# *Real-Time Systems*

# *Lecture 14: Regions and Zones*

*2017-12-21*

Dr. Bernd Westphal

Albert-Ludwigs-Universität Freiburg, Germany

# Content

- **Motivation**:
  Sometimes, regions seem too fine-grained
- **Definition**
  - **Examples**: Zone or Not Zone
- **Zone**-based **Reachability Analysis**
  - The **basic algorithm**.
  - Building blocks:
    - **Post**-operator,
    - **subsumption check**
  - A **symbolic Post**-operator
- **Difference-Bounds-Matrices** (DBMs)
- **Discussion**: **Zones** vs. **Regions**

# *Zones*

*(Presentation following Fränzle (2007))*

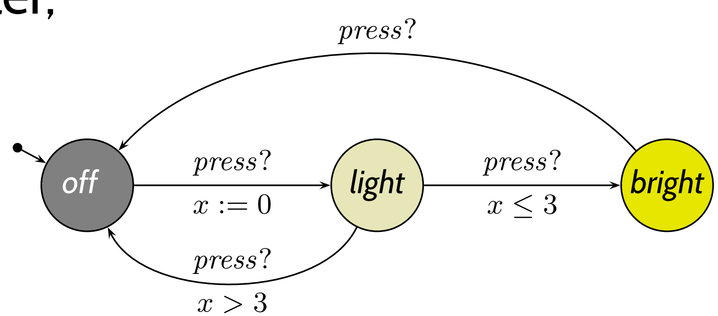
# *Recall: Number of Regions*

**Lemma 4.28.** Let $X$ be a set of clocks, $c_x \in \mathbb{N}_0$ the maximal constant for each $x \in X$, and $c = \max\{c_x \mid x \in X\}$. Then

$$(2c+2)^{|X|} \cdot (4c+3)^{\frac{1}{2}|X|\cdot(|X|-1)}$$

is an **upper bound** on the **number of regions**.

- In the desk lamp controller,




many regions are reachable in $\mathcal{R}(\mathcal{L})$, but we convinced ourselves that it's **actually** only important whether $\nu(x) \in [0,3]$ or $\nu(x) \in (3,\infty)$.

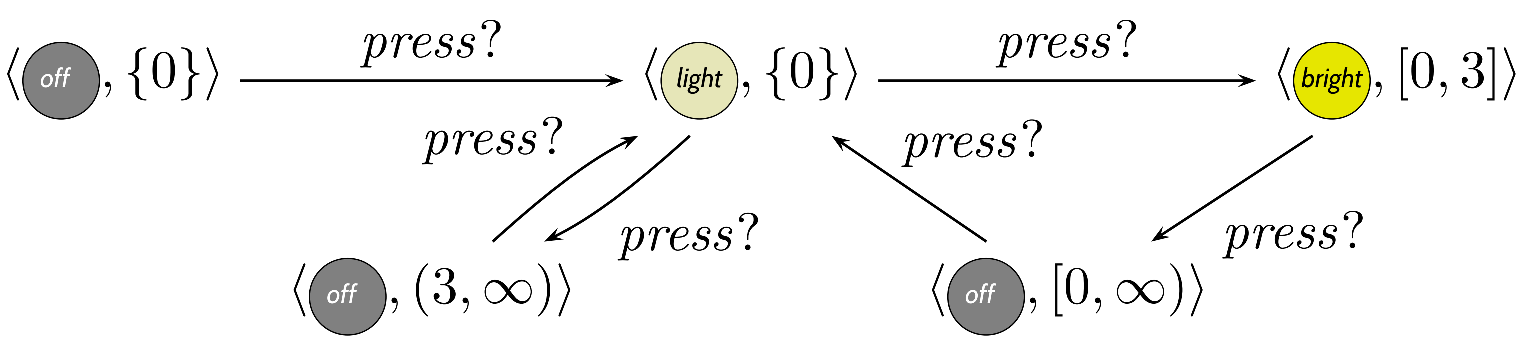
So: it seems like there are even **equivalence classes** of **undistinguishable regions** in certain timed automata.

# *Wanted: Zones instead of Regions*

- In $\mathcal{R}(\mathcal{L})$ we have transitions:
  - $\langle \text{light}, \{0\}\rangle \xrightarrow{press?} \langle \text{bright}, \{0\}\rangle$, $\quad \langle \text{light}, \{0\}\rangle \xrightarrow{press?} \langle \text{bright}, (0,1)\rangle$,
  - $\dots$,
  - $\langle \text{light}, \{0\}\rangle \xrightarrow{press?} \langle \text{bright}, (2,3)\rangle$, $\quad \langle \text{light}, \{0\}\rangle \xrightarrow{press?} \langle \text{bright}, \{3\}\rangle$
- Which seems to be a complicated way to write just:

$$\langle \text{light}, \{0\}\rangle \xrightarrow{press?} \langle \text{bright}, [0,3]\rangle$$

- Can't we **constructively** abstract $\mathcal{L}$ to:



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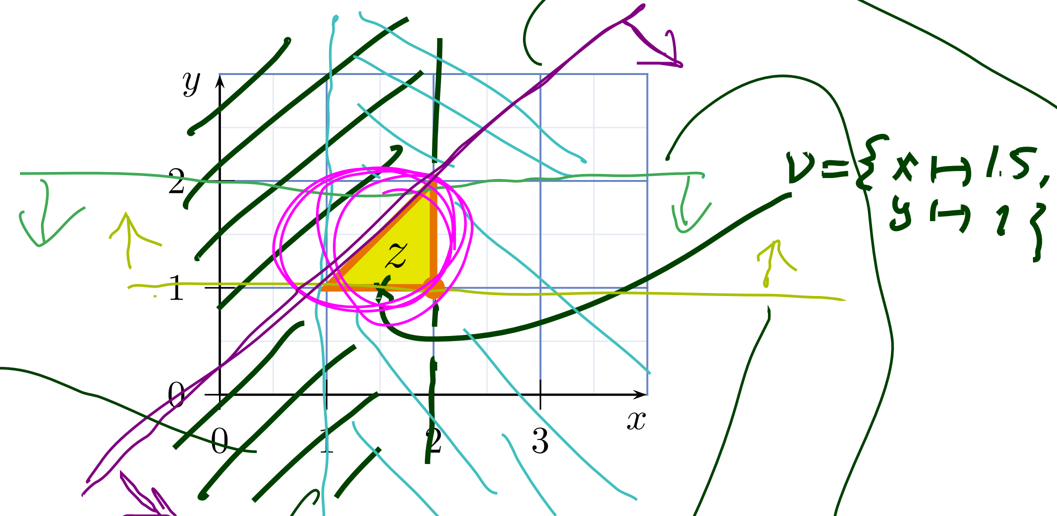
# *What is a Zone?*

**Definition.** A (**clock**) **zone** is a set $z \subseteq (X \to \mathrm{Time})$ of valuations of clocks $X$ such that there exists $\varphi \in \Phi(X)$ with

$$\nu \in z \text{ if and only if } \nu \models \varphi.$$

**Example**:




is a clock zone by

$$\varphi = x > 1 \wedge x \leq 2 \wedge y \geq 1 \wedge y < 2 \wedge (x - y \geq 0)$$

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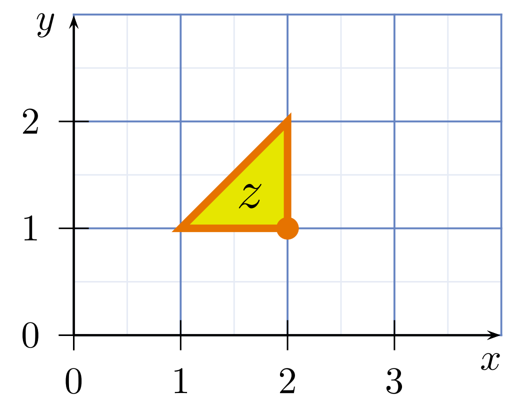


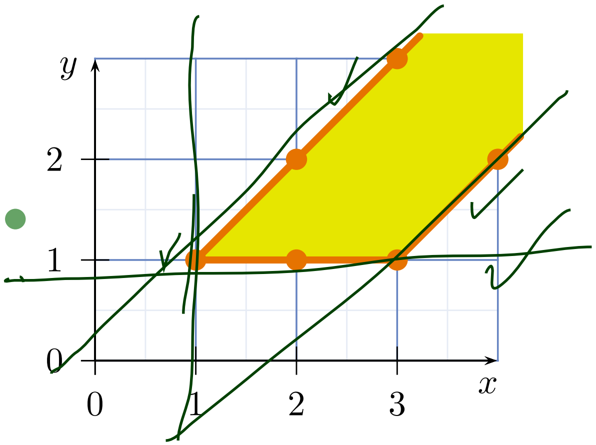

is a clock zone by

$$\varphi = (x \leq 2) \wedge (x > 1) \wedge (y \geq 1) \wedge (y < 2) \wedge (x - y \geq 0)$$

- Note: Each clock constraint $\varphi$ is a **symbolic representation** of a zone.
- But: There's no one-on-one correspondence between clock constraints and zones. The zone $z = \emptyset$ corresponds to $(x > 1 \wedge x < 1)$, $(x > 2 \wedge x < 2)$, ...

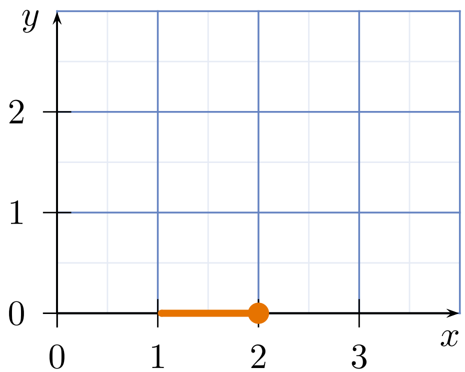
# More Examples: Zone or Not?

$z$ is a zone iff there is $\varphi \in \Phi(X)$ s.t. $z = \{\nu \mid \nu \models \varphi\}$.

- 

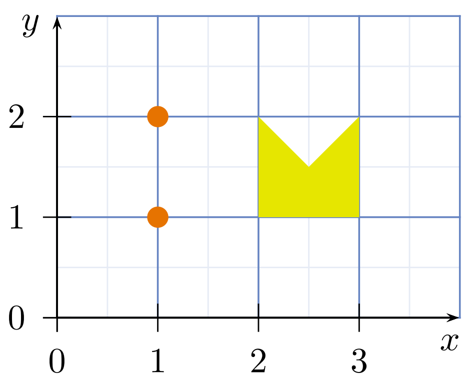
  YES

  $x \geq 1 \wedge x - y \geq 0 \wedge x - y \leq 2 \wedge y \geq 1$

- 

  YES

  $y \geq 0 \wedge y \leq 0 \wedge x > 1 \wedge x \leq 2$ $(\sim (1,2) \cup \{2\})$

- 

  NO

  (not convex)

$z$ is zone $\implies \exists\, r_1, \dots, r_n$ regions. $z = \bigcup_{i=1}^{n} r_i$

$z = \bigcup_{i=1}^{n} r_i$, $r_i$ region $\not\Rightarrow$ $z$ is zone

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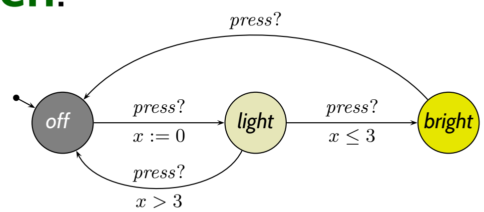
# *Zone-based Reachability Analysis*

**Given**:

- 

and initial configuration $\langle$ off $, \{0\}\rangle$

Assume a function

$$\mathrm{Post}_e : (L \times \text{Zones}) \to (L \times \text{Zones})$$

such that $\mathrm{Post}_e(\langle \ell, z\rangle)$ yields the configuration $\langle \ell', z'\rangle$ such that

- zone $z'$ denotes exactly those clock valuations $\nu'$
  - which are reachable from a configuration $\langle \ell, \nu\rangle$, $\nu \in z$,
    - by taking edge $e = (\ell, \alpha, \varphi, Y, \ell') \in E$.

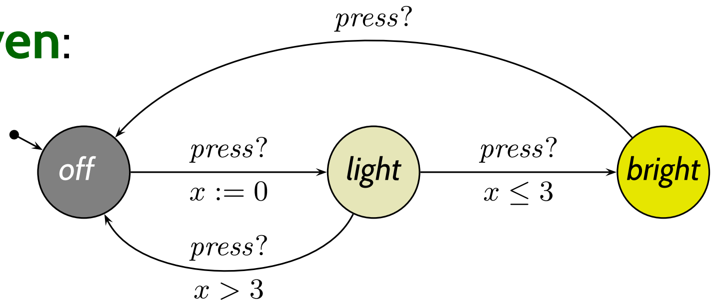
Then $\ell \in L$ is reachable in $\mathcal{A}$ if and only if

$$\mathrm{Post}_{e_n}(\dots(\mathrm{Post}_{e_1}(\langle \ell_{ini}, z_{ini}\rangle)\dots)) = \langle \ell, z\rangle$$

for some $e_1, \dots, e_n \in E$ and some $z$.

# *Zone-based Reachability: In Other Words*

**Given**:

- 

and initial configuration ⟨off, $\{0\}$⟩

**Wanted**: A procedure to compute the set

- ⟨light, $\{0\}$⟩
- ⟨bright, $[0, 3]$⟩
- ⟨off, $[0, \infty)$⟩

- Set $R := \{\langle \ell_{ini}, z_{ini} \rangle\} \subset L \times$ Zones
- Repeat
  - pick
    - a pair $\langle \ell, z \rangle$ from $R$ and
    - an edge $e \in E$ with source $\ell$

    such that $\mathrm{Post}_e(\langle \ell, z \rangle)$ is not already subsumed by $R$
  - add $\mathrm{Post}_e(\langle \ell, z \rangle)$ to $R$

  until no more such $\langle \ell, z \rangle \in R$ and $e \in E$ are found.

# *Stocktaking: What's Missing?*

- Set $R := \{\langle \ell_{ini}, z_{ini} \rangle\} \subset L \times$ Zones
- Repeat
  - pick
    - a pair $\langle \ell, z \rangle$ from $R$ and
    - an edge $e \in E$ with source $\ell$

    such that $\mathrm{Post}_e(\langle \ell, z \rangle)$ is not already **subsumed** by $R$
  - add $\mathrm{Post}_e(\langle \ell, z \rangle)$ to $R$

  until no more such $\langle \ell, z \rangle \in R$ and $e \in E$ are found.

**Missing**:

- Algorithm to effectively compute $\mathrm{Post}_e(\langle \ell, z \rangle)$
  for a given configuration $\langle \ell, z \rangle \in L \times$ Zones and an edge $e \in E$.
- Decision procedure for whether
  configuration $\langle \ell', z' \rangle$ is **subsumed** by a given subset of $L \times$ Zones.

**Note**: The algorithm in general **terminates only if** we apply **widening** to zones, that is, roughly, to take maximal constants $c_x$ into account (not in lecture).

# *What is a Good "Post"?*

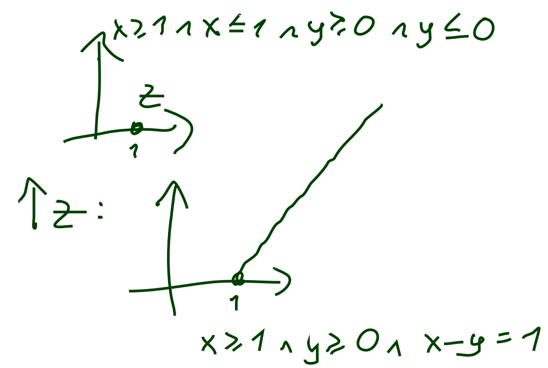
- If $z$ is given by a constraint $\varphi \in \Phi(X)$, (write: $z = [\![\varphi]\!]$) then the zone component $z'$ of $\mathrm{Post}_e(\ell, z) = \langle \ell', z' \rangle$ should also be a constraint from $\Phi(X)$.

  (We want to **manipulate constraints**, not those unhandy sets of clock valuations.)

**Good news**: the following operations can be carried out by manipulating $\varphi$.

(1) The **elapse time** operation:

$$\begin{aligned} \uparrow \quad : \quad & \text{Zones} \to \text{Zones} \\ & z \mapsto \{\nu + t \mid t \in \text{Time}\} \end{aligned}$$




can be carried out **symbolically** as follows:

- Let $z = [\![\varphi]\!]$.
- Obtain $\varphi'$ by removing all upper bounds $x \leq c$, $x < c$, from $\varphi$ and adding diagonals.
- Then $[\![\varphi']\!] = z\uparrow$.

This procedure defines $\uparrow\colon \Phi(X) \to \Phi(X)$ (a function on **clock constraints**!), such that $[\![\varphi\uparrow]\!] = z\uparrow$ if $z = [\![\varphi]\!]$.

# *Good News Cont'd*

**Good news**: the following operations can be carried out by manipulating $\varphi$.

(1) **elapse time**: $\varphi\uparrow$ with $[\![\varphi\uparrow]\!] = z\uparrow$ if $z = [\![\varphi]\!]$.

(2) **zone intersection**: if $z_1 = [\![\varphi_1]\!]$ and $z_2 = [\![\varphi_2]\!]$, then $[\![\varphi_1 \wedge \varphi_2]\!] = z_1 \cap z_2$.

(3) **clock reset**:

$$\begin{aligned} \cdot\,[\,\cdot\; := 0] \quad : \quad & \text{Zones} \times X \to \text{Zones} \\ & (z, x) \mapsto \{\nu[x := 0] \mid \nu \in z\} \end{aligned}$$

can be carried out **symbolically** by setting

$$\begin{aligned} \cdot\,[\,\cdot\; := 0] \quad : \quad & \Phi \times X \to \Phi \\ & (\varphi, x) \mapsto (x = 0) \wedge (\exists\, x.\varphi) \end{aligned}$$

$x = y \wedge x = 2$

$x = 0 \wedge x = y \wedge x = 2$ ☹

$x = 0 \wedge (\exists \tilde{x}.\ \tilde{x} = y \wedge \tilde{x} = 2)$

using **clock hiding** (existential quantification);

$$[\![\exists\, x.\varphi]\!] = \{\nu \mid \text{there is } t \in \text{Time such that } \nu[x := t] \models \varphi\}$$

# *This is Good News...*

...because given $\langle \ell, z \rangle = \langle \ell, [\![\varphi_0]\!] \rangle$ and $e = (\ell, \alpha, \varphi, \{y_1, \dots, y_n\}, \ell') \in E$ we have

$$\text{Post}_e(\langle \ell, z \rangle) = \langle \ell', [\![\varphi_5]\!] \rangle \qquad (\textbf{symbolical}: \text{Post}_e(\langle \ell, \varphi_0 \rangle) = \langle \ell', \varphi_5 \rangle)$$

where

- $\varphi_1 = \varphi_0 \uparrow$

  let **time elapse** starting from $\varphi_0$:
  $\varphi_1$ represents all valuations reachable by waiting in $\ell$ for an arbitrary amount of time.

- $\varphi_2 = \varphi_1 \wedge I(\ell)$

  **intersect with invariant** of $\ell$: $\varphi_2$ represents the "good" valuations reachable from $\varphi_1$.

- $\varphi_3 = \varphi_2 \wedge \varphi$

  **intersect with guard**: in $\varphi_3$ are the reachable "good" valuations where $e$ is enabled.

- $\varphi_4 = \varphi_3[y_1 := 0] \dots [y_n := 0]$

  **reset clocks**: $\varphi_4$ are all possible outcomes of taking $e$ from $\varphi_3$.

- $\varphi_5 = \varphi_4 \wedge I(\ell')$

  **intersect with invariant** of $\ell'$: $\varphi_5$ are the "good" outcomes of taking $e$ from $\varphi_3$.

# *Example*

- $\varphi_1 = \varphi_0 \uparrow$ let **time elapse**.
- $\varphi_2 = \varphi_1 \wedge I(\ell)$ **intersect with invariant** of $\ell$
- $\varphi_3 = \varphi_2 \wedge \varphi$ **intersect with guard**
- $\varphi_4 = \varphi_3[y_1 := 0] \dots [y_n := 0]$ **reset clocks**
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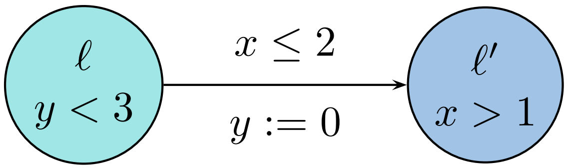

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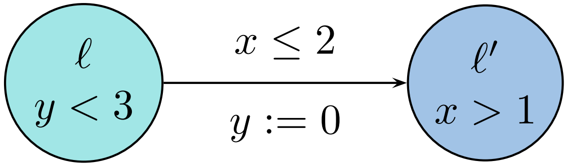


$$\varphi_0 = 1 \leq y \leq 2 \\ \wedge 1 \leq x \leq 3 \wedge x \geq y$$

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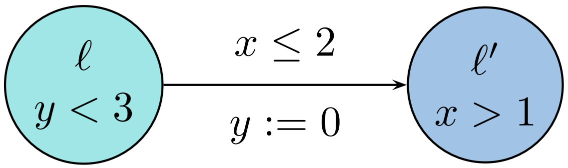


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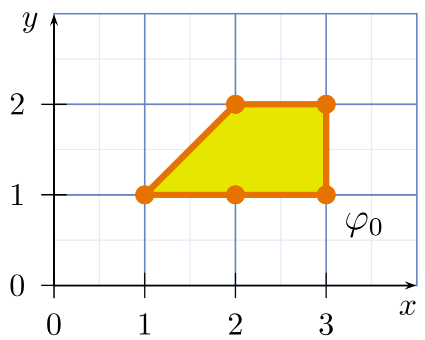

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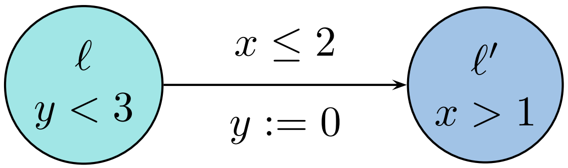
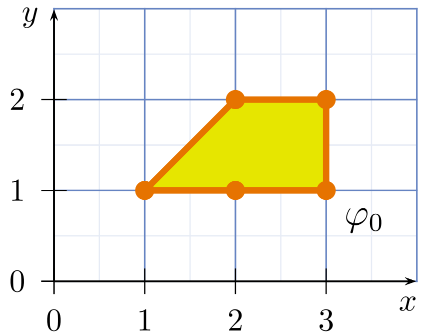



$$\varphi_0 = 1 \le y \le 2 \wedge 1 \le x \le 3 \wedge x \ge y$$

$$\varphi_1 = 1 \le y \wedge 1 \le x \wedge x \ge y \wedge x \le y + 2$$



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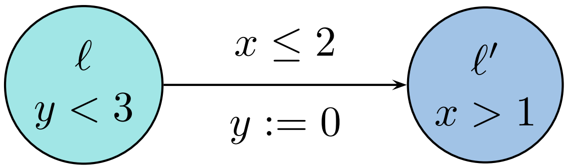


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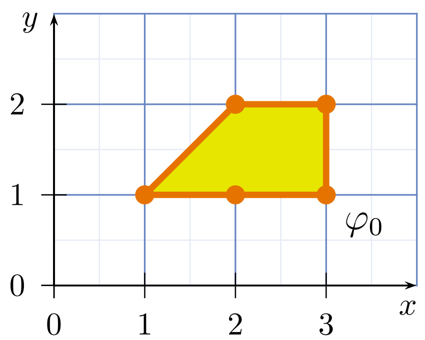


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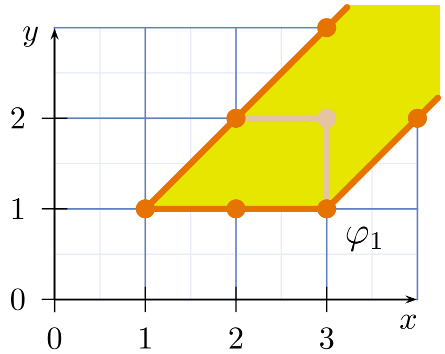

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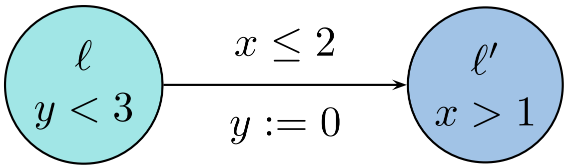


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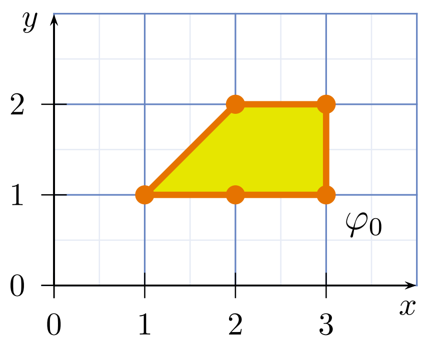


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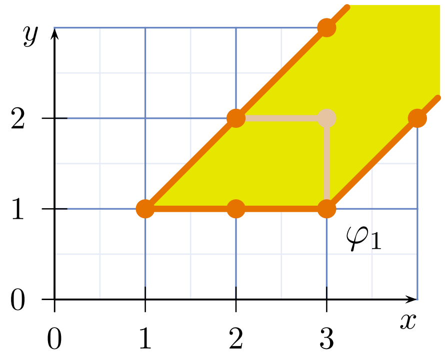


$\varphi_2 = 1 \leq y < 3 \wedge 1 \leq x$
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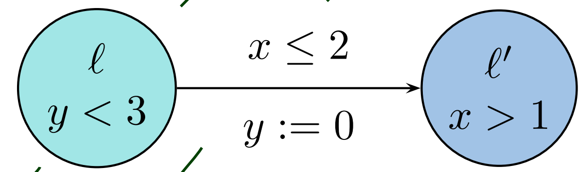


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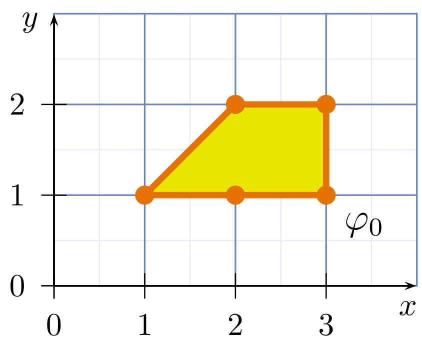


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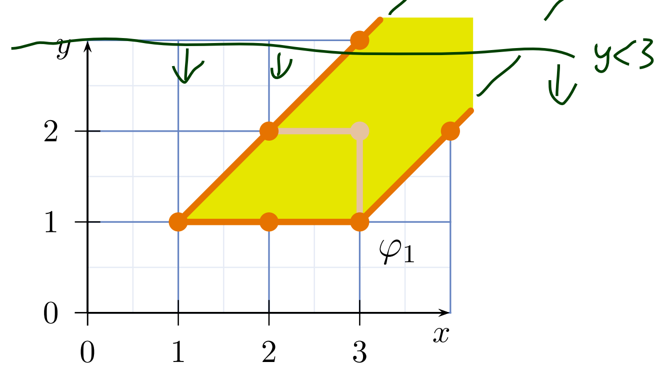


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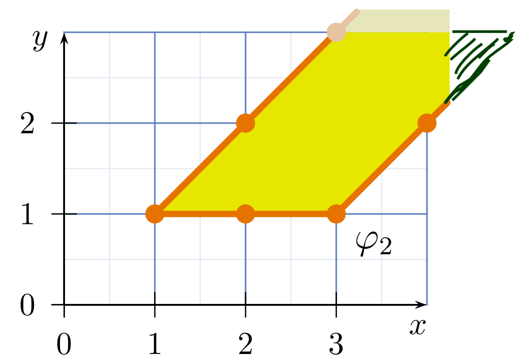

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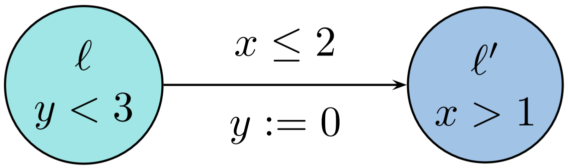


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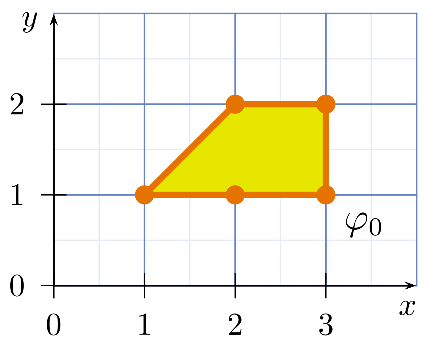


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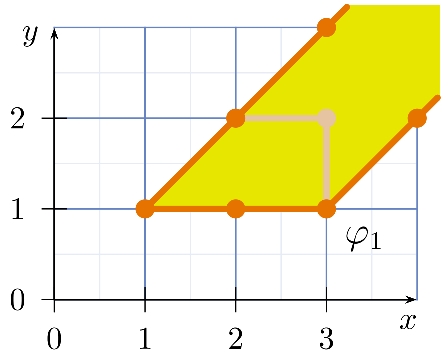


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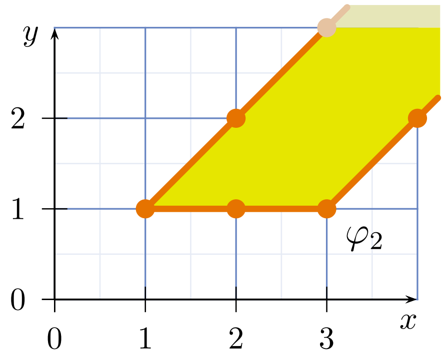


$\varphi_3 = 1 \leq y < 3$
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# Example

- $\varphi_1 = \varphi_0 \uparrow$ — let **time elapse**.
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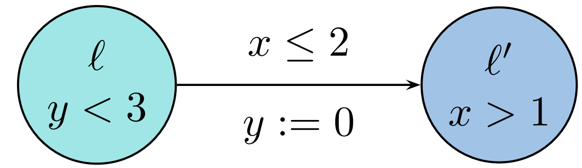


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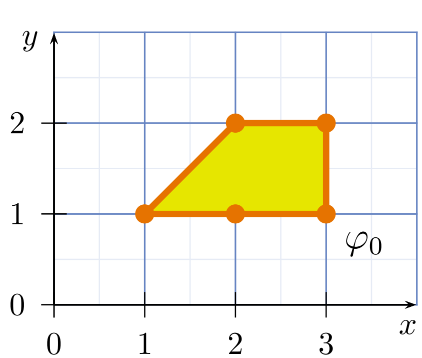


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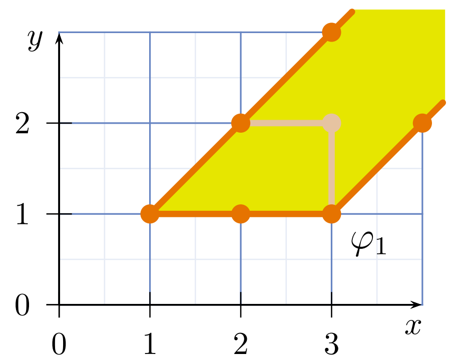


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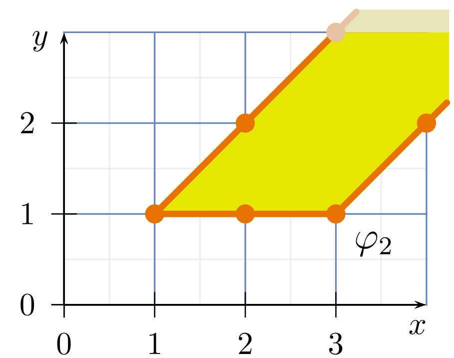


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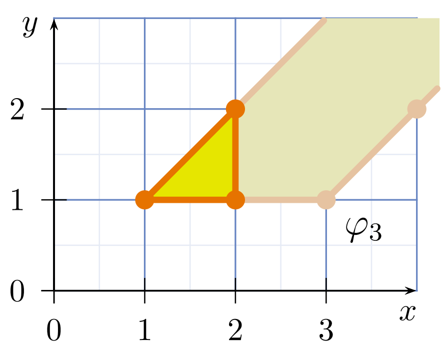

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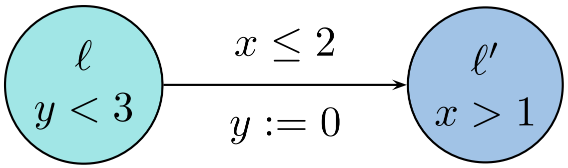


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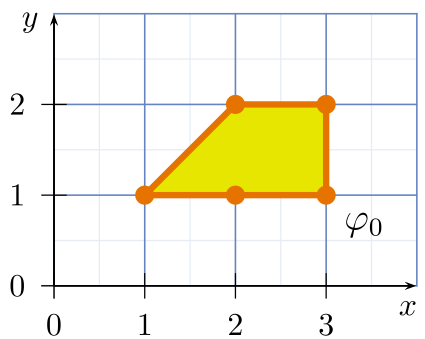


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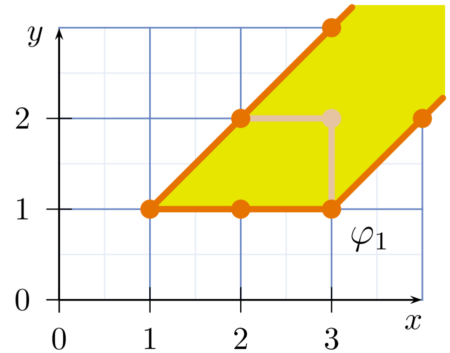


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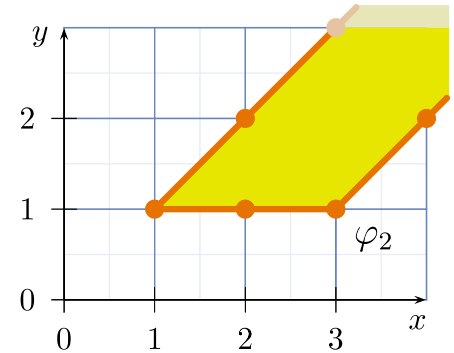


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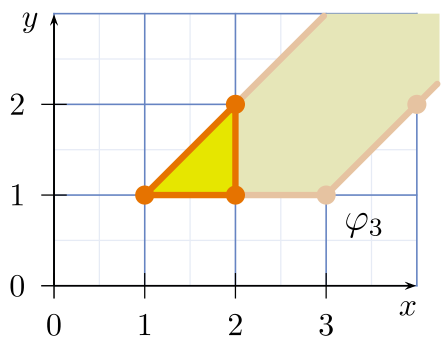


$\varphi_4 \equiv y = 0 \wedge$
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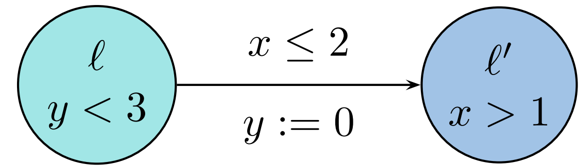


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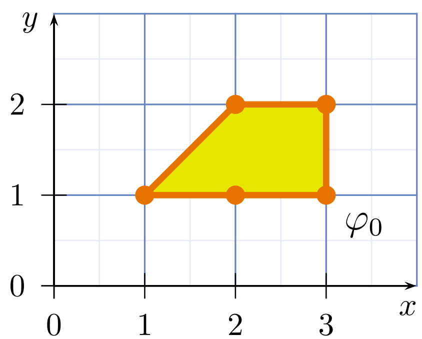


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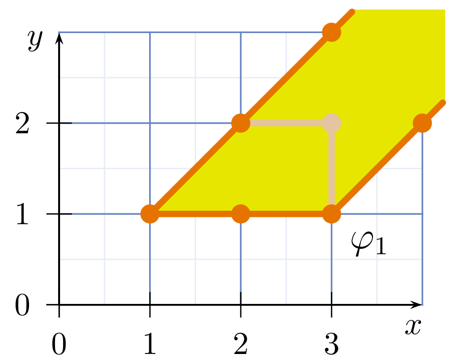


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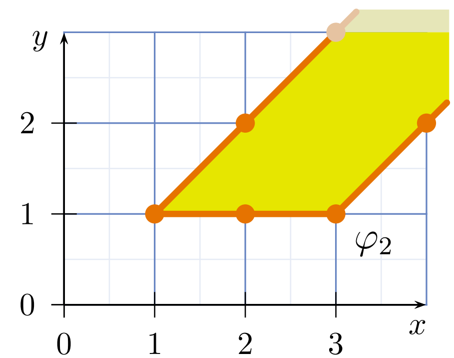


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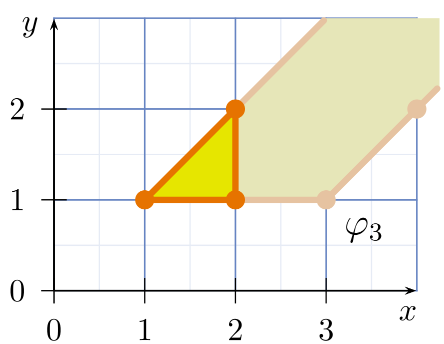


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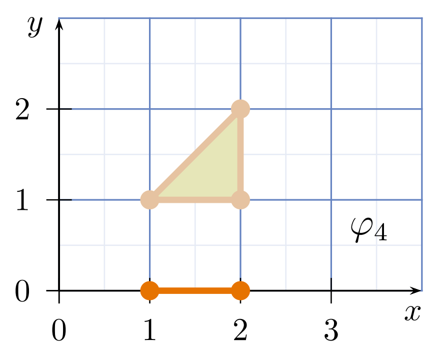

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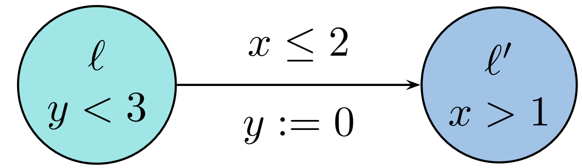


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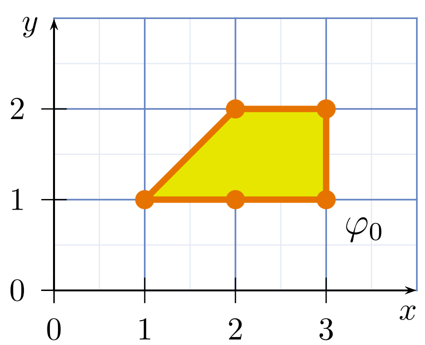


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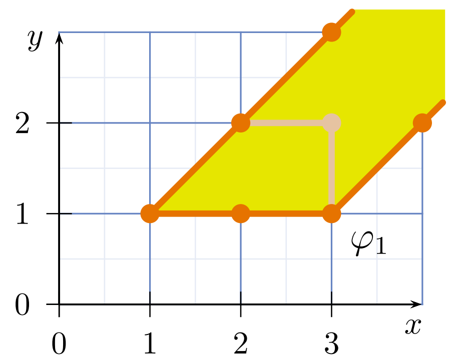


$\varphi_2 = 1 \leq y < 3 \wedge 1 \leq x$
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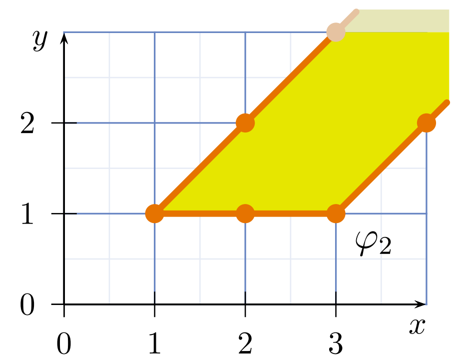


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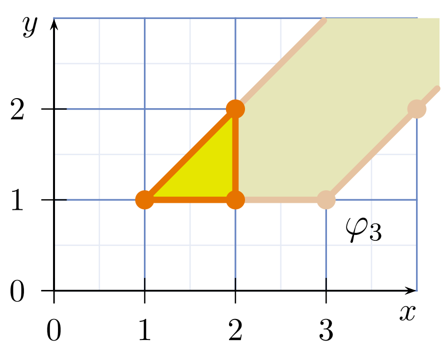


$\varphi_4 = y = 0 \wedge$
$\exists\, y.1 \leq y < 3 \wedge 1 \leq x \leq$
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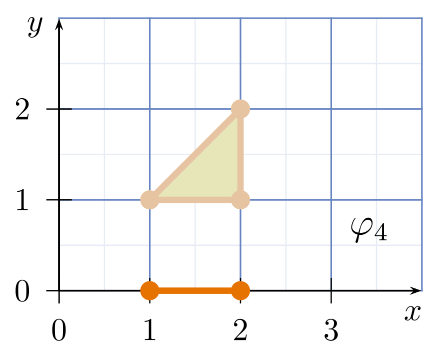


$\varphi_5 = x > 1 \wedge y = 0 \wedge$
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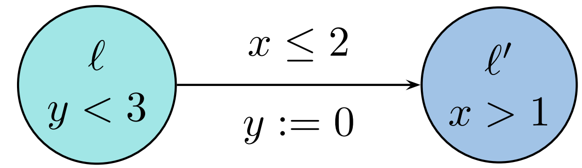


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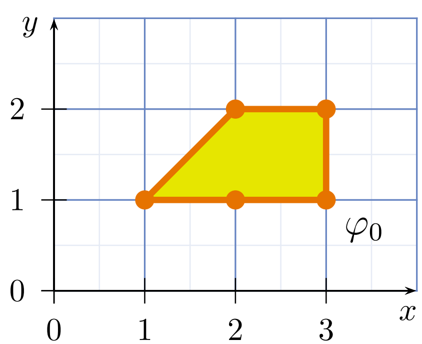


$\varphi_1 \equiv 1 \leq y \wedge 1 \leq x$
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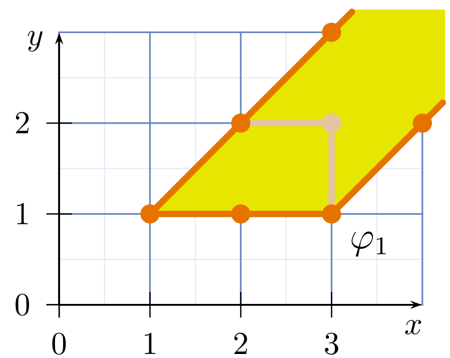


$\varphi_2 \equiv 1 \leq y < 3 \wedge 1 \leq x$
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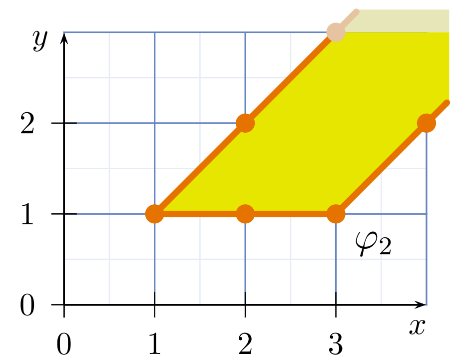


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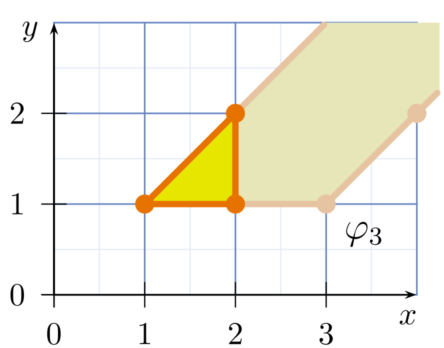


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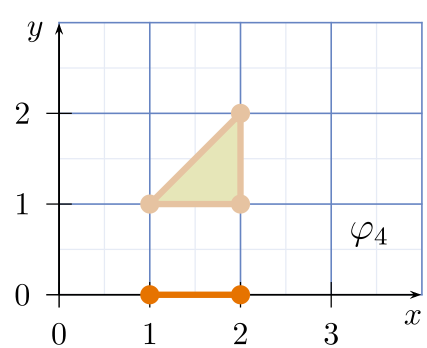


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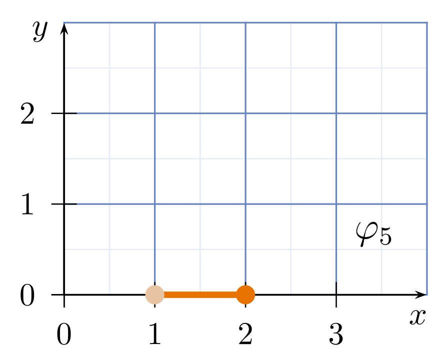

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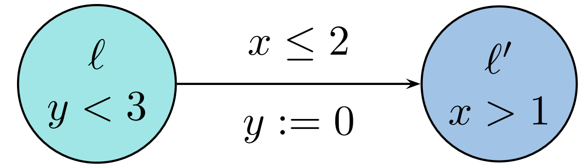
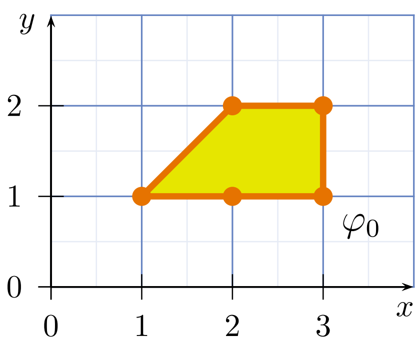



$\varphi_0 \equiv 1 \le y \le 2 \wedge 1 \le x \le 3 \wedge x \ge y$

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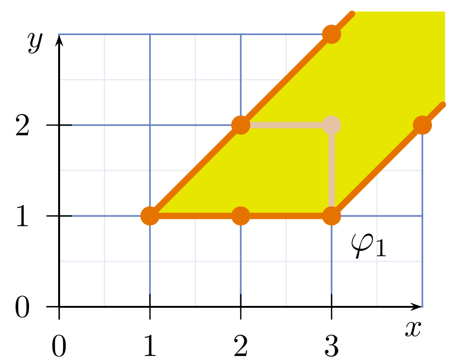


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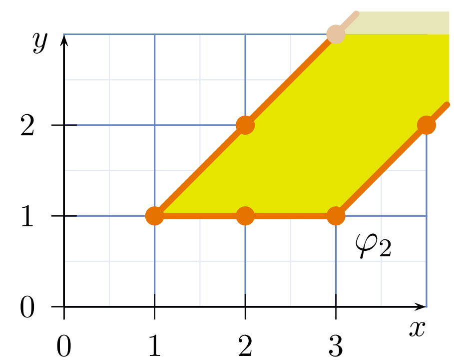


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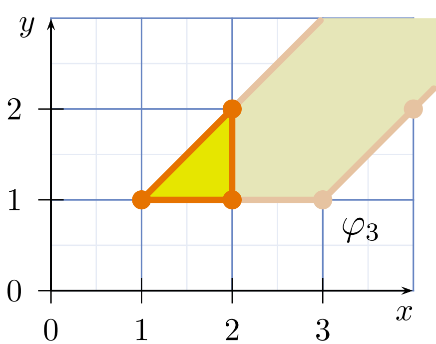
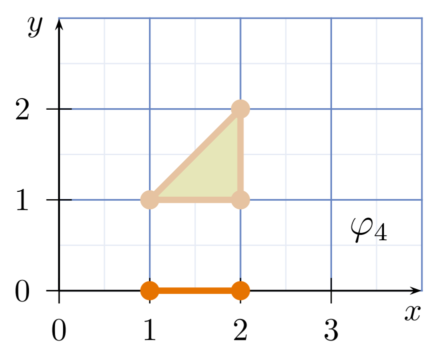
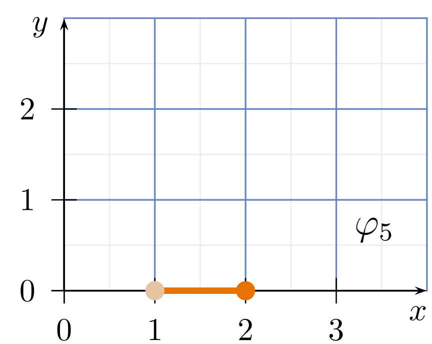



$\varphi_4 \equiv y = 0 \wedge \exists\, y . 1 \le y < 3 \wedge 1 \le x \le 2 \wedge x \ge y \wedge x \le y + 2$




$\varphi_5 \iff 1 < x \le 2 \wedge y = 0$



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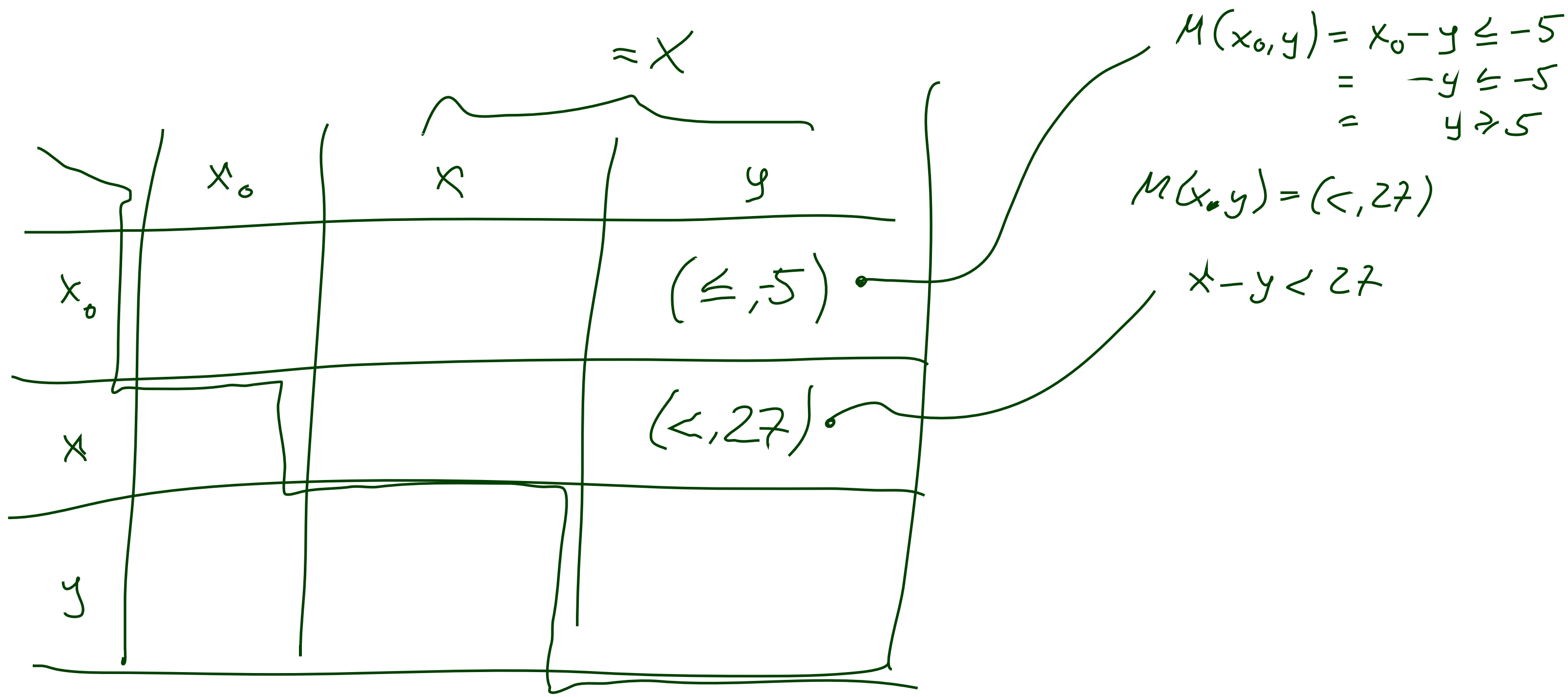
  Sometimes, regions seem too fine-grained
- **Definition**
  - **Examples**: Zone or Not Zone
- **Zone**-based **Reachability Analysis**
  - The **basic algorithm**.
  - Building blocks:
    - **Post**-operator,
    - **subsumption check**
  - A **symbolic Post**-operator
- **Difference-Bounds-Matrices** (DBMs)
- **Discussion**: **Zones** vs. **Regions**

# Difference Bound Matrices

- Given a finite set of clocks $X$, a **DBM** over $X$ is a mapping

$$M : (X \,\dot{\cup}\, \{x_0\}) \times (X \,\dot{\cup}\, \{x_0\}) \to (\{<, \leq\} \times \mathbb{Z}) \cup \{(<, \infty)\}$$

- $M(x, y) = (\sim, c)$ encodes the conjunct $x - y \sim c$ ($x$ and $y$ can be $x_0$).



# *Difference Bound Matrices*

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- $M(x, y) = (\sim, c)$ encodes the conjunct $x - y \sim c$    ($x$ and $y$ can be $x_0$).
- If $M$ and $N$ are **DBMs encoding** $\varphi_1$ and $\varphi_2$ (representing zones $z_1$ and $z_2$), then we can efficiently compute $M \uparrow$, $M \wedge N$, $M[x := 0]$ such that
  - all three are **again DBM**,
  - $M \uparrow$   **encodes**   $\varphi_1 \uparrow$,
  - $M \wedge N$   **encodes**   $\varphi_1 \wedge \varphi_2$, and
  - $M[x := 0]$   **encodes**   $\varphi_1[x := 0]$.
- And there is a **canonical form** of DBM.

  (Canonisation of DBM can be done in cubic time (**Floyd-Warshall** algorithm)).
- Thus: we can define our '$\mathrm{Post}$' on DBM, and let our algorithm run on DBM.

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# *Pros and cons*

- **Zone-based**
  reachability analysis usually is explicit wrt. discrete locations:
  - maintains a list of location/zone pairs (or location/DBM pairs)
  - **confined wrt. size of discrete state space**
  - **avoids blowup by number of clocks and size of clock constraints through symbolic representation of clocks**

- **Region-based**
  analysis provides a finite-state abstraction,
  amenable to finite-state symbolic model-checking
  - **less dependent on size of discrete state space**
  - **exponential in number of clocks**

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# *Tell Them What You've Told Them...*

- A **zone** is **a set of clock valuations**
  which can be characterised by a **clock constraint**.
- Each **zone** is a union of **regions**,
  not every union of **regions** is a **zone**.
- There is an **effectively computable**
  **Post**-operation for TA edges on **zones**.
  - based on: **time elapse**, **intersection**, **reset**
  - so there is a **fully symbolic**
    **decision procedure** for location reachability
    (if we ensure **termination** by **widening**)
  - even more convenient: using DBMs
    - since DBMs have a **normal form**
- For a **given model**, sometimes the **region**-based /
  sometimes the **zone**-based approach is faster.
  Not so many region-based tools are "on the market" these days.

*References*

# References

Fränzle, M. (2007). Formale methoden eingebetteter systeme. Lecture, Summer Semester 2007, Carl-von-Ossietzky Universität Oldenburg.

Olderog, E.-R. and Dierks, H. (2008). *Real-Time Systems - Formal Specification and Automatic Verification*. Cambridge University Press.