Automata Theory

Nested Word Automata

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June 4th, 2012

Overview

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Regular language

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procedure foo()
{
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}
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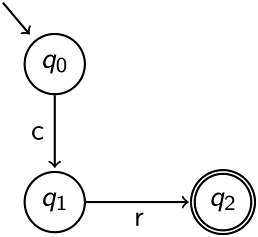

(det.) Context-free language

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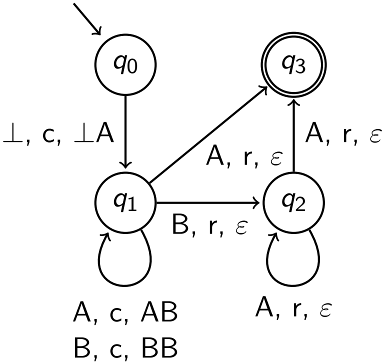

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comparison of numbers	constants	two variables
closure		
decidability		
determinize		

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Question: Is there some class of languages in between that is more expressive than regular languages, but keeps their nice properties?

Answer (Alur & Madhusudan 2004): yes, at least in some sense

Visibly pushdown languages (VPLs)

A *visibly pushdown language* (VPL) is the language accepted by a *visibly pushdown automaton* (VPA).

A VPA $\mathcal{A} = \langle Q, q_0, Q_f, \Sigma, \Gamma, \perp, \delta \rangle$ is a deterministic PDA with special rules: Determined by the input symbol, only one symbol per **push** is allowed and reading the stack implies a **pop**.

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- $\delta = \delta_i \uplus \delta_c \uplus \delta_r$,
    - $\delta_i \subseteq Q \times \Sigma_i \to Q$
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Note: pops occur implicitly, $\perp$ never popped, no ε

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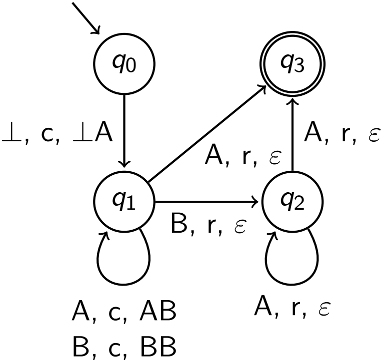

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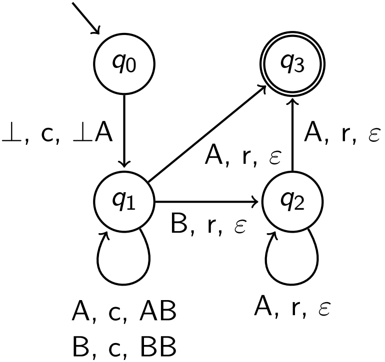

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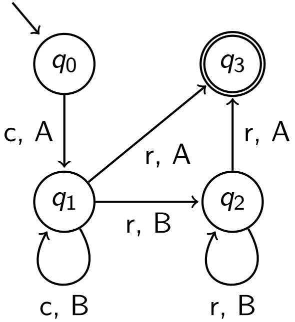

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$$\begin{aligned}\delta_c = \{\ & (q_0, c, A, q_1),\\ & (q_1, c, B, q_1)\ \ \}\\ \delta_r = \{\ & (q_1, r, A, q_3),\\ & (q_1, r, B, q_2),\\ & (q_2, r, A, q_3),\\ & (q_2, r, B, q_2)\ \ \}\end{aligned}$$

From VPAs to NWAs

- main differences between VPAs and PDAs:
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  $\rightarrow$ NWAs $\preceq$ deterministic PDAs
- main idea: call and return symbols are matched in the input

Overview

Well nested sequences

A sequence of symbols is *well nested* if calls and returns are matched without crossing, i.e., for any different call-return-pairs (c_i, r_i), (c_j, r_j), $c_i < c_j < r_i < r_j$ is forbidden.

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Note: Every sequence has a unique well nesting.

Nested words

A relation $\rightsquigarrow \subset \{-\infty, 1, 2, \ldots, \ell\} \times \{1, 2, \ldots, \ell, \infty\}$ of length $\ell \geq 0$ is a *matching relation* if the following holds:

- I if $i \rightsquigarrow j$, then $i < j$ (monotone)
- II if $i_1 \rightsquigarrow j$ and $i_2 \rightsquigarrow j$, then $i_1 = i_2$ (left-unique)
  if $i \rightsquigarrow j_1$ and $i \rightsquigarrow j_1$, then $j_1 = j_2$ (right-unique)
- III if $i_1 \rightsquigarrow j_1$ and $i_2 \rightsquigarrow j_2$, then we have not $i_1 < i_2 < j_1 < j_2$ (well nested)

Explanation:

- I not **r c**, not reflexive
- II not **c c r**, not **c r r**
- III not **c c r r**

$\boxed{\text{ex post note: } (-\infty, \infty) \notin \rightsquigarrow, \; \pm\infty \text{ excluded from uniqueness}}$

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If $i \rightsquigarrow j$, i is a *call position* and j is a *return position*. All the rest is an *internal position*. If $i \neq -\infty$ and $j \neq \infty$, they are *well-matched*, otherwise *pending*. $e \in \rightsquigarrow$ is a *nesting edge*.

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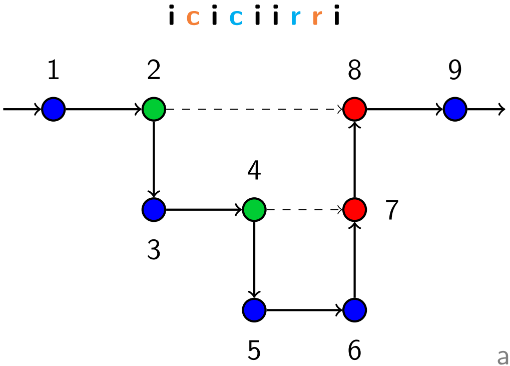
A *nested word* n over Σ is a pair $(a_1 \cdots a_\ell, \rightsquigarrow)$, where $a_i \in \Sigma$ and $\rightsquigarrow$ is a matching relation of length ℓ.

Example 1

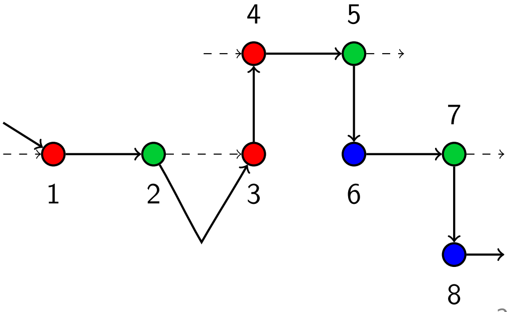


adapted from [1]

Here: $2 \rightsquigarrow 8$, $4 \rightsquigarrow 7$ and the whole word is well-matched.

Example 2

r c r r c i c i



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Here: $-\infty \rightsquigarrow 1$, $2 \rightsquigarrow 3$, $-\infty \rightsquigarrow 4, 5 \rightsquigarrow \infty, 7 \rightsquigarrow \infty$ and only $2 \rightsquigarrow 3$ is well-matched.

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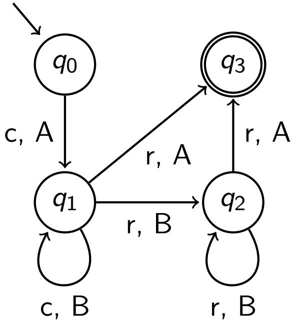
acceptance via both Q_f and P_f
as VPAs: at return implicitly go to hierarchical state before matching call

$\mathcal{L}_2$ as NWA

Consider again $\mathcal{L}_2 = \{\mathrm{c}^n\ \mathrm{r}^n \mid n > 0\}$.

We construct an NWA for $\mathcal{L}_2' := \{(\langle\mathrm{c})^n\ (\mathrm{r}\rangle)^n \mid n > 0\}$.



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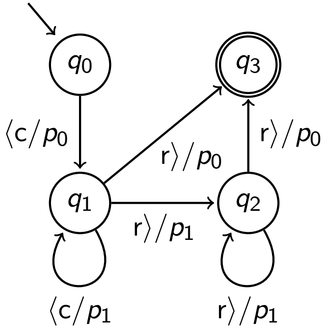

$P = \{p_0, p_1\}$, $P_f \subseteq \{p_0\}$

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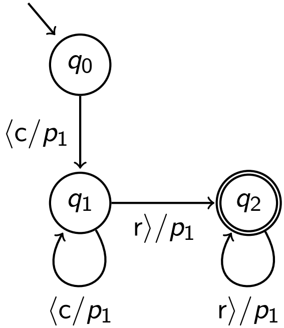
We can also use hierarchical states for acceptance.



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  possibly exponentially more states for deterministic NWAs
- not all sets of NWs acceptable by NWAs
  $\{(\langle a)^n(b\rangle)^n \mid n > 0\}$ vs. $\{a^n b^n \mid n > 0\}$

Comparison of properties

	DFA	DNWA	PDA	DPDA
pre-/suffix	✓	✓	✓	✓
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emptiness	NLOGSPACE	PTIME	PTIME	PTIME
equivalence	NLOGSPACE	PTIME	undecidable	decidable
inclusion	NLOGSPACE	PTIME	undecidable	undecidable

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emptiness	NLOGSPACE	PTIME	PTIME	PTIME
equivalence	NLOGSPACE	PTIME	undecidable	decidable
inclusion	NLOGSPACE	PTIME	undecidable	undecidable

Note: Equivalence and inclusion problem are EXPTIME-complete for nondeterministic NWAs.
Implication: determinization $\in \Omega(\text{EXPTIME})$ if at all possible

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- handle hierarchical proceeding when reading return symbols

The states: definition

Consider the NNWA $\mathcal{A} = \langle Q, Q_0, Q_f, P, P_0, \overbrace{P_f}^{\stackrel{\text{wlog}}{=} P}, \delta_i, \delta_c, \delta_r \rangle$.
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The states: semantics

Consider a nested word n with k pending calls. We can write this

$$n = n_1 \langle c_1 n_2 \langle c_2 \cdots n_k \langle c_k n_{k+1}$$

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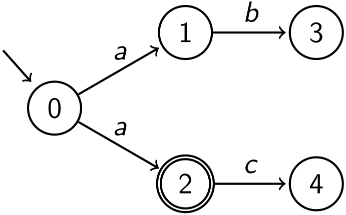
Example



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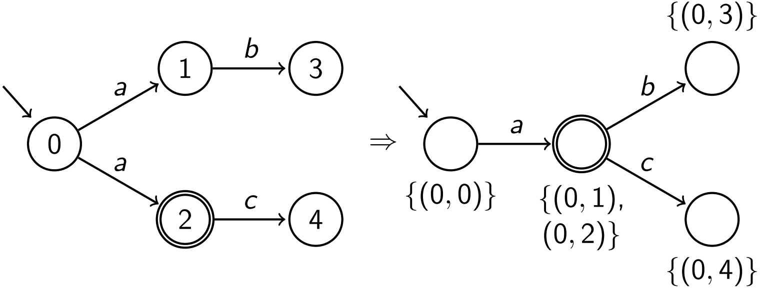

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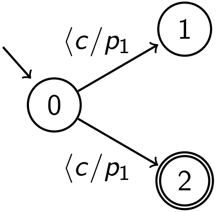

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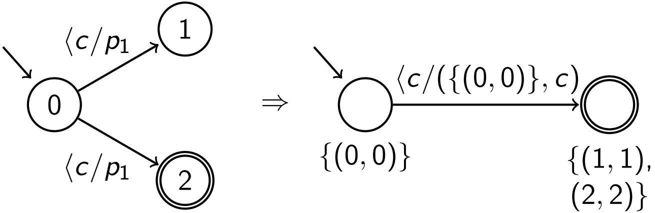

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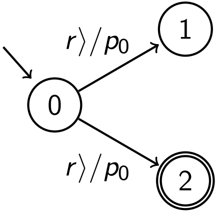

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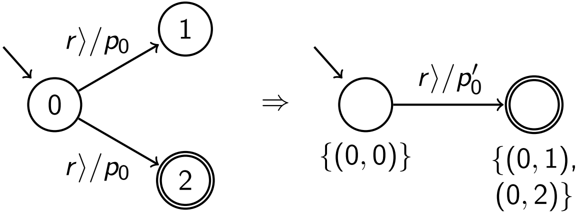

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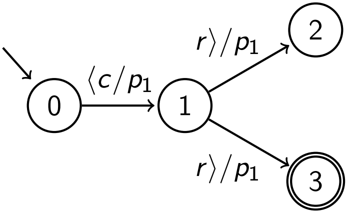

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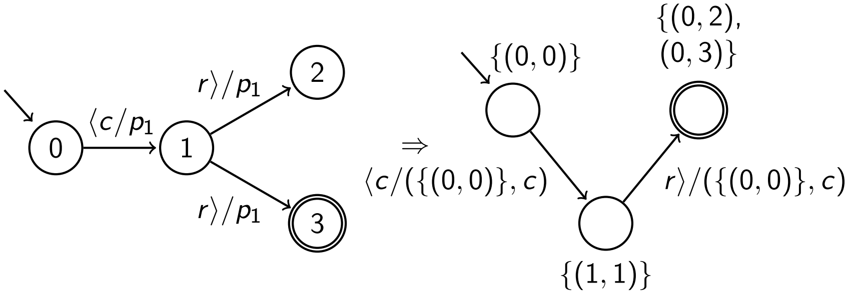

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  This is succinct, so there exists an example where the DNWA cannot have less states.

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- recent concept, time will show the relevance

References

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