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Tutorial for Cyber-Physical Systems - Discrete Models
Exercise Sheet 2

The goal of this sheet to prepare notions that we will use in the context of transition systems.

Exercise 1: Set Notation for Valuations

This exercise refers to Sect. A.3 Propositional Logic in the appendix of the book. The goal of this exercise is to train your understanding of valuations and when a valuation satisfies a Boolean formula (and when it does not). Given a valuation μ and a Boolean formula ϕ, it is easy to see whether the valuation satisfies the Boolean formula, formally $\mu \models \phi$. It is perhaps more difficult to think about all valuations that satisfy a given Boolean formula.

Let $AP = \{a_1, \ldots, a_n\}$ be a set of atomic propositions (if you prefer, you can say Boolean variable instead of atomic proposition).

Intuitively, a valuation gives a value to each atomic proposition. The value is a truth value, which we here denote by 0 or 1. The book uses the term "evaluation" instead of "valuation". Formally, a valuation can be represented as a function $\mu : AP \to \{0, 1\}$. If $\mu(a_i)$ is 0, then the truth value of the atomic proposition a_i is 0. Alternatively, a valuation can be represented by a subset A of atomic propositions. If $a_i \in A$, then the truth value of the atomic proposition a_i is 1 (and 0, otherwise). The connection between the two representations can be stated by $A_\mu = \{a \in AP \mid \mu(a) = 1\}$. See also the book.

(a) Give a description of all valuations μ such that $\mu \models \phi$,
once expressed in terms of functions and once in terms of sets, with

- $\phi = a_1 \wedge \cdots \wedge a_i$
- $\phi = a_1 \vee \cdots \vee a_i$

where i is some number smaller than n, i.e. , $i \leq n$.

(b) Let $AP = \{a_1, a_2, a_3, a_4, a_5\}$. Find a formula ϕ such that μ_A with $A = \{a_2, a_3\}$ is the unique satisfying valuation for ϕ.

Exercise 2: Equivalence Relations I

The goal of this exercise is to obtain an intuition about the terms "reflexive", "symmetric" and "transitive".

Which of the following relationships between people is an equivalence relation?

(a) sibling (We assume that everyone is a sibling of himself or herself and that siblings have the same pair of parents, i.e., no half sister, no half brother.)

(b) friendship

(c) brother-of

(d) have the same father

(e) have the same mother

(f) have the same father or have the same mother

Is the intersection of two equivalence relations again an equivalence relation?
Is the union of two equivalence relations again an equivalence relation?

Exercise 3: Equivalence Relations II

The goal of this exercise to obtain an intuition about the connection between equivalence relations and functions.

In the lecture, we discussed an equivalence relation between pairs of integer numbers where each equivalence class corresponds to a rational number. We will now introduce an equivalence relation between pairs of natural numbers such that each equivalence class corresponds to an integer, and vice versa (each integer corresponds to an equivalence class).

The equivalence relation $\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$ is defined by

$$(a, b) \sim (c, d) \quad \text{if} \quad a + d = b + c \ .$$

(a) Show that $\sim$ is indeed an equivalence relation.

(b) Define the function from $\mathbb{N}^2$ into the integers that corresponds to the equivalence relation $\sim$.

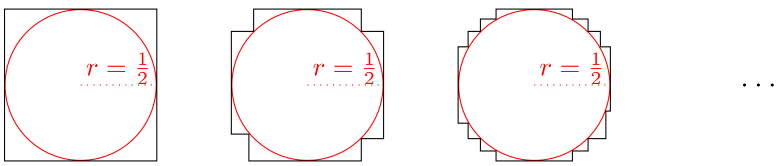
Exercise 4*: Correct Proofs

The goal of this task is to initiate philosophical discussions about mathematics.

Consider the following proof sketch.

We use the fact that the perimeter of a circle of radius $r = 0.5$ is π to prove that π is equal to 4. Consider the sequence of polygons depicted below. It converges to the circle. Since each polygon has the perimeter $P_i = 4$ for $i = 1, 2, \ldots$ and the limit of a constant sequence is the corresponding constant, the perimeter of the circle must be 4. Hence, π is equal to 4.

Do you trust the proof?



$P_1 = 4$ $P_2 = 4$ $P_3 = 4$