mode into account.

(iv) define language $\mathcal{L}(\mathcal{M})$ of a UML model.

$\bullet$ Then $\mathcal{M} \models \mathcal{L}$ (universal) if and only if $\mathcal{L}(\mathcal{M}) \subseteq \mathcal{L}(\mathcal{L})$.

And $\mathcal{M} \models \mathcal{L}$ (existential) if and only if $\mathcal{L}(\mathcal{M}) \cap \mathcal{L}(\mathcal{L}) \neq \emptyset$. 
Live Sequence Charts — TBA Construction
Definition.
Let \((L, \preceq, \sim), \mathcal{I}, \text{Msg}, \text{Cond}, \text{LocInv}, \Theta)\) be an LSC body.
A non-empty set \(\emptyset \neq C \subseteq L\) is called a cut of the LSC body iff

- it is **downward closed**, i.e. \(\forall l, l' \bullet l', l' \in C \land l \preceq l' \implies l \in C\),
- it is **closed under simultaneity**, i.e.
  \[\forall l, l' \bullet l' \in C \land l \sim l' \implies l \in C\]
  and
- it comprises at least **one location per instance line**, i.e.
  \[\forall i \in I \exists l \in C \bullet i_l = i\].

The **temperature function** is extended to cuts as follows:

\[
\Theta(C) = \begin{cases} 
\text{hot} & \text{if } \exists l \in C \bullet (\# l' \in C \bullet l \prec l') \land \Theta(l) = \text{hot} \\
\text{cold} & \text{otherwise}
\end{cases}
\]

that is, \(C\) is **hot** if and only if at least one of its maximal elements is hot.
$\emptyset \neq C \subseteq L$ – downward closed – simultaneity closed – at least one loc. per instance line
\[ \emptyset \neq C \subseteq L \] – downward closed – simultaneity closed – at least one loc. per instance line
The partial order “≤” and the simultaneity relation “∼” of locations induce a **direct successor relation** on cuts of an LSC body as follows:

**Definition.**
Let \( C \subseteq L \) be a cut of LSC body \( ((L, \preceq, \sim), \mathcal{I}, \text{Msg}, \text{Cond}, \text{LocInv}, \Theta) \).

A set \( \emptyset \neq F \subseteq L \) of locations is called **fired-set** \( F \) of cut \( C \) if and only if

- \( C \cap F = \emptyset \) and \( C \cup F \) is a cut, i.e. \( F \) is closed under simultaneity,
- all locations in \( F \) are **direct \( \prec \)-successors** of the front of \( C \), i.e.
  \[
  \forall l \in F \exists l' \in C \bullet l' \prec l \land (\nexists l'' \in C \bullet l' \prec l'' \prec l),
  \]
- locations in \( F \), that lie on the same instance line, are **pairwise unordered**, i.e.
  \[
  \forall l \neq l' \in F \bullet (\exists I \in \mathcal{I} \bullet \{l, l'\} \subseteq I) \implies l \npreceq l' \land l' \npreceq l,
  \]
- for each asynchronous (!) message reception in \( F \), the corresponding **sending is already in** \( C \),
  \[
  \forall (l, E, l') \in \text{Msg} \bullet l' \in F \implies l \in C.
  \]

The cut \( C' = C \cup F \) is called **direct successor of** \( C \) via \( F \), denoted by \( C \sim_F C' \).
$C \cap F = \emptyset - C \cup F$ is a cut – only direct $\prec$-successors – same instance line on front pairwise unordered – sending of asynchronous reception already in $(\xi)$.
\( C \cap F = \emptyset \) - \( C \cup F \) is a cut – only direct \textless\-successors – same instance line on front pairwise unordered – sending of asynchronous reception already in
The TBA $B_L$ of LSC $L$ over $\Phi$ and $q$ is $(Expr_B(X), X, Q, q_{ini}, \rightarrow, Q_F)$ with

- $Q$ is the set of cuts of $L$, $q_{ini}$ is the instance heads cut,
- $Expr_B(X) = Expr_{\mathcal{S}}(\mathcal{S}, X)$ (for considered signature $\mathcal{S}$),
- $\rightarrow$ consists of loops, progress transitions (by $\rightsquigarrow_F$), and legal exits (cold cond./local inv.),
- $Q_F = \{ C \in Q \mid \Theta(C) = \text{cold} \lor C = L \}$ is the set of cold cuts and the maximal cut.
Signal and Attribute Expressions

- Let $\mathcal{S} = (T, C, V, atr, E)$ be a signature and $X$ a set of logical variables,

- The signal and attribute expressions $Expr(\mathcal{S}, X)$ are defined by the grammar:

  $$\psi ::= true \mid E_{x,y}^! \mid E_{x,y}^? \mid \neg \psi \mid \psi_1 \lor \psi_2$$

  where $expr : \text{Bool} \in Expr(\mathcal{S}), E \in \mathcal{E}, x, y \in X$ (or keyword $env$).

- We use

  $$\mathcal{E}?(X) := \{E_{x,y}^!, E_{x,y}^? \mid E \in \mathcal{E}, x, y \in X\}$$

  to denote the set of event expressions over $\mathcal{E}$ and $X$. 
Recall: The TBA $\mathcal{B}(\mathcal{L})$ of LSC $\mathcal{L}$ is $(\text{Expr}_\mathcal{B}(X), X, Q, q_{\text{ini}}, \rightarrow, Q_F)$ with

- $Q$ is the set of cuts of $\mathcal{L}$, $q_{\text{ini}}$ is the instance heads cut,
- $\text{Expr}_\mathcal{B} = \Phi \cup \mathcal{E}_i? (X)$,
- $\rightarrow$ consists of loops, progress transitions (from $\sim_F$), and legal exits (cold cond./local inv.),
- $F = \{ C \in Q \mid \Theta(C) = \text{cold} \lor C = L \}$ is the set of cold cuts.

So in the following, we “only” need to construct the transitions’ labels:

$$\rightarrow = \{(q, \psi_{\text{loop}}(q), q) \mid q \in Q\} \cup \{(q, \psi_{\text{prog}}(q, q'), q') \mid q \sim_F q'\} \cup \{(q, \psi_{\text{exit}}(q), L) \mid q \in Q\}$$
“Only” construct the transitions’ labels:

\[
\rightarrow = \{(q, \psi_{\text{loop}}(q), q) \mid q \in Q\} \cup \{(q, \psi_{\text{prog}}(q, q'), q') \mid q \sim_F q'\} \cup \{(q, \psi_{\text{exit}}(q), L) \mid q \in Q\}
\]

\[
\psi_{\text{loop}}(q) = \psi_{\text{Msg}}(q) \land \psi_{\text{hot}}^{\text{loop}}(q) \land \psi_{\text{LocInv}}^{\text{cold}}(q)
\]

\[
\psi_{\text{exit}}(q) = (\psi_{\text{loop}}^{\text{hot}}(q) \land \neg \psi_{\text{LocInv}}^{\text{cold}}(q)) \\
\lor \bigvee_{1 \leq i \leq n} (\psi_{\text{prog}}^{\text{hot}}(q, q_i) \\
\land (\neg \psi_{\text{LocInv}}^{\text{cold}}(q, q_i) \lor \neg \psi_{\text{Cond}}^{\text{cold}}(q, q_i)))
\]

\[
\psi_{\text{prog}}(q, q_n) = \psi_{\text{Prog}}^{\text{hot}}(q, q_n) \\
\land \psi_{\text{LocInv}}^{\text{cold}}(q, q_n) \land \psi_{\text{Cond}}^{\text{cold}}(q, q_n)
\]

\[
C_1 : C_2 \quad C_3
\]

\[
\begin{align*}
&D \quad E \\
&x \geq 3
\end{align*}
\]
### Loop Condition

$$\psi_{\text{loop}}(q) = \psi_{\text{Msg}}(q) \land \psi_{\text{LocInv}}^{\text{hot}}(q) \land \psi_{\text{LocInv}}^{\text{cold}}(q)$$

- $$\psi_{\text{Msg}}(q) = \neg \bigvee_{1 \leq i \leq n} \psi_{\text{Msg}}(q, q_i) \land \left( \text{strict} \implies \bigwedge_{\psi \in \text{Msg}(L)} \neg \psi \right) =: \psi_{\text{strict}}(q)$$

- $$\psi_{\theta}^{\text{LocInv}}(q) = \bigwedge_{\ell = (l, i, \phi, l', i')} \in \text{LocInv}, \Theta(\ell) = \theta, \ell \text{ active at } q \phi$$

A location $$l$$ is called **front location** of cut $$C$$ if and only if $$\not\exists l' \in L \bullet l \prec l'$$.

Local invariant $$(l_0, i_0, \phi, l_1, i_1)$$ is **active** at cut (!) $$q$$ if and only if $$l_0 \preceq l \prec l_1$$ for some front location $$l$$ of cut $$q$$ (or $$l_1 \in q \land i_1 \bullet$$).

- $$\text{Msg}(F) = \{E^l_{x_i, x_i'} \mid (l, E, l') \in \text{Msg}, l \in F\} \cup \{E^l_{x_i, x_i'} \mid (l, E, l') \in \text{Msg}, l' \in F\}$$

- $$x_l \in X$$ is the logical variable associated with the instance line $$I$$ which includes $$l$$, i.e. $$l \in I$$.

- $$\text{Msg}(F_1, \ldots, F_n) = \bigcup_{1 \leq i \leq n} \text{Msg}(F_i)$$

![Diagram](diagram.png)
Progress Condition

\[ \psi_{\text{hot}}(q, q_i) = \psi_{\text{Msg}}(q, q_n) \land \psi_{\text{hot}}(q, q_n) \land \psi_{\text{LocInv}, \bullet}(q_n) \]

- \[ \psi_{\text{Msg}}(q, q_i) = \bigwedge_{\psi \in \text{Msg}(q_i \setminus q)} \psi \land \bigwedge_{j \neq i} \bigwedge_{\psi \in \text{Msg}(q_j \setminus q)} \text{Msg}(q_i \setminus q) \neg \psi \land (\text{strict} \implies \bigwedge_{\psi \in \text{Msg}(L) \setminus \text{Msg}(F_i)} \neg \psi) \]

- \[ \psi_{\text{Cond}}(q, q_i) = \bigwedge_{\gamma = (L, \phi) \in \text{Cond}} \Theta(\gamma) = \theta, L \cap (q_i \setminus q) \neq \emptyset \phi \]

- \[ \psi_{\text{LocInv}, \bullet}(q, q_i) = \bigwedge_{\lambda = (l, \nu, l', \nu', \nu', \nu') \in \text{LocInv}} \Theta(\lambda) = \theta, \lambda \bullet\text{-active at } q_i \phi \]

Local invariant \((l_0, \nu_0, \phi, l_1, \nu_1)\) is \(\bullet\text{-active} at q \) if and only if

- \( l_0 < l < l_1 \), or
- \( l = l_0 \land \nu_0 = \bullet, \) or
- \( l = l_1 \land \nu_1 = \bullet \)

for some front location \(l\) of cut (!) \(q\).
Using logical variables $x, y, z$ for the instances lines (from left to right).
\( \Phi \in \text{OCL} \)

\[ \mathcal{L} = (\mathcal{T}, \mathcal{C}, V, \text{attr}), \text{SM} \]

\[ M = (\Sigma_{\mathcal{L}}, A_{\mathcal{L}}, \rightarrow_{\text{SM}}) \]

\[ \pi = (\sigma_0, \varepsilon_0) \xrightarrow{(\text{cons}_0, \text{Snd}_0)} u_0 (\sigma_1, \varepsilon_1) \cdots \]

\[ G = (N, E, f) \]

\[ B = (Q_{SD}, q_0, A_{\mathcal{L}}, \rightarrow_{SD}, F_{SD}) \]

\( u_0 \)
Excursion: Büchi Automata
From Finite Automata to Symbolic Büchi Automata

$A$: $\Sigma = \{0, 1\}$

$B$: $\Sigma = \{0, 1\}$

$A_{\text{sym}}$: $\Sigma = \{x\} \rightarrow \mathbb{N}$

$B_{\text{sym}}$: $\Sigma = \{x\} \rightarrow \mathbb{N}$

$\text{Lang}(A) = 0.(1.0)^*$

$\text{Lang}(B) = (0.1)^\infty$

$\text{Lang}(A_{\text{sym}})$

$\text{Lang}(B_{\text{sym}})$
Symbolic Büchi Automata

Definition. A Symbolic Büchi Automaton (TBA) is a tuple

$$\mathcal{B} = (\text{Expr}_B(X), X, Q, q_{ini}, \rightarrow, Q_F)$$

where

- $X$ is a set of logical variables,
- $\text{Expr}_B(X)$ is a set of Boolean expressions over $X$,
- $Q$ is a finite set of states,
- $q_{ini} \in Q$ is the initial state,
- $\rightarrow \subseteq Q \times \text{Expr}_B(X) \times Q$ is the transition relation. Transitions $(q, \psi, q')$ from $q$ to $q'$ are labelled with an expression $\psi \in \text{Expr}_B(X)$.
- $Q_F \subseteq Q$ is the set of fair (or accepting) states.
**Definition.** Let $X$ be a set of logical variables and let $Expr_{\mathcal{B}}(X)$ be a set of Boolean expressions over $X$.

A set $(\Sigma, \cdot \models \cdot)$ is called an **alphabet** for $Expr_{\mathcal{B}}(X)$ if and only if

- for each $\sigma \in \Sigma$,
  - for each expression $expr \in Expr_{\mathcal{B}}$, and
    - for each valuation $\beta : X \rightarrow \mathcal{D}(X)$ of logical variables,
      
      either \( \sigma \models_\beta expr \) or \( \sigma \not\models_\beta expr \).

($\sigma$ satisfies (or does not satisfy) $expr$ under valuation $\beta$)

An **infinite sequence**

$$w = (\sigma_i)_{i \in \mathbb{N}_0} \in \Sigma^\omega$$

over $(\Sigma, \cdot \models \cdot)$ is called **word** (for $Expr_{\mathcal{B}}(X)$).
**Definition.** Let $\mathcal{B} = (\text{Expr}_\mathcal{B}(X), X, Q, q_{\text{ini}}, \rightarrow, Q_F)$ be a TBA and

$$w = \sigma_1, \sigma_2, \sigma_3, \ldots$$

a word for $\text{Expr}_\mathcal{B}(X)$. An infinite sequence

$$q = q_0, q_1, q_2, \ldots \in Q^\omega$$

is called **run of $\mathcal{B}$ over $w$** under valuation $\beta : X \rightarrow \mathcal{P}(X)$ if and only if

- $q_0 = q_{\text{ini}}$,
- for each $i \in \mathbb{N}_0$ there is a transition
  $$\left(q_i, \psi_i, q_{i+1}\right) \in \rightarrow$$

such that $\sigma_i \models_\beta \psi_i$.

**Example:**

$\mathcal{B}_{\text{sym}}$:

$\Sigma = \{x\} \rightarrow \mathbb{N}$
Definition.
We say TBA $B = (Expr_B(X), X, Q, q_{ini}, \rightarrow, Q_F)$ accepts the word

$$w = (\sigma_i)_{i \in \mathbb{N}_0} \in (Expr_B \rightarrow \mathbb{B})^\omega$$

if and only if $B$ has a run

$$\varrho = (q_i)_{i \in \mathbb{N}_0}$$

over $w$ such that fair (or accepting) states are visited infinitely often by $\varrho$, i.e., such that

$$\forall i \in \mathbb{N}_0 \exists j > i : q_j \in Q_F.$$

We call the set $L(B) \subseteq (Expr_B \rightarrow \mathbb{B})^\omega$ of words that are accepted by $B$ the language of $B$.

Example:

$$B_{sym}: \quad \Sigma = \{x\} \rightarrow \mathbb{N}$$

$$\begin{array}{cccc}
q_1 & \xrightarrow{even(x)} & q_2 \\
odd(x) & & odd(x)
\end{array} \quad \begin{array}{cccc}
q_2 & \xrightarrow{even(x)} & q_1 \\
odd(x) & & odd(x)
\end{array}$$
References
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