Automata Theory Seminar

Büchi Complementation via Alternating Automata

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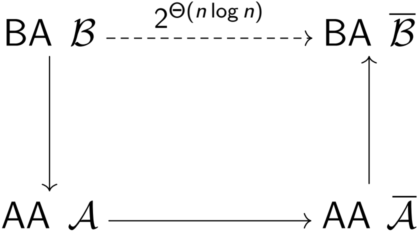
Büchi Complementation



BA: Büchi Automaton

AA: Alternating Automaton

- Expensive: If $\mathcal{B}$ has n states, $\overline{\mathcal{B}}$ has $2^{\Theta(n \log n)}$ states in the worst case (Michel 1988, Safra 1988).
- Complicated: Direct approaches are rather involved.

Consider indirect approach: detour over **alternating automata**.

Transition Modes (1)

Existential: some run is accepting

$$q_0 \longrightarrow q_{1_a} \longrightarrow q_{2_a} \longrightarrow q_{3_a} \longrightarrow q_{4_a} \longrightarrow q_{5_a} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_b} \longrightarrow q_{2_b} \longrightarrow q_{3_b} \longrightarrow q_{4_b} \longrightarrow q_{5_b} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_c} \longrightarrow q_{2_c} \longrightarrow q_{3_c} \longrightarrow q_{4_c} \longrightarrow q_{5_c} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_d} \longrightarrow q_{2_d} \longrightarrow q_{3_d} \longrightarrow q_{4_d} \longrightarrow q_{5_d} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_e} \longrightarrow q_{2_e} \longrightarrow q_{3_e} \longrightarrow q_{4_e} \longrightarrow q_{5_e} \longrightarrow \cdots$$

Universal: every run is accepting

$$q_0 \longrightarrow q_{1_a} \longrightarrow q_{2_a} \longrightarrow q_{3_a} \longrightarrow q_{4_a} \longrightarrow q_{5_a} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_b} \longrightarrow q_{2_b} \longrightarrow q_{3_b} \longrightarrow q_{4_b} \longrightarrow q_{5_b} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_c} \longrightarrow q_{2_c} \longrightarrow q_{3_c} \longrightarrow q_{4_c} \longrightarrow q_{5_c} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_d} \longrightarrow q_{2_d} \longrightarrow q_{3_d} \longrightarrow q_{4_d} \longrightarrow q_{5_d} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_e} \longrightarrow q_{2_e} \longrightarrow q_{3_e} \longrightarrow q_{4_e} \longrightarrow q_{5_e} \longrightarrow \cdots$$

Transition Modes (2)

Alternating: in <u>some</u> set of runs <u>every</u> run is accepting

$$q_0 \longrightarrow q_{1_a} \longrightarrow q_{2_a} \longrightarrow q_{3_a} \longrightarrow q_{4_a} \longrightarrow q_{5_a} \longrightarrow \cdots$$
$$q_0 \longrightarrow q_{1_b} \longrightarrow q_{2_b} \longrightarrow q_{3_b} \longrightarrow q_{4_b} \longrightarrow q_{5_b} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_c} \longrightarrow q_{2_c} \longrightarrow q_{3_c} \longrightarrow q_{4_c} \longrightarrow q_{5_c} \longrightarrow \cdots$$
$$q_0 \longrightarrow q_{1_d} \longrightarrow q_{2_d} \longrightarrow q_{3_d} \longrightarrow q_{4_d} \longrightarrow q_{5_d} \longrightarrow \cdots$$
$$q_0 \longrightarrow q_{1_e} \longrightarrow q_{2_e} \longrightarrow q_{3_e} \longrightarrow q_{4_e} \longrightarrow q_{5_e} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_f} \longrightarrow q_{2_f} \longrightarrow q_{3_f} \longrightarrow q_{4_f} \longrightarrow q_{5_f} \longrightarrow \cdots$$

$$q_0 \longrightarrow q_{1_g} \longrightarrow q_{2_g} \longrightarrow q_{3_g} \longrightarrow q_{4_g} \longrightarrow q_{5_g} \longrightarrow \cdots$$
$$q_0 \longrightarrow q_{1_h} \longrightarrow q_{2_h} \longrightarrow q_{3_h} \longrightarrow q_{4_h} \longrightarrow q_{5_h} \longrightarrow \cdots$$
$$q_0 \longrightarrow q_{1_i} \longrightarrow q_{2_i} \longrightarrow q_{3_i} \longrightarrow q_{4_i} \longrightarrow q_{5_i} \longrightarrow \cdots$$

SPECIAL CASE: $\mathcal{A}$ in existential mode

- $\mathcal{A}$ accepts iff $\exists$ run $\rho : \rho$ fulfills acceptance condition of $\mathcal{A}$
- $\overline{\mathcal{A}}$ accepts iff $\forall$ run $\rho : \neg(\rho$ fulfills acceptance condition of $\mathcal{A})$
  iff $\forall$ run ρ: ρ fulfills **dual** acceptance condition of $\mathcal{A}$

$\Rightarrow$ complementation $\widehat{=}$ dualization of:

- transition mode
- acceptance condition

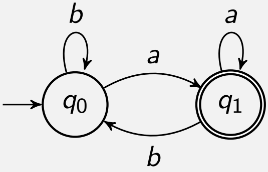
Want acceptance condition that is **closed under dualization**.

Outline

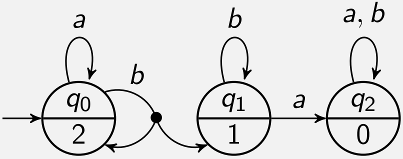
Outline

Example ($(b^* a)^\omega$)

Büchi automaton $\mathcal{B}$:



Equivalent WAPA $\mathcal{A}$:



Weak Alternating Parity Automaton ☺

Definition (Weak Alternating Parity Automaton)

A weak alternating parity automaton (WAPA) is a tuple

$$\mathcal{A} := \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$$

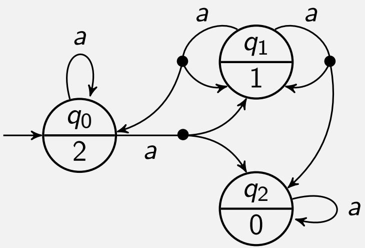
where

- Q finite set of states
- Σ finite alphabet
- $\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$ transition function
- q_{in} initial state
- $\pi : Q \to \mathbb{N}$ parity function

(Thomas and Löding, $\sim$2000)

$\mathbb{B}^+(Q)$: set of all positive Boolean formulae over Q
(built only from elements in $Q \cup \{\wedge, \vee, \top, \bot\}$)

Example (a^ω)



$\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$

$\langle q_0, a \rangle \mapsto q_0 \vee (q_1 \wedge q_2)$

$\langle q_1, a \rangle \mapsto (q_0 \wedge q_1) \vee (q_1 \wedge q_2)$

$\langle q_2, a \rangle \mapsto q_2$

Definition (Minimal Models)

$\mathrm{Mod}_{\downarrow}(\theta) \subseteq 2^Q$: set of minimal models of $\theta \in \mathbb{B}^+(Q)$, i.e. the set of minimal subsets $M \subseteq Q$ s.t. θ is satisfied by

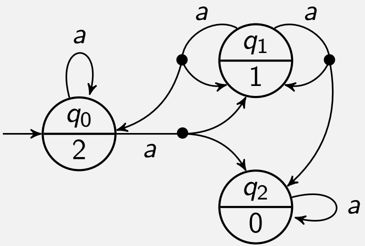
$$q \mapsto \begin{cases} \mathit{true} & \text{if } q \in M \\ \mathit{false} & \text{otherwise} \end{cases}$$

Example

$$\mathrm{Mod}_{\downarrow}(q_0 \vee (q_1 \wedge q_2)) = \{\{q_0\}, \{q_1, q_2\}\}$$

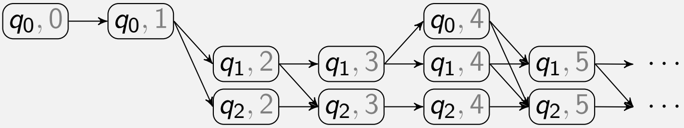
Run Graph (1)

Example (a^ω)



Accepting run:

Rejecting run:



Definition (Run)

A run of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ on a word $a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed acyclic graph

$$R := \langle V, E \rangle$$

where

- $V \subseteq Q \times \mathbb{N}$ with $\langle q_{in}, 0 \rangle \in V$
- V contains only vertices reachable from $\langle q_{in}, 0 \rangle$.
- E contains only edges of the form $\big\langle \langle p, i \rangle, \langle q, i+1 \rangle \big\rangle$.
- For every vertex $\langle p, i \rangle \in V$ the set of successors is a minimal model of $\delta(p, a_i)$

$$\big\{ q \in Q \mid \big\langle \langle p, i \rangle, \langle q, i+1 \rangle \big\rangle \in E \big\} \in \mathrm{Mod}_{\downarrow}(\delta(p, a_i))$$

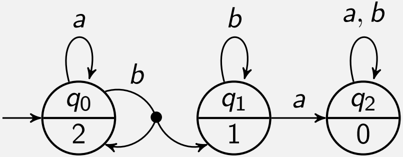
DEFINITION (Acceptance)

Let $\mathcal{A}$ be a WAPA, $w \in \Sigma^\omega$ and $R = \langle V, E \rangle$ a run of $\mathcal{A}$ on w.

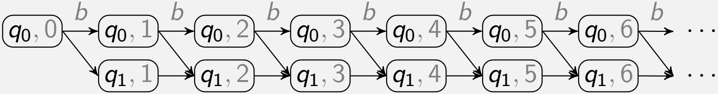
- An infinite path ρ in R satisfies the **acceptance condition** of $\mathcal{A}$ iff the smallest occurring parity is even, i.e.
$$\min\{\pi(q) \mid \exists\, i \in \mathbb{N} : \langle q, i \rangle \text{ occurs in } \rho\} \text{ is even.}$$
- R is an **accepting run** iff every infinite path ρ in R satisfies the acceptance condition.
- $\mathcal{A}$ **accepts** w iff there is some accepting run of $\mathcal{A}$ on w.

INFINITELY MANY a'S

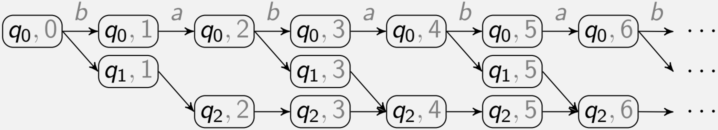
EXAMPLE ($(b^* a)^\omega$)



Run on b^ω:



Run on $(ba)^\omega$:



Dual Automaton (1)

☺

Definition (Dual Automaton)

The dual of a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ is

$$\overline{\mathcal{A}} := \langle Q, \Sigma, \overline{\delta}, q_{in}, \overline{\pi} \rangle$$

where

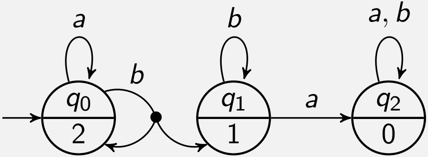
- $\overline{\delta}(q, a)$ is obtained from $\delta(q, a)$ by exchanging $\wedge, \vee$ and $\top, \bot$
- $\overline{\pi}(q) := \pi(q) + 1$

for all $q \in Q$ and $a \in \Sigma$

Dual Automaton (2)

Example ($(b^* a)^\omega$)

WAPA $\mathcal{A}$:

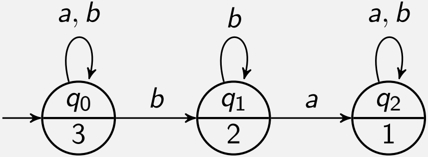


$\delta(q_0, a) = q_0$
$\delta(q_0, b) = q_0 \wedge q_1$

$\delta(q_1, a) = q_2$
$\delta(q_1, b) = q_1$

$\delta(q_2, a) = q_2$
$\delta(q_2, b) = q_2$

Dual $\overline{\mathcal{A}}$:



$\overline{\delta}(q_0, a) = q_0$
$\overline{\delta}(q_0, b) = q_0 \vee q_1$

$\overline{\delta}(q_1, a) = q_2$
$\overline{\delta}(q_1, b) = q_1$

$\overline{\delta}(q_2, a) = q_2$
$\overline{\delta}(q_2, b) = q_2$

Complementation Theorem

Main statement of this talk:

Theorem (Complementation)

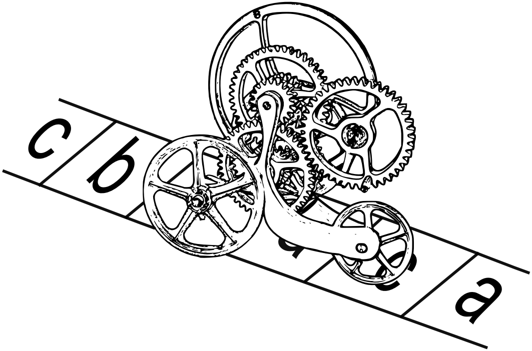
The dual $\overline{\mathcal{A}}$ of a WAPA $\mathcal{A}$ accepts its complement, i.e.

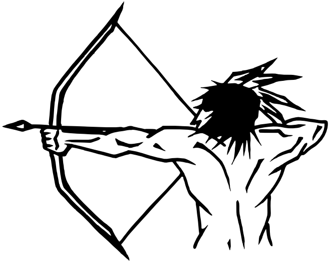
$$\mathcal{L}(\overline{\mathcal{A}}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$$

(Thomas and Löding, $\sim$2000)

Outline

Automaton vs. Pathfinder





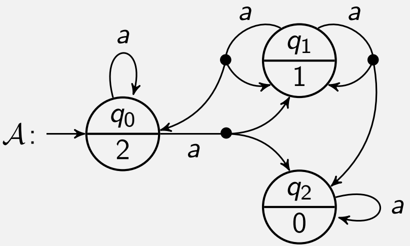
player A

find accepting run R

player P

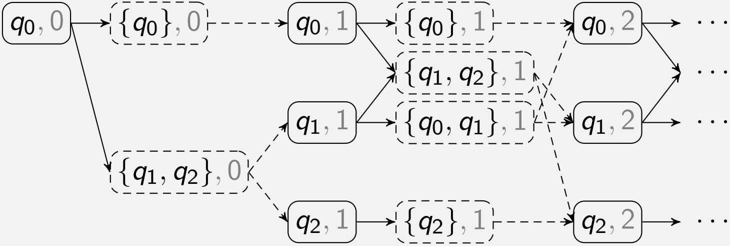
find rejecting path in R

Example (a^ω)



$w = a^\omega$

Game $G_{\mathcal{A},w}$:



Infinite Parity Game (2) ☺

Definition (Game)

A game for a WAPA $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$ and $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$ is a directed graph

$$G_{\mathcal{A},w} := \langle V_A \,\dot{\cup}\, V_P, E \rangle$$

where

- $V_A := Q \times \mathbb{N}$ (decision nodes of player A)
- $V_P := 2^Q \times \mathbb{N}$ (decision nodes of player P)
- $E \subseteq (V_A \times V_P) \cup (V_P \times V_A)$
  s.t. the only contained edges are
  - $\big\langle \langle q, i \rangle, \langle M, i \rangle \big\rangle$ iff $M \in \mathrm{Mod}_{\downarrow}(\delta(q, a_i))$
  - $\big\langle \langle M, i \rangle, \langle q, i+1 \rangle \big\rangle$ iff $q \in M$

  for $q \in Q$, $M \subseteq Q$, $i \in \mathbb{N}$

(Thomas and Löding, $\sim$2000)

DEFINITION (Play)

A **play** γ in a game $G_{\mathcal{A},w}$ is an infinite path starting with $\langle q_{in}, 0 \rangle$.

DEFINITION (Winner)

The **winner** of a play γ is

- player A iff the smallest parity of occurring V_A-nodes is even
- player P · odd

$X \in \{A, P\}$: a player, $\overline{X}$: its opponent

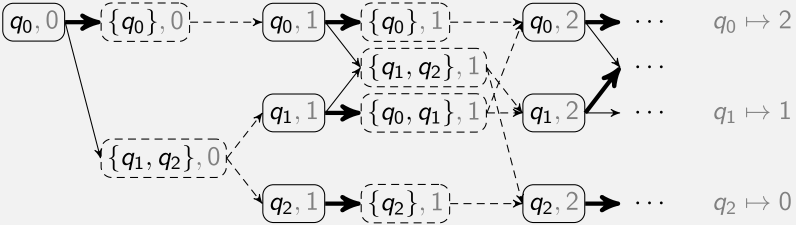
DEFINITION (Strategy)

- A **strategy** $f_X : V_X \to V_{\overline{X}}$ for player X selects for every decision node of player X one of its successor nodes in $G_{\mathcal{A},w}$.
- f_X is a **winning strategy** iff player X wins every play γ that is played according to f_X.

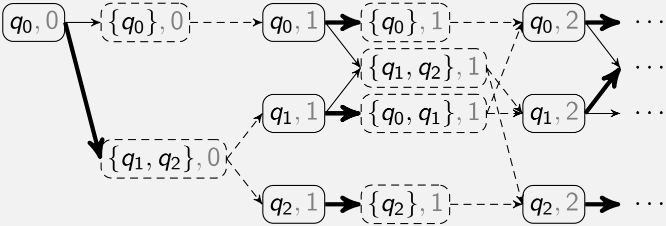
EXAMPLE

Winning strategy for player A (so far):

parities



Not a winning strategy for player A:



Lemma 1

Let $\mathcal{A}$ be a WAPA and $w \in \Sigma^\omega$.

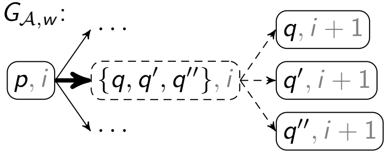
Lemma 1

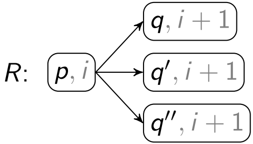
Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ accepts w.

Explanation (oral):

Player A wins every play γ played according to f_A.

There is a run graph R in which every path ρ is accepting.





Lemma 2

Let $\mathcal{A}$ be a WAPA and $w \in \Sigma^\omega$.

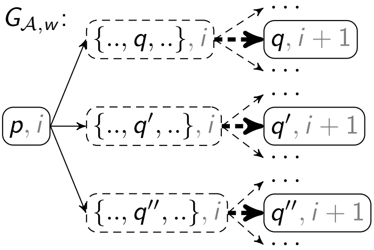
Lemma 2

Player P has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ does *not* accept w.

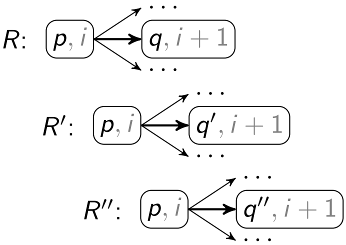
(pointed out by Jan Leike)

Explanation (oral):

Player P wins every play γ played according to f_P.



Every run graph R contains a rejecting path ρ.



Let $\theta \in \mathbb{B}^+(Q)$ be a formula over Q.

SUBLEMMA

$S \subseteq Q$ is a model of $\overline{\theta}$ **iff** for all $M \in \mathrm{Mod}_{\downarrow}(\theta)$: $S \cap M \neq \emptyset$.

PROOF:

- W.l.o.g. θ is in DNF, i.e.

$$\theta = \bigvee_{M \in \mathrm{Mod}_{\downarrow}(\theta)} \bigwedge_{q \in M} q$$

- Then $\overline{\theta}$ is in CNF, i.e.

$$\overline{\theta} = \bigwedge_{M \in \mathrm{Mod}_{\downarrow}(\theta)} \bigvee_{q \in M} q$$

- Thus $S \subseteq Q$ is a model of $\overline{\theta}$ **iff** it contains at least one element from each disjunct of θ.

Lemma 3 (1)

Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w = a_0 a_1 a_2 \ldots \in \Sigma^\omega$.

Lemma 3

Player A has a winning strategy in $G_{\mathcal{A},w}$
iff player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$.

Proof:

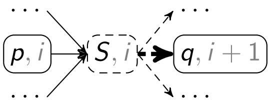
- $\Rightarrow$ Construct a winning strategy $\overline{f_P}$ for player P in $G_{\overline{\mathcal{A}},w}$.
  ...
- $\Leftarrow$ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.
  ...

Lemma 3 (2)

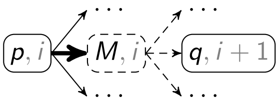
⇒ Construct a winning strategy $\overline{f_P}$ for player P in $G_{\overline{\mathcal{A}},w}$.

At position $\langle S, i\rangle \in V_P$

in $G_{\overline{\mathcal{A}},w}$:



in $G_{\mathcal{A},w}$:



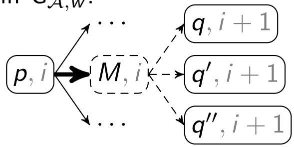
- f_A: winning strategy for player A in $G_{\mathcal{A},w}$
- Assume there is $\langle p, i\rangle \in V_A$ occurring in a play γ in $G_{\mathcal{A},w}$ played according to f_A s.t. $S \in \mathrm{Mod}_{\downarrow}(\overline{\delta}(p, a_i))$ (otherwise don't care).
- $f_A\big(\langle p, i\rangle\big) = \langle M, i\rangle \;\Rightarrow\; M \in \mathrm{Mod}_{\downarrow}(\delta(p, a_i))$
- $\overset{\text{(sublemma)}}{\Longrightarrow}$ There exists a $q \in S \cap M$.
- Define $\overline{f_P}\big(\langle S, i\rangle\big) := \langle q, i+1\rangle$
- $\forall\, \overline{\gamma}$: play in $G_{\overline{\mathcal{A}},w}$ played according to $\overline{f_P}$
  $\exists\, \gamma$: play in $G_{\mathcal{A},w}$ played according to f_A
  s.t. $\overline{\gamma}$ and γ contain the same V_A-nodes.
  - Player A wins γ in $G_{\mathcal{A},w}$.
  - $\forall\, q \in Q : \overline{\pi}(q) = \pi(q) + 1$

  $\Rightarrow$ Player P wins $\overline{\gamma}$ in $G_{\overline{\mathcal{A}},w}$.

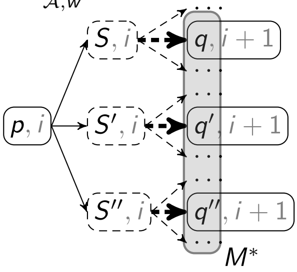
⇐ Construct a winning strategy f_A for player A in $G_{\mathcal{A},w}$.

At position $\langle p, i\rangle \in V_A$

in $G_{\mathcal{A},w}$:



in $G_{\overline{\mathcal{A}},w}$:



- $\overline{f_P}$: winning strategy for player P in $G_{\overline{\mathcal{A}},w}$
- $M^* := \{ q \in Q \mid \exists\, S \in \mathrm{Mod}_{\downarrow}(\overline{\delta}(p, a_i)) : \overline{f_P}(\langle S, i\rangle) = \langle q, i+1\rangle \}$
  $\overset{\text{(sublemma)}}{\Longrightarrow}$ M^* is a model of $\delta(p, a_i)$.
- M: subset of M^* that is a minimal model
  $M \subseteq M^*, \quad M \in \mathrm{Mod}_{\downarrow}(\delta(p, a_i))$
- Define $f_A(\langle p, i\rangle) := \langle M, i\rangle$
- $\forall\, \gamma$: play in $G_{\mathcal{A},w}$ played according to f_A
  $\exists\, \overline{\gamma}$: play in $G_{\overline{\mathcal{A}},w}$ played according to $\overline{f_P}$
  s.t. γ and $\overline{\gamma}$ contain the same V_A-nodes.
  - Player P wins $\overline{\gamma}$ in $G_{\overline{\mathcal{A}},w}$.
  - $\forall\, q \in Q : \pi(q) = \overline{\pi}(q) - 1$

  $\Rightarrow$ Player A wins γ in $G_{\mathcal{A},w}$.

All Three Lemmas

☺

Let $\mathcal{A}$ be a WAPA, $\overline{\mathcal{A}}$ its dual and $w \in \Sigma^\omega$.

Lemma 1

Player A has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ accepts w.

Lemma 2

Player P has a winning strategy in $G_{\mathcal{A},w}$ **iff** $\mathcal{A}$ does *not* accept w.

Lemma 3

Player A has a winning strategy in $G_{\mathcal{A},w}$
iff player P has a winning strategy in $G_{\overline{\mathcal{A}},w}$.

THEOREM (Complementation)

The dual $\overline{\mathcal{A}}$ of a WAPA $\mathcal{A}$ accepts its complement, i.e.

$$\mathcal{L}(\overline{\mathcal{A}}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$$

(Thomas and Löding, $\sim$2000)

PROOF:

$$\mathcal{A} \text{ accepts } w \overset{\text{(lemma 1)}}{\iff} \text{player } A \text{ has a winning strategy in } G_{\mathcal{A},w}$$

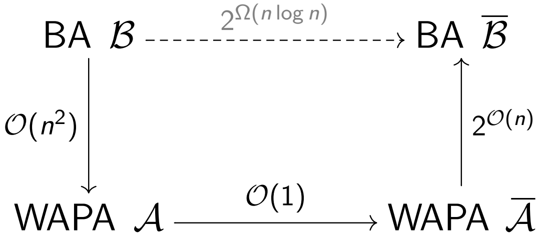
$$\overset{\text{(lemma 3)}}{\iff} \text{player } P \text{ has a winning strategy in } G_{\overline{\mathcal{A}},w}$$

$$\overset{\text{(lemma 2)}}{\iff} \overline{\mathcal{A}} \text{ does } \textit{not} \text{ accept } w$$

□

Outline

Büchi Complementation Algorithm



- Total complexity: $2^{\mathcal{O}(n^2)}$
- Can reach $2^{\mathcal{O}(n \log n)}$ (lower bound) by improving $\overline{\mathcal{A}} \to \overline{\mathcal{B}}$.

References

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Appendix

From BA to WAPA

☺

Given:

- $\mathcal{B} = \langle Q, \Sigma, \delta, q_{in}, F \rangle$: BA
- $n = |Q|$

Construction (BA → WAPA)

$$\mathcal{A} := \langle \underbrace{Q \times \{0, \dots, 2n\}}_{\mathcal{O}(n^2)},\ \Sigma,\ \delta',\ \langle q_{in}, 2n \rangle,\ \pi \rangle$$

where

- $$\delta'(\langle p, i \rangle, a) := \begin{cases} \bigvee_{q \in \delta(p,a)} \langle q, 0 \rangle & \text{if } i = 0 \\ \bigvee_{q \in \delta(p,a)} \langle q, i \rangle \wedge \langle q, i-1 \rangle & \text{if } i \text{ even, } i > 0 \\ \bigvee_{q \in \delta(p,a)} \langle q, i \rangle & \text{if } i \text{ odd, } p \notin F \\ \bigvee_{q \in \delta(p,a)} \langle q, i-1 \rangle & \text{if } i \text{ odd, } p \in F \end{cases}$$
- $\pi(\langle p, i \rangle) := i$

for $p \in Q$, $a \in \Sigma$, $i \in \{0, \dots, 2n\}$

(Thomas and Löding, ∼2000)

From WAPA to BA ☺

Given:

- $\mathcal{A} = \langle Q, \Sigma, \delta, q_{in}, \pi \rangle$: **stratified** WAPA, i.e.
$$\forall p \in Q \;\; \forall a \in \Sigma \;:\; \delta(p, a) \in \mathbb{B}^+\big(\{q \in Q \mid \pi(p) \geq \pi(q)\}\big)$$
- $E \subseteq Q$: all states with even parity

Construction (WAPA → BA)

$$\mathcal{B} := \langle \underbrace{2^Q \times 2^Q}_{2^{\mathcal{O}(n)}},\; \Sigma,\; \delta',\; \langle \{q_{in}\}, \emptyset \rangle,\; 2^Q \times \{\emptyset\} \rangle$$

where

- $\delta'(\langle M, \emptyset \rangle, a) \;:= \Big\{ \langle M', M' \setminus E \rangle \;\Big|\; M' \in \mathrm{Mod}_{\downarrow}\big(\bigwedge_{q \in M} \delta(q, a) \big) \Big\}$
- $\delta'(\langle M, O \rangle, a) := \Big\{ \langle M', O' \setminus E \rangle \;\Big|\; M' \in \mathrm{Mod}_{\downarrow}\big(\bigwedge_{q \in M} \delta(q, a) \big),\; O' \subseteq M',\; O' \in \mathrm{Mod}_{\downarrow}\big(\bigwedge_{q \in O} \delta(q, a) \big) \Big\}$

for $a \in \Sigma$, $M, O \subseteq Q$, $O \neq \emptyset$

(Miyano and Hayashi, 1984)