Automata Theory Seminar

Büchi Complementation via Alternating Automata

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Weak Alternating Parity Automaton

Definition (Weak Alternating Parity Automaton)

A weak alternating parity automaton (WAPA) is a tuple

$$\mathcal{A} := \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$$

where

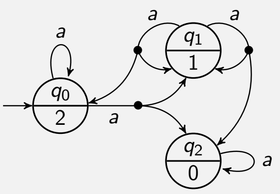
- Q finite set of states
- Σ finite alphabet
- $\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$ transition function
- q_{in} initial state
- $\pi : Q \to \mathbb{N}$ parity function

(Thomas and Löding, ∼2000)

$\mathbb{B}^+(Q)$: set of all positive Boolean formulae over Q (built only from elements in $Q \cup \{\wedge, \vee, \top, \bot\}$)

Transitions

Example (a^ω)



$\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$

$\langle q_0, a\rangle \mapsto q_0 \vee (q_1 \wedge q_2)$

$\langle q_1, a\rangle \mapsto (q_0 \wedge q_1) \vee (q_1 \wedge q_2)$

$\langle q_2, a\rangle \mapsto q_2$

Definition (Minimal Models)

$\mathrm{Mod}_\downarrow(\theta) \subseteq 2^Q$: set of minimal models of $\theta \in \mathbb{B}^+(Q)$, i.e. the set of minimal subsets $M \subseteq Q$ s.t. θ is satisfied by

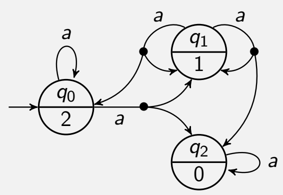
$$q \mapsto \begin{cases} true & \text{if } q \in M \\ false & \text{otherwise} \end{cases}$$

Example

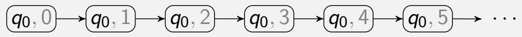
$\mathrm{Mod}_\downarrow(q_0 \vee (q_1 \wedge q_2)) = \{\{q_0\}, \{q_1, q_2\}\}$

Run Graph (1)

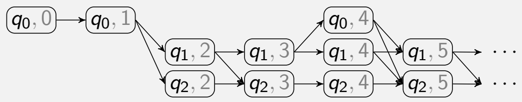
Example (a^ω)



Accepting run:



Rejecting run:



Run Graph (2)

Definition (Run)

A run of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ on a word $a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed acyclic graph

$$R := \langle V, E \rangle$$

where

- $V \subseteq Q \times \mathbb{N}$ with $\langle q_{in}, 0\rangle \in V$
- V contains only vertices reachable from $\langle q_{in}, 0\rangle$.
- E contains only edges of the form $\big\langle \langle p, i\rangle, \langle q, i+1\rangle \big\rangle$.
- For every vertex $\langle p, i\rangle \in V$ the set of successors is a minimal model of $\delta(p, a_i)$
  $$\big\{ q \in Q \mid \big\langle \langle p, i\rangle, \langle q, i+1\rangle \big\rangle \in E \big\} \in \mathrm{Mod}_\downarrow(\delta(p, a_i))$$

Acceptance

Definition (Acceptance)

Let $\mathcal{A}$ be a WAPA, $w \in \Sigma^\omega$ and $R = \langle V, E\rangle$ a run of $\mathcal{A}$ on w.

- An infinite path ρ in R satisfies the **acceptance condition** of $\mathcal{A}$ iff the smallest occurring parity is even, i.e.
  $$\min\{\pi(q) \mid \exists i \in \mathbb{N} : \langle q, i\rangle \text{ occurs in } \rho\} \text{ is even.}$$
- R is an **accepting run** iff every infinite path ρ in R satisfies the acceptance condition.
- $\mathcal{A}$ **accepts** w iff there is some accepting run of $\mathcal{A}$ on w.

Dual Automaton (1)

Definition (Dual Automaton)

The dual of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ is

$$\overline{\mathcal{A}} := \langle Q, \Sigma, \overline{\delta}, q_{in}, \overline{\pi} \rangle$$

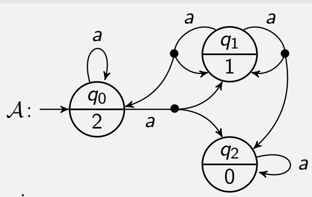
where

- $\overline{\delta}(q, a)$ is obtained from $\delta(q, a)$ by exchanging $\wedge, \vee$ and $\top, \bot$
- $\overline{\pi}(q) := \pi(q) + 1$

for all $q \in Q$ and $a \in \Sigma$

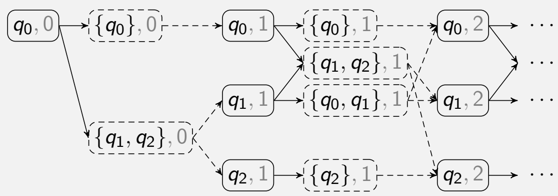
Infinite Parity Game (1)

Example (a^ω)



$w = a^\omega$

Game $G_{\mathcal{A},w}$:



Infinite Parity Game (2)

Definition (Game)

A game for a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed graph

$$G_{\mathcal{A},w} := \langle V_A \dot\cup V_P, E \rangle$$

where

- $V_A := Q \times \mathbb{N}$ (decision nodes of player A)
- $V_P := 2^Q \times \mathbb{N}$ (decision nodes of player P)
- $E \subseteq (V_A \times V_P) \cup (V_P \times V_A)$ s.t. the only contained edges are
  - $\langle \langle q, i \rangle, \langle M, i \rangle \rangle$ iff $M \in \mathrm{Mod}_\downarrow(\delta(q, a_i))$
  - $\langle \langle M, i \rangle, \langle q, i+1 \rangle \rangle$ iff $q \in M$

  for $q \in Q$, $M \subseteq Q$, $i \in \mathbb{N}$

(Thomas and Löding, ∼2000)

Playing a Game

Definition (Play)

A **play** γ in a game $G_{\mathcal{A},w}$ is an infinite path starting with $\langle q_{in}, 0 \rangle$.

Definition (Winner)

The **winner** of a play γ is

- player A iff the smallest parity of occurring V_A-nodes is even
- player P · odd

$X \in \{A, P\}$: a player, $\overline{X}$: its opponent

Definition (Strategy)

- A **strategy** $f_X : V_X \to V_{\overline{X}}$ for player X selects for every decision node of player X one of its successor nodes in $G_{\mathcal{A},w}$.
- f_X is a **winning strategy** iff player X wins every play γ that is played according to f_X.

Sublemma

Let $\theta \in \mathbb{B}^+(Q)$ be a formula over Q.

Sublemma

$S \subseteq Q$ is a model of $\overline{\theta}$ **iff** for all $M \in \mathrm{Mod}_\downarrow(\theta)$: $S \cap M \neq \emptyset$.

Proof:

- W.l.o.g. θ is in DNF, i.e.
$$\theta = \bigvee_{M \in \mathrm{Mod}_\downarrow(\theta)} \bigwedge_{q \in M} q$$
- Then $\overline{\theta}$ is in CNF, i.e.
$$\overline{\theta} = \bigwedge_{M \in \mathrm{Mod}_\downarrow(\theta)} \bigvee_{q \in M} q$$
- Thus $S \subseteq Q$ is a model of $\overline{\theta}$ **iff** it contains at least one element from each disjunct of θ.

All Three Lemmas

Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w \in \Sigma^\omega$.

Lemma 1

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ accepts w.

Lemma 2

Player P has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ does *not* accept w.

Lemma 3

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$.

Complementation Theorem

Theorem (Complementation)

The dual $\overline{\mathcal{A}}$ of a WAPA $\mathcal{A}$ accepts its complement, i.e.

$$\mathcal{L}(\overline{\mathcal{A}}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$$

(Thomas and Löding, ∼2000)

Proof:

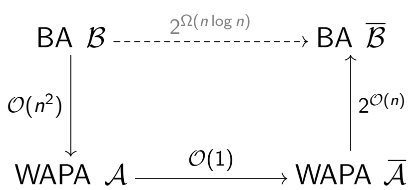
$\mathcal{A}$ accepts w $\overset{\text{(lemma 1)}}{\iff}$ player A has a winning strategy in $G_{\mathcal{A},w}$

$\overset{\text{(lemma 3)}}{\iff}$ player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$

$\overset{\text{(lemma 2)}}{\iff}$ $\overline{\mathcal{A}}$ does *not* accept w

□

Büchi Complementation Algorithm



- Total complexity: $2^{\mathcal{O}(n^2)}$
- Can reach $2^{\mathcal{O}(n \log n)}$ (lower bound) by improving $\overline{\mathcal{A}} \to \overline{\mathcal{B}}$.

From BA to WAPA

Given:

- $\mathcal{B} = \langle Q, \Sigma, \delta, q_{in}, F \rangle$: BA
- $n = |Q|$

Construction (BA → WAPA)

$$\mathcal{A} := \langle \underbrace{Q \times \{0, \ldots, 2n\}}_{\mathcal{O}(n^2)}, \Sigma, \delta', \langle q_{in}, 2n \rangle, \pi \rangle$$

where

- $\delta'(\langle p, i \rangle, a) := \begin{cases} \bigvee_{q \in \delta(p,a)} \langle q, 0 \rangle & \text{if } i = 0 \\ \bigvee_{q \in \delta(p,a)} \langle q, i \rangle \wedge \langle q, i-1 \rangle & \text{if } i \text{ even, } i > 0 \\ \bigvee_{q \in \delta(p,a)} \langle q, i \rangle & \text{if } i \text{ odd, } p \notin F \\ \bigvee_{q \in \delta(p,a)} \langle q, i-1 \rangle & \text{if } i \text{ odd, } p \in F \end{cases}$
- $\pi(\langle p, i \rangle) := i$

for $p \in Q$, $a \in \Sigma$, $i \in \{0, \ldots, 2n\}$

(Thomas and Löding, ∼2000)

From WAPA to BA

Given:

- $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$: **stratified** WAPA, i.e.
$$\forall p \in Q \ \forall a \in \Sigma : \ \delta(p, a) \in \mathbb{B}^+(\{q \in Q \mid \pi(p) \geq \pi(q)\})$$
- $E \subseteq Q$: all states with even parity

Construction (WAPA → BA)

$$\mathcal{B} := \langle \underbrace{2^Q \times 2^Q}_{2^{\mathcal{O}(n)}}, \Sigma, \delta', \langle \{q_{in}\}, \emptyset \rangle, 2^Q \times \{\emptyset\} \rangle$$

where

- $\delta'(\langle M, \emptyset \rangle, a) := \left\{ \langle M', M' \setminus E \rangle \;\middle|\; M' \in \mathrm{Mod}_\downarrow\left(\bigwedge_{q \in M} \delta(q, a) \right) \right\}$
- $\delta'(\langle M, O \rangle, a) := \left\{ \langle M', O' \setminus E \rangle \;\middle|\; M' \in \mathrm{Mod}_\downarrow\left(\bigwedge_{q \in M} \delta(q, a) \right), O' \subseteq M', O' \in \mathrm{Mod}_\downarrow\left(\bigwedge_{q \in O} \delta(q, a) \right) \right\}$

for $a \in \Sigma$, $M, O \subseteq Q$, $O \neq \emptyset$

(Miyano and Hayashi, 1984)