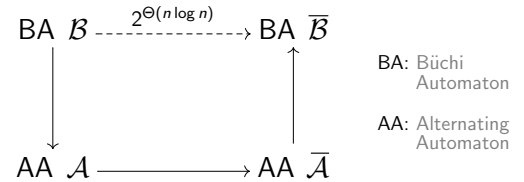


BÜCHI COMPLEMENTATION VIA ALTERNATING AUTOMATA

Fabian Reiter

July 16, 2012

BÜCHI COMPLEMENTATION

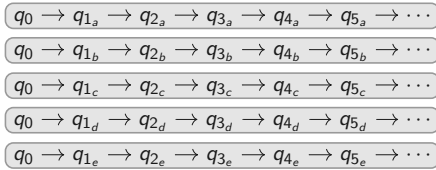


- **Expensive:** If \mathcal{B} has n states, $\bar{\mathcal{B}}$ has $2^{O(n \log n)}$ states in the worst case (Michel 1988, Safra 1988).
- **Complicated:** Direct approaches are rather involved.

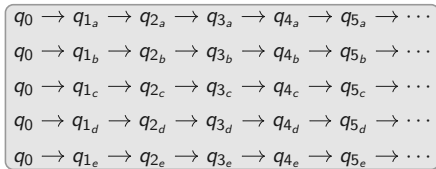
Consider indirect approach: detour over **alternating automata**.

TRANSITION MODES (1)

Existential: some run is accepting

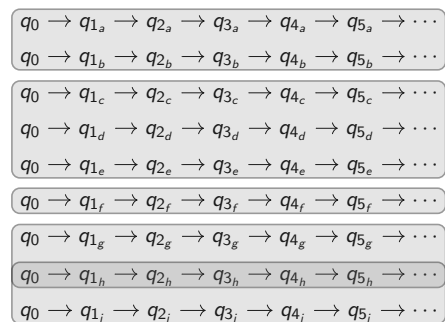


Universal: every run is accepting



TRANSITION MODES (2)

Alternating: in some set of runs every run is accepting



ALTERNATION AND COMPLEMENTATION

SPECIAL CASE: \mathcal{A} in existential mode

- \mathcal{A} accepts iff \exists run ρ : ρ fulfills acceptance condition of \mathcal{A}
- $\bar{\mathcal{A}}$ accepts iff \forall run ρ : $\neg(\rho$ fulfills acceptance condition of $\mathcal{A})$
 iff \forall run ρ : ρ fulfills **dual** acceptance condition of \mathcal{A}

\Rightarrow complementation $\hat{=}$ dualization of:

- transition mode
- acceptance condition

Want acceptance condition that is **closed under dualization**.

OUTLINE

- 1 WEAK ALTERNATING PARITY AUTOMATA
- 2 INFINITE PARITY GAMES
- 3 PROOF OF THE COMPLEMENTATION THEOREM
- 4 BÜCHI COMPLEMENTATION ALGORITHM

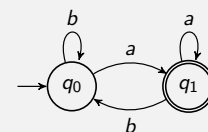
OUTLINE

- 1 WEAK ALTERNATING PARITY AUTOMATA
 - Definitions and Examples
 - Dual Automaton
- 2 INFINITE PARITY GAMES
- 3 PROOF OF THE COMPLEMENTATION THEOREM
- 4 BÜCHI COMPLEMENTATION ALGORITHM

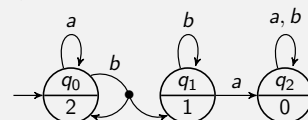
PREVIEW

EXAMPLE $((b^*a)^\omega)$

Büchi automaton \mathcal{B} :



Equivalent WAPA \mathcal{A} :





DEFINITION (Weak Alternating Parity Automaton)

A weak alternating parity automaton (WAPA) is a tuple

$$\mathcal{A} := \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$$

where

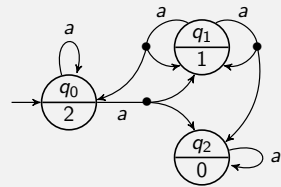
- Q finite set of states
- Σ finite alphabet
- $\delta : Q \times \Sigma \rightarrow \mathbb{B}^+(Q)$ transition function
- q_{in} initial state
- $\pi : Q \rightarrow \mathbb{N}$ parity function

(Thomas and Löding, ~2000)

$\mathbb{B}^+(Q)$: set of all positive Boolean formulae over Q
(built only from elements in $Q \cup \{\wedge, \vee, \top, \perp\}$)



EXAMPLE (a^ω)



$$\delta : Q \times \Sigma \rightarrow \mathbb{B}^+(Q)$$

$$\langle q_0, a \rangle \mapsto q_0 \vee (q_1 \wedge q_2)$$

$$\langle q_1, a \rangle \mapsto (q_0 \wedge q_1) \vee (q_1 \wedge q_2)$$

$$\langle q_2, a \rangle \mapsto q_2$$

DEFINITION (Minimal Models)

$\text{Mod}_\downarrow(\theta) \subseteq 2^Q$: set of minimal models of $\theta \in \mathbb{B}^+(Q)$, i.e. the set of minimal subsets $M \subseteq Q$ s.t. θ is satisfied by

$$q \mapsto \begin{cases} \text{true} & \text{if } q \in M \\ \text{false} & \text{otherwise} \end{cases}$$

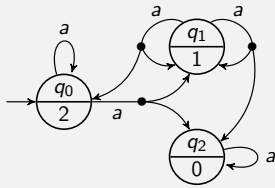
EXAMPLE

$$\text{Mod}_\downarrow(q_0 \vee (q_1 \wedge q_2)) = \{\{q_0\}, \{q_1, q_2\}\}$$

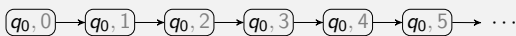
RUN GRAPH (1)



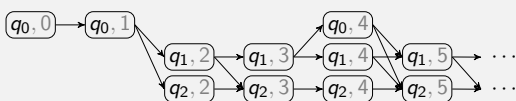
EXAMPLE (a^ω)



Accepting run:



Rejecting run:



RUN GRAPH (2)



DEFINITION (Run)

A run of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ on a word $a_0 a_1 a_2 \dots \in \Sigma^\omega$ is a directed acyclic graph

$$R := \langle V, E \rangle$$

where

- $V \subseteq Q \times \mathbb{N}$ with $\langle q_{in}, 0 \rangle \in V$
- V contains only vertices reachable from $\langle q_{in}, 0 \rangle$.
- E contains only edges of the form $\langle \langle p, i \rangle, \langle q, i + 1 \rangle \rangle$.
- For every vertex $\langle p, i \rangle \in V$ the set of successors is a minimal model of $\delta(p, a_i)$

$$\{q \in Q \mid \langle \langle p, i \rangle, \langle q, i + 1 \rangle \rangle \in E\} \in \text{Mod}_\downarrow(\delta(p, a_i))$$

ACCEPTANCE



DEFINITION (Acceptance)

Let \mathcal{A} be a WAPA, $w \in \Sigma^\omega$ and $R = \langle V, E \rangle$ a run of \mathcal{A} on w .

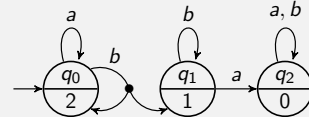
- An infinite path ρ in R satisfies the **acceptance condition** of \mathcal{A} iff the smallest occurring parity is even, i.e. $\min\{\pi(q) \mid \exists i \in \mathbb{N} : \langle q, i \rangle \text{ occurs in } \rho\}$ is even.

- R is an **accepting run** iff every infinite path ρ in R satisfies the acceptance condition.

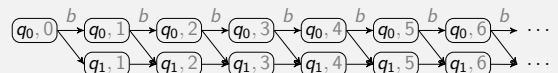
- \mathcal{A} **accepts** w iff there is some accepting run of \mathcal{A} on w .

INFINITELY MANY a 's

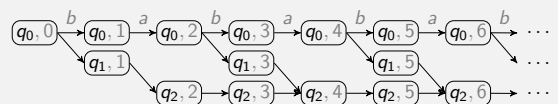
EXAMPLE ($(b^* a)^\omega$)



Run on b^ω :



Run on $(ba)^\omega$:



DUAL AUTOMATON (1)



DEFINITION (Dual Automaton)

The dual of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ is

$$\bar{\mathcal{A}} := \langle Q, \Sigma, \bar{\delta}, q_{in}, \bar{\pi} \rangle$$

where

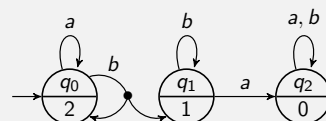
- $\bar{\delta}(q, a)$ is obtained from $\delta(q, a)$ by exchanging \wedge, \vee and \top, \perp
- $\bar{\pi}(q) := \pi(q) + 1$

for all $q \in Q$ and $a \in \Sigma$

DUAL AUTOMATON (2)

EXAMPLE ($(b^* a)^\omega$)

WAPA \mathcal{A} :



$$\delta(q_0, a) = q_0$$

$$\delta(q_0, b) = q_0 \wedge q_1$$

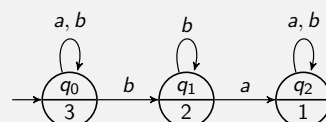
$$\delta(q_1, a) = q_2$$

$$\delta(q_1, b) = q_1$$

$$\delta(q_2, a) = q_2$$

$$\delta(q_2, b) = q_2$$

Dual $\bar{\mathcal{A}}$:



$$\bar{\delta}(q_0, a) = q_0$$

$$\bar{\delta}(q_0, b) = q_0 \vee q_1$$

$$\bar{\delta}(q_1, a) = q_2$$

$$\bar{\delta}(q_1, b) = q_1$$

$$\bar{\delta}(q_2, a) = q_2$$

$$\bar{\delta}(q_2, b) = q_2$$

COMPLEMENTATION THEOREM

Main statement of this talk:

THEOREM (Complementation)

The dual $\bar{\mathcal{A}}$ of a WAPA \mathcal{A} accepts its complement, i.e.

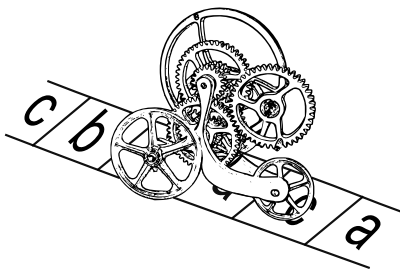
$$\mathcal{L}(\bar{\mathcal{A}}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$$

(Thomas and Löding, ~2000)

OUTLINE

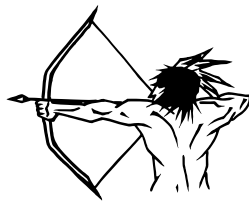
- 1 WEAK ALTERNATING PARITY AUTOMATA
- 2 INFINITE PARITY GAMES
- 3 PROOF OF THE COMPLEMENTATION THEOREM
- 4 BÜCHI COMPLEMENTATION ALGORITHM

AUTOMATON VS. PATHFINDER



player A

find accepting run R



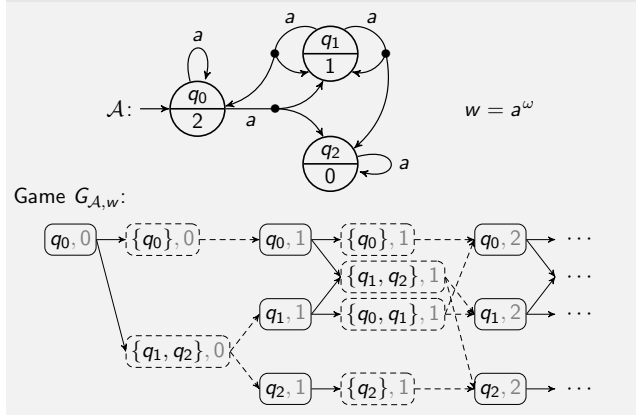
player P

find rejecting path in R

INFINITE PARITY GAME (1)



EXAMPLE (a^ω)



INFINITE PARITY GAME (2)



DEFINITION (Game)

A game for a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0 a_1 a_2 \dots \in \Sigma^\omega$ is a directed graph

$$G_{\mathcal{A},w} := \langle V_A \dot{\cup} V_P, E \rangle$$

where

- $V_A := Q \times \mathbb{N}$ (decision nodes of player A)
 - $V_P := 2^Q \times \mathbb{N}$ (decision nodes of player P)
 - $E \subseteq (V_A \times V_P) \cup (V_P \times V_A)$
s.t. the only contained edges are
 - $\langle \langle q, i \rangle, \langle M, i \rangle \rangle$ iff $M \in \text{Mod}_i(\delta(q, a_i))$
 - $\langle \langle M, i \rangle, \langle q, i+1 \rangle \rangle$ iff $q \in M$
- for $q \in Q, M \subseteq Q, i \in \mathbb{N}$

(Thomas and Löding, ~2000)

PLAYING A GAME



DEFINITION (Play)

A **play** γ in a game $G_{\mathcal{A},w}$ is an infinite path starting with $\langle q_{in}, 0 \rangle$.

DEFINITION (Winner)

The **winner** of a play γ is

- player A iff the smallest parity of occurring V_A -nodes is even
- player P

$X \in \{A, P\}$: a player, \bar{X} : its opponent

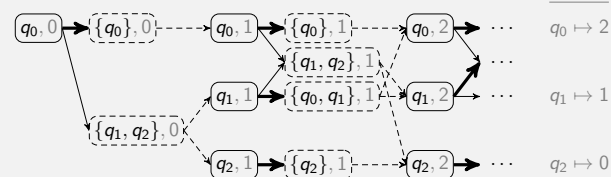
DEFINITION (Strategy)

- A **strategy** $f_X : V_X \rightarrow V_{\bar{X}}$ for player X selects for every decision node of player X one of its successor nodes in $G_{\mathcal{A},w}$.
- f_X is a **winning strategy** iff player X wins every play γ that is played according to f_X .

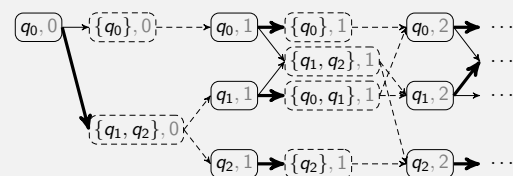
STRATEGIES

EXAMPLE

Winning strategy for player A (so far):



Not a winning strategy for player A:



OUTLINE

- 1 WEAK ALTERNATING PARITY AUTOMATA
- 2 INFINITE PARITY GAMES
- 3 PROOF OF THE COMPLEMENTATION THEOREM
 - Lemma 1
 - Lemma 2
 - Lemma 3
 - Sublemma
 - Putting it All Together
- 4 BÜCHI COMPLEMENTATION ALGORITHM

LEMMA 1

Let \mathcal{A} be a WAPA and $w \in \Sigma^\omega$.

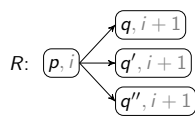
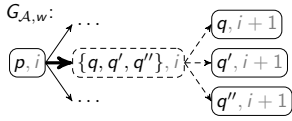
LEMMA 1

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** \mathcal{A} accepts w .

EXPLANATION (oral):

Player A wins every play γ played according to f_A .

There is a run graph R in which every path ρ is accepting.



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LEMMA 2

Let \mathcal{A} be a WAPA and $w \in \Sigma^\omega$.

LEMMA 2

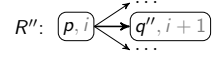
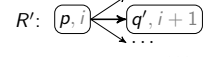
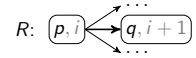
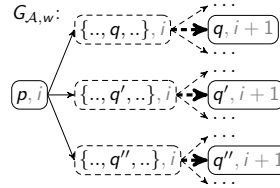
Player P has a winning strategy in $G_{\mathcal{A},w}$ **iff** \mathcal{A} does *not* accept w .

(pointed out by Jan Leike)

EXPLANATION (oral):

Player P wins every play γ played according to f_P .

Every run graph R contains a rejecting path ρ .



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SUBLEMMA



Let $\theta \in \mathbb{B}^+(Q)$ be a formula over Q .

SUBLEMMA

$S \subseteq Q$ is a model of $\bar{\theta}$ **iff** for all $M \in \text{Mod}_\downarrow(\theta)$: $S \cap M \neq \emptyset$.

PROOF:

- W.l.o.g. θ is in DNF, i.e.

$$\theta = \bigvee_{M \in \text{Mod}_\downarrow(\theta)} \bigwedge_{q \in M} q$$

- Then $\bar{\theta}$ is in CNF, i.e.

$$\bar{\theta} = \bigwedge_{M \in \text{Mod}_\downarrow(\theta)} \bigvee_{q \in M} q$$

- Thus $S \subseteq Q$ is a model of $\bar{\theta}$ **iff** it contains at least one element from each disjunct of θ .

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LEMMA 3 (1)

Let \mathcal{A} be a WAPA, $\bar{\mathcal{A}}$ its dual and $w = a_0 a_1 a_2 \dots \in \Sigma^\omega$.

LEMMA 3

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** player P has a winning strategy in $G_{\bar{\mathcal{A}},w}$.

PROOF:

\Rightarrow Construct a winning strategy \bar{f}_P for player P in $G_{\bar{\mathcal{A}},w}$.

...

\Leftarrow Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

...

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LEMMA 3 (2)

\Rightarrow Construct a winning strategy \bar{f}_P for player P in $G_{\bar{\mathcal{A}},w}$.

At position $\langle S, i \rangle \in V_P$

- f_A : winning strategy for player A in $G_{\mathcal{A},w}$

- Assume there is $\langle p, i \rangle \in V_A$ occurring in a play γ in $G_{\mathcal{A},w}$ played according to f_A s.t. $S \in \text{Mod}_\downarrow(\delta(p, a_i))$ (otherwise don't care).

- $f_A(\langle p, i \rangle) = \langle M, i \rangle \Rightarrow M \in \text{Mod}_\downarrow(\delta(p, a_i))$

- (sublemma) \Rightarrow There exists a $q \in S \cap M$.

- Define $\bar{f}_P(\langle S, i \rangle) := \langle q, i+1 \rangle$

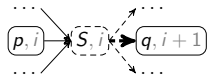
- $\forall \bar{\gamma}$: play in $G_{\bar{\mathcal{A}},w}$ played according to \bar{f}_P
- $\exists \gamma$: play in $G_{\mathcal{A},w}$ played according to f_A s.t. $\bar{\gamma}$ and γ contain the same V_A -nodes.

- Player A wins γ in $G_{\mathcal{A},w}$.

- $\forall q \in Q : \bar{\pi}(q) = \pi(q) + 1$

\Rightarrow Player P wins $\bar{\gamma}$ in $G_{\bar{\mathcal{A}},w}$.

in $G_{\bar{\mathcal{A}},w}$:



in $G_{\mathcal{A},w}$:



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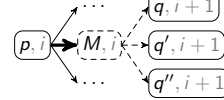
LEMMA 3 (3)

\Leftarrow Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i \rangle \in V_A$

- \bar{f}_P : winning strategy for player P in $G_{\bar{\mathcal{A}},w}$

in $G_{\mathcal{A},w}$:



- $M^* := \{q \in Q \mid \exists S \in \text{Mod}_\downarrow(\delta(p, a_i)) : \bar{f}_P(\langle S, i \rangle) = \langle q, i+1 \rangle\}$
- (sublemma) M^* is a model of $\delta(p, a_i)$.

- M : subset of M^* that is a minimal model $M \subseteq M^*, M \in \text{Mod}_\downarrow(\delta(p, a_i))$

- Define $f_A(\langle p, i \rangle) := \langle M, i \rangle$

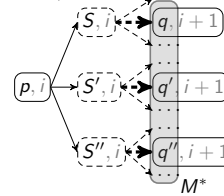
- $\forall \gamma$: play in $G_{\mathcal{A},w}$ played according to f_A
- $\exists \bar{\gamma}$: play in $G_{\bar{\mathcal{A}},w}$ played according to \bar{f}_P s.t. γ and $\bar{\gamma}$ contain the same V_A -nodes.

- Player P wins $\bar{\gamma}$ in $G_{\bar{\mathcal{A}},w}$.

- $\forall q \in Q : \bar{\pi}(q) = \pi(q) - 1$

\Rightarrow Player A wins γ in $G_{\mathcal{A},w}$.

in $G_{\bar{\mathcal{A}},w}$:



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ALL THREE LEMMAS



Let \mathcal{A} be a WAPA, $\bar{\mathcal{A}}$ its dual and $w \in \Sigma^\omega$.

LEMMA 1

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** \mathcal{A} accepts w .

LEMMA 2

Player P has a winning strategy in $G_{\mathcal{A},w}$ **iff** \mathcal{A} does *not* accept w .

LEMMA 3

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** player P has a winning strategy in $G_{\bar{\mathcal{A}},w}$.

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COMPLEMENTATION THEOREM



THEOREM (Complementation)

The dual $\bar{\mathcal{A}}$ of a WAPA \mathcal{A} accepts its complement, i.e.

$$\mathcal{L}(\bar{\mathcal{A}}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$$

(Thomas and Löding, ~2000)

PROOF:

\mathcal{A} accepts $w \stackrel{(\text{lemma 1})}{\iff}$ player A has a winning strategy in $G_{\mathcal{A},w}$

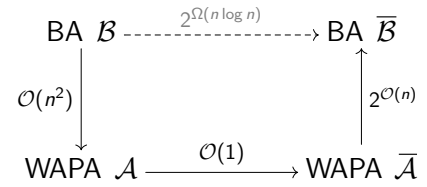
$\stackrel{(\text{lemma 3})}{\iff}$ player P has a winning strategy in $G_{\bar{\mathcal{A}},w}$

$\stackrel{(\text{lemma 2})}{\iff} \bar{\mathcal{A}}$ does *not* accept w

□

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- 1 WEAK ALTERNATING PARITY AUTOMATA
- 2 INFINITE PARITY GAMES
- 3 PROOF OF THE COMPLEMENTATION THEOREM
- 4 BÜCHI COMPLEMENTATION ALGORITHM



- Total complexity: $2^{\mathcal{O}(n^2)}$
- Can reach $2^{\mathcal{O}(n \log n)}$ (lower bound) by improving $\overline{\mathcal{A}} \rightarrow \overline{\mathcal{B}}$.

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APPENDIX

FROM BA TO WAPA



- GIVEN:
- $\mathcal{B} = \langle Q, \Sigma, \delta, q_{in}, F \rangle$: BA
 - $n = |Q|$

CONSTRUCTION (BA \rightarrow WAPA)

$$\mathcal{A} := \langle \underbrace{Q \times \{0, \dots, 2n\}}_{\mathcal{O}(n^2)}, \Sigma, \delta', \langle q_{in}, 2n \rangle, \pi \rangle$$

where

$$\delta'(\langle p, i \rangle, a) := \begin{cases} \bigvee_{q \in \delta(p, a)} \langle q, 0 \rangle & \text{if } i = 0 \\ \bigvee_{q \in \delta(p, a)} \langle q, i \rangle \wedge \langle q, i - 1 \rangle & \text{if } i \text{ even, } i > 0 \\ \bigvee_{q \in \delta(p, a)} \langle q, i \rangle & \text{if } i \text{ odd, } p \notin F \\ \bigvee_{q \in \delta(p, a)} \langle q, i - 1 \rangle & \text{if } i \text{ odd, } p \in F \end{cases}$$

$$\pi(\langle p, i \rangle) := i$$

for $p \in Q, a \in \Sigma, i \in \{0, \dots, 2n\}$

(Thomas and Löding, ~2000)

FROM WAPA TO BA



- GIVEN:
- $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$: stratified WAPA, i.e.
 $\forall p \in Q \forall a \in \Sigma : \delta(p, a) \in \mathbb{B}^+(\{q \in Q \mid \pi(p) \geq \pi(q)\})$
 - $E \subseteq Q$: all states with even parity

CONSTRUCTION (WAPA \rightarrow BA)

$$\mathcal{B} := \langle \underbrace{2^Q \times 2^Q}_{2^{\mathcal{O}(n)}}, \Sigma, \delta', \langle \{q_{in}\}, \emptyset \rangle, 2^Q \times \{\emptyset\} \rangle$$

where

$$\delta'(\langle M, \emptyset \rangle, a) := \left\{ \langle M', M' \setminus E \rangle \mid M' \in \text{Mod}_\downarrow(\bigwedge_{q \in M} \delta(q, a)) \right\}$$

$$\delta'(\langle M, O \rangle, a) := \left\{ \langle M', O' \setminus E \rangle \mid \begin{array}{l} M' \in \text{Mod}_\downarrow(\bigwedge_{q \in M} \delta(q, a)), \\ O' \subseteq M', \\ O' \in \text{Mod}_\downarrow(\bigwedge_{q \in O} \delta(q, a)) \end{array} \right\}$$

for $a \in \Sigma, M, O \subseteq Q, O \neq \emptyset$

(Miyano and Hayashi, 1984)