Recall: Restricted DC (RDC)

\[ F ::= \lceil P \rceil \mid \neg F_1 \mid F_1 \lor F_2 \mid F_1 ; F_2 \mid \ell = 1 \mid \ell = x \mid \forall x \cdot F_1 \]

where \( P \) is a state assertion with boolean observables only.

From now on: "RDC + \( \ell = x, \forall x \)"

Decidability Results for RDC

in Continuous Time

Decidability Results for DC

in Discrete Time

<table>
<thead>
<tr>
<th>Fragment</th>
<th>Continuous Time</th>
<th>Decidable</th>
<th>Decidable</th>
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<tbody>
<tr>
<td>RDC</td>
<td>Continuous Time</td>
<td>Decidable</td>
<td>Decidable</td>
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<tr>
<td>RDC + ( \ell = r )</td>
<td>( r \in \mathbb{N} )</td>
<td>Decidable</td>
<td>Undecidable</td>
</tr>
<tr>
<td>RDC + ( \int P_1 = \int P_2 )</td>
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</table>

Theorem 3.10. The realisability from 0 problem for DC with continuous time is undecidable, not even semi-decidable.

Theorem 3.11. The satisfiability problem for DC with continuous time is undecidable.
Reduction to 2CM: Idea

If realizability from 0 was (semi-)decidable, divergence of two-counter machines would be (which it isn't).

We assume that at most one command starts in each deterministic machine program where

\[ \text{encoding } M \]

Given a two-counter machine, its encoding is

\[ \text{Sketch: Proof of Theorem 3.10} \]
Reducing Divergence to DC realisability: Idea

Examples

\[ \begin{align*}
\text{Formula Construction for Given 2-CM} & : \\
\text{We use } \pi \text{ by a DC formula } \\
\text{then } \pi \text{ diverges } \\
\text{there exists } \exists n \in \mathbb{N} \\
\text{each two subsequent intervals } [0, n) \text{ and } [n, n+1) \\
\text{there is reached, we stay there } \pi \text{ diverges } \\
\text{in transition relation } K \vdash \\
\text{iff } \vdash \\
\text{characterised } \end{align*} \]

Reducing Divergence to DC realisability: Idea

\[ \begin{align*}
\text{Examples } & : \\
\text{we use } \pi \text{ by a DC formula } \\
\text{then } \pi \text{ diverges } \\
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\text{there is reached, we stay there } \pi \text{ diverges } \\
\text{in transition relation } K \vdash \\
\text{iff } \vdash \\
\text{characterised } \end{align*} \]
Thus it is semi-decidable whether $\forall \ell \in \mathbb{R}$ which we denote by $\text{RDC}$ is not decidable, this is $\text{RDC}$ augmented by "if and only if $C$ is a theorem in $F$"

Furthermore, by taking the contraposition, we see $\forall \ell \in \mathbb{R}$ is not necessary — we can use $1 \geq \ell$ instead, with fresh $\ell$ where $\ell$ is not a state assertion, $\exists \ell \in \mathbb{R}$ if and only if $M$ halts does not $\exists \ell \in \mathbb{R}$.

This yields $\forall \ell \in \mathbb{R}$. This provides us with an alternative proof of Theorem 2.23 ("there is no sound and decidable calculus that is both complete and correct for $\text{DC}$") as follows. Let

$$x = \ell; \quad y = \ell; \quad z = \ell$$

Moreover, $\exists \ell \in \mathbb{R}$ if and only if $M$ halts does not $\exists \ell \in \mathbb{R}$.

The satisfiability problem for $\text{DC}$ with continuous time is undecidable. (Theorem 3.11.)

The satisfiability problem for $\text{DC}$ with continuous time is undecidable. (Theorem 3.11.)
For Restricted DC plus $\ell = x$ and $\forall x$ in continuous time:

- Satisfiability is undecidable.
- Proof idea: reduce to halting problem of two-counter machines.

For full DC, it doesn't get better.