Graph Automata

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Motivation

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We want an automata model that

- operates on graphs,
- generalizes nested words and tree automata, and
- has some nice properties.

Outline

1. Introduce graph automata and related concepts
2. Proof that their emptiness is decidable[1]
3. Show applications

[1] side conditions apply

Definition of Graph Automata

Let Σ be a finite alphabet and $\mathcal{C}$ be a class of Σ-labeled graphs.
A *graph automaton* on $\mathcal{C}$, $GA = (Q, (T_a)_{a\in\Sigma}, type)$, where

- Q is a finite set of states,
- $T_a \subset Q \times Q$ is a tiling relation for every $a \in \Sigma$, and
- $type : Q \to 2^{\Sigma} \times 2^{\Sigma}$ is the type-relation.

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GA accepts a graph $G = (V, (E_a)_{a\in\Sigma})$ iff there is a map $\rho : V \to Q$ such that

- for every $(u, v) \in E_a$, $(\rho(u), \rho(v)) \in T_a$ and
- for every $v \in V$, $type(\rho(v)) = (In, Out)$, where $In = \{a \mid \exists u.(u, v) \in E_a\}$ and $Out = \{a \mid \exists u.(v, u) \in E_a\}$.

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We restrict ourselves to graphs with at most one incoming and at most one outgoing a-labeled edge for each $a \in \Sigma$ at any vertex.

Example (simplified)

Check a graph for 3-colorability.

$$GA_3 = (Q, (T_a)_{a \in \Sigma}) \text{ where}$$
$$\Sigma = \{a\}, Q = \{q_1, q_2, q_3\} \text{ and } T_a = \{(q_i, q_j) \mid i \neq j\}.$$

Example: 3-color the Petersen graph

Example (proper)

Check a graph for 3-colorability.

$$
\begin{aligned}
GA_3 &= (Q, (T_a)_{a \in \Sigma}, type) \text{ where} \\
\Sigma &= \{a_1, \ldots, a_n\}, \\
Q &= \{q_1, q_2, q_3\} \times (2^\Sigma \times 2^\Sigma), \\
T_a &= \{((q_i, t), (q_j, t')) \mid i \neq j\} \text{ for } a \in \Sigma \text{ and} \\
type((q, t)) &= t \text{ for } (q, t) \in Q.
\end{aligned}
$$

This restricts the graph to at most n incoming and outgoing edges at every vertex.

Goal

Theorem (Madhusudan and Parlato 2011 [2])

Let $\mathcal{C}$ be a class of MSO-definable Σ-labeled graphs. The problem of checking, given $k \in \mathbb{N}$ and a graph automaton GA, whether there is some $G \in \mathcal{C}$ of tree-width at most k that is accepted by GA, is decidable, and decidable in time $|GA|^{\mathcal{O}(k)}$.

Monadic second order logic

We use the following syntax for MSO, where x, y are variables, X is a set of vertices and E_a is an a-labeled edge for $a \in \Sigma$.

$$\varphi ::= x = y \mid E_a(x, y) \mid x \in X \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

Definition of tree-width

The *tree-decomposition* of a graph $G = (V, E)$ is a tuple $(\mathcal{T}, (B_t)_{t \in T})$, where $\mathcal{T} = (T, F)$ is a tree and for every node $t \in T$, $B_t \subseteq V$ is a bag of vertices of G such that

- for every $v \in V$, there is a node $t \in T$ such that $v \in B_t$,
- for every edge $(u, v) \in E$, there is a node $t \in T$ such that $u, v \in B_t$, and
- if $v \in B_t$ and $v \in B_{t'}$, for nodes $t, t' \in T$, then for every t'' that lies on the unique path connecting t and t', $v \in B_{t''}$.

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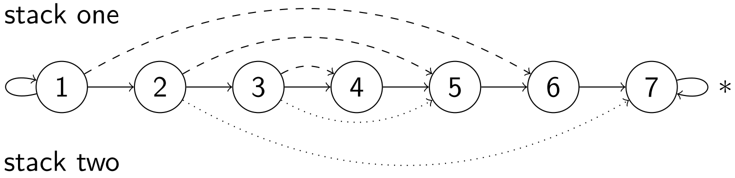
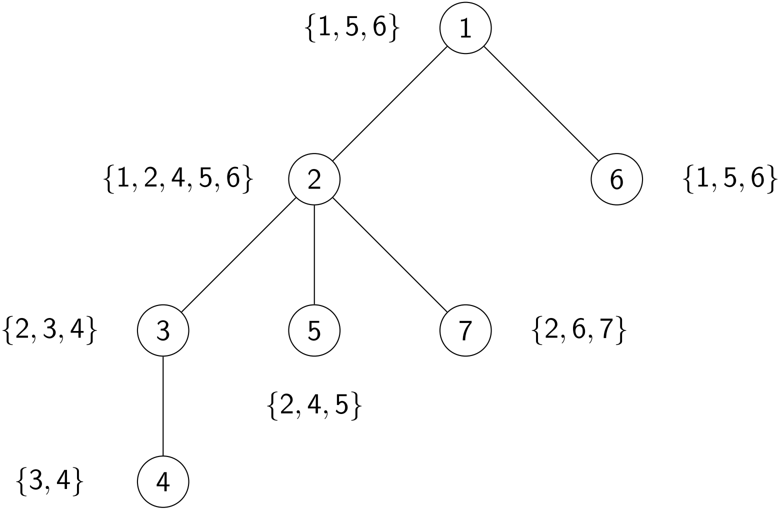
The width of a tree decomposition is the size of the largest bag in it, minus one; i.e. $\max\{\#B_t \mid t \in T\} - 1$.
The *tree-width* of a graph is the smallest of the widths of any of its tree decompositions.

Example



Input word for a 2-NWA.

Example: Tree decomposition



Canonical tree decomposition of the graph. (→ Formal definition)

Some facts about tree-width

- A graph without edges has tree-width 0.
- A tree has tree-width of at most 1.
- A graph with a k-clique has a tree-width of at least $k - 1$.
- A graph with n vertices has a minimal tree decomposition using at most n nodes.
- Many NP-complete problems become tractable on graphs of bounded tree-width.
- Computing tree-widths is NP-hard.

Decidable emptiness

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- Transform $\widehat{\varphi_{\mathcal{C}}}$ into a tree automaton $TA_{\mathcal{C}}$.
- For the graph automaton GA, define a tree automaton TA running over $\mathcal{T}$.
- There is a graph in $\mathcal{C}$ that is accepted by GA iff the intersection of TA and $TA_{\mathcal{C}}$ is not empty.

Labeling of the tree decomposition

The labeling of tree decomposition captures the isomorphism type of the graph.
For a node $v \in T$ and its parent $u \in T$, let the bag $B_v = \{v_1, \dots, v_k\}$ and $B_u = \{u_1, \dots, u_k\}$.
The label for v will be $((L_a)_{a \in \Sigma}, P, W)$ where

- $L_a = \{(i,j) \mid (v_i, v_j) \in E_a\}$
- $P = \{(i,j) \mid v_i = u_j\}$ and
- $W = \{(i,j) \mid v_i = v_j\}$.

Note: Using more careful encoding, this can be achieved using $\mathcal{O}(2^k)$ instead of $\mathcal{O}(2^{k^2})$ many labels [2].

Graph automaton as tree automaton

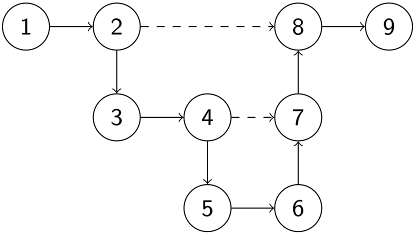
For a graph automaton $GA = (Q, (T_a)_{a\in\Sigma}, type)$ define a bottom-up tree automaton $B = (Labels, Q', Q', \Delta)$ where $Q' = (Q \times 2^{\Sigma} \times 2^{\Sigma})^{k+1}$ and the transition rules

- check that the state at the node respects the tiling requirements T_a and
- accumulate the *In* and *Out* sets for every vertex and cross-references them with the constraints in *type*.

Applications

Nested word automata

- Nested words have a tree-width of at most 2.
- Therefore NWAs have decidable emptiness.



a nested word

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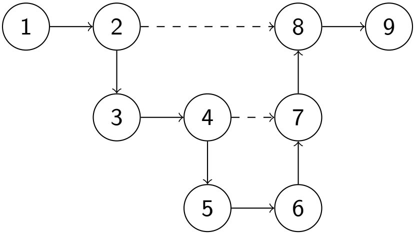

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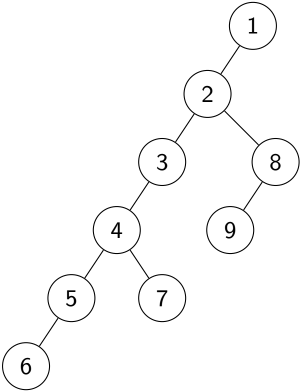

its tree decomposition

n-NWAs

- Generalize NWAs to have n instead of just one nesting relation.
- Corresponds to a PDA with n stacks.
- n-NWAs have undecidable emptiness.
- Therefore n-nested words have unbounded treewidth.

Bounded context switching NWAs

Bounded context switching NWA is an n-NWA where each word is partitioned into at most $k+1$ "contexts". Each context utilizes at most one of the n stacks.

- Tree-width of $k+1$.
- Decidable emptiness.

Other modifications to NWAs

- k-phase n-NWAs: in each phase any stack can be pushed, but only one stack can be popped. Tree-width: $3 \cdot 2^{k-1} + 1$.

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- k-phase n-NWAs: in each phase any stack can be pushed, but only one stack can be popped. Tree-width: $3 \cdot 2^{k-1} + 1$.
- Ordered n-NWAs: any stack can be pushed, but a stack can be popped only if all stacks with with lower index are empty.
  Tree-width: $(n + 1) \cdot 2^{n-1} + 1$.

Further topics

- Formal definition of the canonical tree decomposition
- Efficient coding of tree labels
- Courcelle' theorem
- Simulation of tree automata
- Recognizing connected graphs

Summary

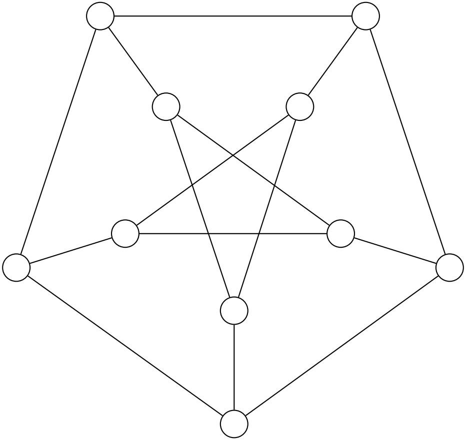
- Graph automata are a powerful automata model.
- Restriction to an MSO-definable class $\mathcal{C}$ of graphs with bounded tree-width yields decidable emptiness.
- Graph automata naturally generalize nested word automata and various modifications thereof.
- However, our definition of graph automata is not particularly useful for problems on graphs.

References

- P. Madhusudan, Gennaro Parlato. *The Tree Width of Auxiliary Storage.* In, POPL, Austin, TX, USA, 26 - 28 Jan 2011. ACM, 283-294.
- P. Madhusudan and G. Parlato. *The tree width of automata with auxiliary storage.* In IDEALS Technical Report, `http://hdl.handle.net/2142/15433`, April 2010.
- W. Thomas. *On logics, tilings, and automata.* In J. L. Albert, B. Monien, and M. Rodriguez-Artalejo, editors, ICALP, volume 510 of Lecture Notes in Computer Science, pages 441-454. Springer, 1991.
- J. Flum and M. Grohe. *Parameterized Complexity Theory (Texts in Theoretical Computer Science. An EATCS Series).* Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2006.

Appendix

3-color the Petersen graph



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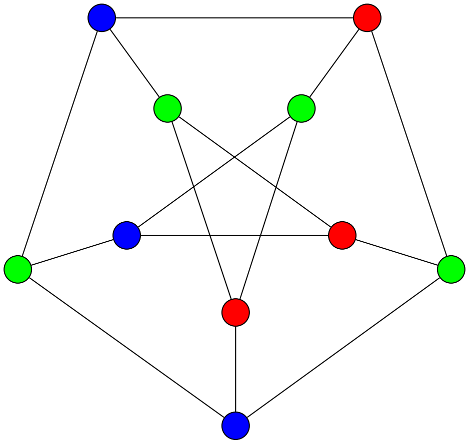


→ Go back

Canonical tree decomposition for nested words

For any n-nested word $N = (V, Init, Final, L, (E_j)_{0 \leq j < n})$, the canonical tree-decomposition of N, $\mathrm{can\text{-}td}(N) = (\mathcal{T}, (B_t)_{t \in T})$ is defined as follows.

- The set of nodes of the tree $\mathcal{T}$ are the vertices V of N.
- If $(u, v) \in E_j$, then v is the right-child of u in $\mathcal{T}$.
- If $(u, v) \in L$ and for all $0 \leq j < n$ and $z \in V$, $(z, v) \notin E_j$, then v is the left-child of u.

The bags B_v associate the minimum set of vertices to the nodes $v \in T$ that satisfy the following.

- For all $v \in V$, $v \in B_v$.
- For every $u, v \in V$, if u is the parent of v in $\mathcal{T}$, then $u \in B_v$.
- For $u, v \in V$, if $(u, v) \in L$ then $u \in B_z$ for all vertices z that are on the unique path from u to v in $\mathcal{T}$.

→ Go back

Labeling of the tree decomposition using $\mathcal{O}(2^k)$ labels

For a node $v \in T$ and its parent $u \in T$, let the bag $B_v \subseteq \{v_1, \ldots, v_k\}$ and $B_u \subseteq \{u_1, \ldots, u_k\}$ where $v_i \neq v_j$ and $u_i \neq u_j$ for $i \neq j$.

Without loss of generality one can assume

- that the vertex $v_i \in B_v$ is equal to a vertex in the parent bag $u_j \in B_u$ iff $i = j$ and
- that every edge node in the tree captures at most one edge in the graph.

The label for v will be $((L_a)_{a \in \Sigma}, P, W)$ where

- $L_a = (i, j)$, where $(v_i, v_j) \in E_a$ and $v_i, v_j \in B_v$,
- $P = \{i \mid v_i = u_i, v_i \in B_v, u_j \in B_u\}$ and
- $W = \{i \mid v_i \in B_v\}$.

This encoding uses $(k^2)^{\#\Sigma} \cdot 2^k \cdot 2^k = \mathcal{O}(2^k)$ many labels [2].

→ Go back

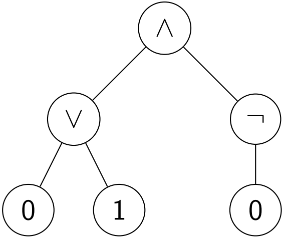
Simulating tree automata

Simulating tree automata induces the following difficulties:

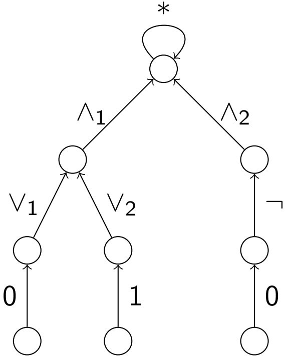
1. Tree automata ignore vertex types.
2. Tree automata have labeled nodes, graph automata labeled edges.
3. Every edge has to specify its position in the predicate.

Simulating tree automata: Example

Consider the tree language of valid propositional logic formulae.



Example tree from the tree automata presentation



Corresponding input for the graph automaton

→ Go back

Courcelle's theorem

Theorem (Courcelle [4])

Every graph property definable in monadic second-order logic can be decided in linear time on graphs of bounded tree-width.

→ Go back