Decision Procedures

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• Syntax and Semantics of First Order Logic (FOL)

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 - Putting it all together (Nelson-Oppen).

First-Order Logic

Also called Predicate Logic or Predicate Calculus

FOL Syntax	
variables	x, y, z, \cdots
<u>constants</u>	a, b, c, \cdots
<u>functions</u>	f, g, h, \cdots with arity $n > 0$



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functions	f, g, h, \cdots with arity $n > 0$
<u>terms</u>	variables, constants or
	n-ary function applied to n terms as arguments
	a, x, f(a), g(x, b), f(g(x, f(b)))



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	a, x, f(a), g(x, b), f(g(x, f(b)))
predicates	p, q, r, \cdots with arity $n \ge 0$
atom	op , ot , or an n-ary predicate applied to n terms
literal	atom or its negation
	$p(f(x),g(x,f(x))), \neg p(f(x),g(x,f(x)))$

Note: 0-ary functions: constant 0-ary predicates: P, Q, R, \dots

quantifiers

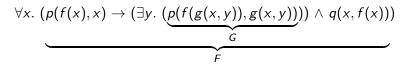
existential quantifier $\exists x.F[x]$ "there exists an x such that F[x]" universal quantifier $\forall x.F[x]$ "for all x, F[x]"

 $\begin{array}{ll} \underline{\text{FOL formula}} & \text{literal, application of logical connectives} \\ (\neg, \lor, \land, \rightarrow, \leftrightarrow) \text{ to formulae,} \\ \text{ or application of a quantifier to a formula} \end{array}$

Example



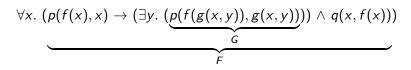
FOL formula



The scope of $\forall x \text{ is } F$. The scope of $\exists y \text{ is } G$. Example



FOL formula



The scope of $\forall x$ is F. The scope of $\exists y$ is G. The formula reads: "for all x, if p(f(x), x)then there exists a y such that p(f(g(x, y)), g(x, y)) and q(x, f(x))"

• The length of one side of a triangle is less than the sum of the lengths of the other two sides

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 $\forall x, y, z. triangle(x, y, z) \rightarrow length(x) < length(y) + length(z)$



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- UNI FREIBURG
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• Fermat's Last Theorem.

$$\forall n. integer(n) \land n > 2 \rightarrow \forall x, y, z. integer(x) \land integer(y) \land integer(z) \land x > 0 \land y > 0 \land z > 0 \rightarrow x^{n} + y^{n} \neq z^{n}$$

Pumping Lemma



Pumping Lemma



For every regular Language *L* there is some $n \ge 0$, such that for all words $z \in L$ with $|z| \ge n$ there is a decomposition z = uvw with $|v| \ge 1$ and $|uv| \le n$, such that for all $i \ge 0$: $uv^i w \in L$.

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$$\begin{array}{l} \forall L. \ regular language(L) \rightarrow \\ \exists n. \ integer(n) \land n \geq 0 \land \\ \forall z. \ z \in L \land |z| \geq n \rightarrow \\ \exists u, v, w. \ word(u) \land word(v) \land word(w) \land \\ z = uvw \land |v| \geq 1 \land |uv| \leq n \land \\ \forall i. \ integer(i) \land i \geq 0 \rightarrow uv^{i}w \in L \end{array}$$

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Predicates: regularlanguage, integer, word, $\cdot \in \cdot, \cdot \leq \cdot, \cdot \geq \cdot, \cdot = \cdot$

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Predicates: regularlanguage, integer, word, $\cdot \in \cdot, \cdot \leq \cdot, \cdot \geq \cdot, \cdot = \cdot$ Constants: 0, 1 Functions: $|\cdot|$ (word length), concatenation, iteration

An interpretation $I : (D_I, \alpha_I)$ consists of:

• Domain D_I

non-empty set of values or objects



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An interpretation $I : (D_I, \alpha_I)$ consists of:

• Domain D_I non-empty set of values or objects for example D_I = playing cards (finite), integers (countable infinite), or reals (uncountable infinite)

• Assignment α_I

• each variable x assigned value $\alpha_I[x] \in D_I$



FOL Semantics

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$$\alpha_I[f]: D_I^n \to D_I$$

In particular, each constant a (0-ary function) assigned value $\alpha_I[a] \in D_I$

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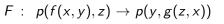
$$\alpha_I[f] : D_I^n \rightarrow D_I$$

In particular, each constant a (0-ary function) assigned value $\alpha_I[a] \in D_I$

• each n-ary predicate p assigned

$$\alpha_I[p]: D_I^n \to \{\top, \bot\}$$

In particular, each propositional variable P (0-ary predicate) assigned truth value $(\top,\,\perp)$



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$$F : p(f(x,y),z) \rightarrow p(y,g(z,x))$$

Interpretation
$$I : (D_I, \alpha_I)$$

 $D_I = \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ integers
 $\alpha_I[f] : D_I^2 \to D_I \qquad \alpha_I[g] : D_I^2 \to D_I$
 $(x, y) \mapsto x + y \qquad (x, y) \mapsto x - y$
 $\alpha_I[p] : D_I^2 \to \{\top, \bot\}$
 $(x, y) \mapsto \begin{cases} \top & \text{if } x < y \\ \bot & \text{otherwise} \end{cases}$
Also $\alpha_I[x] = 13, \alpha_I[y] = 42, \alpha_I[z] = 1$
Compute the truth value of F under I

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Compute the truth value of F under I

1.
$$I \not\models p(f(x,y),z)$$
since $13 + 42 \ge 1$ 2. $I \not\models p(y,g(z,x))$ since $42 \ge 1 - 13$ 3. $I \models F$ by 1, 2, and \rightarrow

F is true under I

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For a variable x:

Definition (x-variant)

An x-variant of interpretation I is an interpretation J : (D_J, α_J) such that

- $D_I = D_J$
- $\alpha_I[y] = \alpha_J[y]$ for all symbols y, except possibly x

That is, I and J agree on everything except possibly the value of x

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Denote $J : I \triangleleft \{x \mapsto v\}$ the x-variant of I in which $\alpha_J[x] = v$ for some $v \in D_I$. Then

•
$$I \models \forall x. F$$
 iff for all $v \in D_I, I \triangleleft \{x \mapsto v\} \models F$

• $I \models \exists x. F$ iff there exists $v \in D_I$ s.t. $I \triangleleft \{x \mapsto v\} \models F$

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Consider

$$F: \forall x. \exists y. 2 \cdot y = x$$

Here $2 \cdot y$ is the infix notatation of the term (2, y), and $2 \cdot y = x$ is the infix notatation of the atom = ((2, y), x).

- 2 is a 0-ary function symbol (a constant).
- · is a 2-ary function symbol.
- = is a 2-ary predicate symbol.
- x, y are variables.

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What is the truth-value of F?





$F: \forall x. \exists y. 2 \cdot y = x$

Let *I* be the standard interpration for integers, $D_I = \mathbb{Z}$. Compute the value of *F* under *I*:





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iff

for all
$$\mathsf{v} \in D_I$$
, $I \triangleleft \{x \mapsto \mathsf{v}\} \models \exists y. \ 2 \cdot y = x$

iff

for all $v \in D_I$, there exists $v_1 \in D_I$, $I \triangleleft \{x \mapsto v\} \triangleleft \{y \mapsto v_1\} \models 2 \cdot y = x$

The latter is false since for $1 \in D_I$ there is no number v_1 with $2 \cdot v_1 = 1$.





$F: \forall x. \exists y. 2 \cdot y = x$

Let *I* be the standard interpration for rational numbers, $D_I = \mathbb{Q}$. Compute the value of *F* under *I*:

Example (\mathbb{Q})



$$F: \forall x. \exists y. 2 \cdot y = x$$

Let *I* be the standard interpration for rational numbers, $D_I = \mathbb{Q}$. Compute the value of *F* under *I*:

$$I \models \forall x. \exists y. 2 \cdot y = x$$

iff

for all
$$\mathsf{v} \in D_I$$
, $I \triangleleft \{x \mapsto \mathsf{v}\} \models \exists y. \ 2 \cdot y = x$

iff

for all $v \in D_I$, there exists $v_1 \in D_I$, $I \triangleleft \{x \mapsto v\} \triangleleft \{y \mapsto v_1\} \models 2 \cdot y = x$

The latter is true since for $v \in D_I$ we can choose $v_1 = \frac{v}{2}$.



Definition (Satisfiability)

F is satisfiable iff there exists an interpretation I such that $I \models F$.

Definition (Validity)

F is valid iff for all interpretations I, $I \models F$.



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Note

F is valid iff $\neg F$ is unsatisfiable

$$F: \forall y. (p(x,y) \rightarrow p(y,x))$$

should be transformed to

$$G : \forall y. (p(a, y) \rightarrow p(y, a))$$

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should be transformed to

$$G: \forall y. (p(a, y) \rightarrow p(y, a))$$

We call the mapping from x to a a substituion denoted as $\sigma : \{x \mapsto a\}$. We write $F\sigma$ for the formula G.

Another convenient notation is F[x] for a formula containing the variable x and F[a] for $F\sigma$.



Definition (Substitution)

A substitution is a mapping from terms to terms, e.g.

$$\sigma : \{t_1 \mapsto s_1, \ldots, t_n \mapsto s_n\}$$

By $F\sigma$ we denote the application of σ to formula F, i.e., the formula F where all occurences of t_1, \ldots, t_n are replaced by s_1, \ldots, s_n .

For a formula named F[x] we write F[t] as shorthand for $F[x]{x \mapsto t}$.

Safe Substitution

Care has to be taken in the presence of quantifiers:

$$F[x] : \exists y. \ y = Succ(x)$$

What is F[y]?

Safe Substitution

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Care has to be taken in the presence of quantifiers:

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What is F[y]? We need to rename bounded variables occuring in the substitution:

$$F[y]$$
 : $\exists y'. y' = Succ(y)$

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What is F[y]? We need to rename bounded variables occuring in the substitution:

$$F[y]$$
 : $\exists y'. y' = Succ(y)$

Bounded renaming does not change the models of a formula:

$$(\exists y. y = Succ(x)) \Leftrightarrow (\exists y'. y' = Succ(x))$$

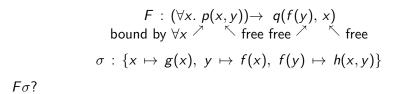
Recursive Definition of Substitution

$$t\sigma = \begin{cases} \sigma(t) & t \in \operatorname{dom}(\sigma) \\ f(t_1\sigma,\ldots,t_n\sigma) & t \notin \operatorname{dom}(\sigma) \wedge t = f(t_1,\ldots,t_n) \\ x & t \notin \operatorname{dom}(\sigma) \wedge t = x \end{cases}$$
$$p(t_1,\ldots,t_n)\sigma = p(t_1\sigma,\ldots,t_n\sigma) \\ (\neg F)\sigma = \neg(F\sigma) \\ (F \wedge G)\sigma = (F\sigma) \wedge (G\sigma) \\ \cdots$$

$$(\forall x. F)\sigma = \begin{cases} \forall x. F\sigma & x \notin Vars(\sigma) \\ \forall x'. ((F\{x \mapsto x'\})\sigma) & \text{otherwise and } x' \text{ is fresh} \end{cases}$$
$$(\exists x. F)\sigma = \begin{cases} \exists x. F\sigma & x \notin Vars(\sigma) \\ \exists x'. ((F\{x \mapsto x'\})\sigma) & \text{otherwise and } x' \text{ is fresh} \end{cases}$$

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Example: Safe Substitution $F\sigma$



Example: Safe Substitution $F\sigma$

$$F: (\forall x. \ p(x, y)) \rightarrow q(f(y), x)$$

bound by $\forall x \nearrow free free \nearrow free$
$$\sigma : \{x \mapsto g(x), \ y \mapsto f(x), \ f(y) \mapsto h(x, y)\}$$

$$F\sigma?$$

Rename

$$F': \forall x'. \ p(x', y) \rightarrow q(f(y), x)$$

$$\uparrow \qquad \uparrow$$

where x' is a fresh variable

Semantic Tableaux

Recall rules from propositional logic:

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The following additional rules are used for quantifiers:

$$\frac{I \models \forall x.F[x]}{I \models F[t]} \text{ for any term } t \qquad \frac{I \not\models \forall x.F[x]}{I \not\models F[a]} \text{ for a fresh constant } a$$
$$\frac{I \not\models \forall x.F[x]}{I \not\models F[a]} \text{ for a fresh constant } a$$
$$\frac{I \not\models \exists x.F[x]}{I \not\models F[t]} \text{ for any term } t$$

(We assume that there are infinitely many constant symbols.)

The formula F[t] is created from the formula F[x] by the substitution $\{x \mapsto t\}$ (roughly, replace every x by t).



Show that $(\exists x. \forall y. p(x, y)) \rightarrow (\forall x. \exists y. p(y, x))$ is valid.



Show that $(\exists x. \forall y. p(x, y)) \rightarrow (\forall x. \exists y. p(y, x))$ is valid. Assume otherwise.

1.
$$I \not\models (\exists x. \forall y. p(x, y)) \rightarrow (\forall x. \exists y. p(y, x))$$
assumption2. $I \models \exists x. \forall y. p(x, y)$ 1 and \rightarrow 3. $I \not\models \forall x. \exists y. p(y, x)$ 1 and \rightarrow 4. $I \models \forall y. p(a, y)$ 2, $\exists (x \mapsto a \text{ fresh})$ 5. $I \not\models \exists y. p(y, b)$ 3, $\forall (x \mapsto b \text{ fresh})$ 6. $I \models p(a, b)$ 4, $\forall (y \mapsto b)$ 7. $I \not\models p(a, b)$ 5, $\exists (y \mapsto a)$ 8. $I \models \bot$ 6,7 contradictory

Thus, the formula is valid.



Is F : $(\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))$ valid?.



Is F : $(\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))$ valid?.

Assume I is a falsifying interpretation for F and apply semantic argument:

1.
$$I \not\models (\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))$$

2. $I \models \forall x. p(x, x)$
3. $I \not\models \exists x. \forall y. p(x, y)$
4. $I \models p(a_1, a_1)$
5. $I \not\models \forall y.p(a_1, y)$
6. $I \not\models p(a_2, a_2)$
7. $I \models p(a_2, a_2)$
8. $I \not\models p(a_2, a_3)$
9. $I \not\models p(a_2, a_3)$
1 and \rightarrow
2, \forall
3, \exists
3, \exists
4, $I \models p(a_2, a_2)$
5, \forall
3, \exists
4, $I \models p(a_2, a_3)$
5, \forall
5,

No contradiction.



Is F : $(\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))$ valid?.

Assume I is a falsifying interpretation for F and apply semantic argument:

1.
$$I \not\models (\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))$$

2. $I \models \forall x. p(x, x)$
3. $I \not\models \exists x. \forall y. p(x, y)$
4. $I \models p(a_1, a_1)$
5. $I \not\models \forall y.p(a_1, y)$
6. $I \not\models p(a_2, a_2)$
7. $I \models p(a_2, a_2)$
8. $I \not\models p(a_2, a_3)$
9. $I \not\models p(a_2, a_3)$
1 and \rightarrow
2, \forall
3, \exists
3, \exists
9. $I \not\models p(a_2, a_3)$
1 and \rightarrow
2, \forall
3, \exists
9, $I \not\models p(a_2, a_3)$
1 and \rightarrow
1 and

No contradiction. Falsifying interpretation I can be "read" from proof:

$$D_I = \mathbb{N}, \quad p_I(x, y) = \begin{cases} \text{true} & y = x, \\ \text{false} & y = x + 1, \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$



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Non-termination

For an invalid formula F the method is not guaranteed to terminate. Thus, the semantic argument is not a decision procedure for validity.



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If for interpretation I the assumption of the proof hold then there is an interpretation I' and a branch such that all statements on that branch hold.

I' differs from I in the values $\alpha_I[a_i]$ of fresh constants a_i .

If all branches of the proof end with $I \models \bot$, then the assumption was wrong. Thus, if the assumption was $I \not\models F$, then F must be valid.



Consider (finite or infinite) proof trees starting with $I \not\models F$. We assume that

- all possible proof rules were applied in all non-closed branches.
- the ∀ and ∃ rules were applied for all terms.
 This is possible since the terms are countable.

If every branch is closed, the tree is finite (Kőnig's Lemma) and we have a finite proof for F.

Otherwise, the proof tree has at least one open branch *P*. We show that *F* is not valid.

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The statements on that branch *P* form a Hintikka set:

•
$$I \models F \land G \in P$$
 implies $I \models F \in P$ and $I \models G \in P$.

•
$$I \not\models F \land G \in P$$
 implies $I \not\models F \in P$ or $I \not\models G \in P$.

- $I \models \forall x. F[x] \in P$ implies for all terms $t, I \models F[t] \in P$.
- $I \not\models \forall x. F[x] \in P$ implies for some term $a, I \not\models F[a] \in P$.

• Similarly for
$$\lor, \rightarrow, \leftrightarrow, \exists$$
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2 Choose $D_I := \{t \mid t \text{ is term}\}, \alpha_I[f](t_1, \dots, t_n) = f(t_1, \dots, t_n),$

$$\alpha_{I}[x] = x, \quad \alpha_{I}[p](t_{1}, \dots, t_{n}) = \begin{cases} \text{true} & I \models p(t_{1}, \dots, t_{n}) \in P \\ \text{false} & \text{otherwise} \end{cases}$$

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I satisfies all statements on the branch. In particular, I is a falsifying interpretation of F, thus F is not valid.



Also in first-order logic normal forms can be used:

- Devise an algorithm to convert a formula to a normal form.
- Then devise an algorithm for satisfiability/validity that only works on the normal form.

Negation Normal Forms (NNF)

Negations appear only in literals. (only $\neg, \land, \lor, \exists, \forall$) To transform *F* to equivalent *F'* in NNF use recursively the following template equivalences (left-to-right):

$$\neg \neg F_1 \Leftrightarrow F_1 \quad \neg \top \Leftrightarrow \bot \quad \neg \bot \Leftrightarrow \top$$
$$\neg (F_1 \land F_2) \Leftrightarrow \neg F_1 \lor \neg F_2 \\ \neg (F_1 \lor F_2) \Leftrightarrow \neg F_1 \land \neg F_2 \\ \end{bmatrix}$$
De Morgan's Law
$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \lor F_2 \\ F_1 \leftrightarrow F_2 \Leftrightarrow (F_1 \rightarrow F_2) \land (F_2 \rightarrow F_1)$$

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$$F_{1} \leftrightarrow F_{2} \Leftrightarrow (F_{1} \rightarrow F_{2}) \land (F_{2} \rightarrow F_{1})$$
$$\neg \forall x. \ F[x] \Leftrightarrow \exists x. \ \neg F[x]$$
$$\neg \exists x. \ F[x] \Leftrightarrow \forall x. \ \neg F[x]$$

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$G : \forall x. (\exists y. p(x, y) \land p(x, z)) \rightarrow \exists w. p(x, w) .$



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$$\forall x. (\exists y. p(x, y) \land p(x, z)) \rightarrow \exists w. p(x, w)$$

$$\forall x. \neg (\exists y. p(x, y) \land p(x, z)) \lor \exists w. p(x, w)$$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \lor F_2$$

$$\forall x. (\forall y. \neg (p(x, y) \land p(x, z))) \lor \exists w. p(x, w)$$

$$\neg \exists x. F[x] \Leftrightarrow \forall x. \neg F[x]$$

$$\forall x. (\forall y. \neg p(x, y) \lor \neg p(x, z)) \lor \exists w. p(x, w)$$

$$Q_1 x_1 \cdots Q_n x_n$$
. $F[x_1, \cdots, x_n]$

where $Q_i \in \{\forall, \exists\}$ and F is quantifier-free.

Every FOL formula F can be transformed to formula F' in PNF s.t. $F' \Leftrightarrow F$:

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- Move all quantifiers to the front

Example: PNF



Find equivalent PNF of

 $F : \forall x. ((\exists y. p(x, y) \land p(x, z)) \rightarrow \exists y. p(x, y))$



Find equivalent PNF of

 $F : \forall x. ((\exists y. p(x, y) \land p(x, z)) \rightarrow \exists y. p(x, y))$

• Write F in NNF

$$F_1$$
: $\forall x. (\forall y. \neg p(x, y) \lor \neg p(x, z)) \lor \exists y. p(x, y)$

• Rename quantified variables to fresh names

$$F_2 : \forall x. (\forall y. \neg p(x, y) \lor \neg p(x, z)) \lor \exists w. p(x, w)$$

 ^ in the scope of $\forall x$

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Example: PNF

• Move all quantifiers to the front

$$F_3$$
: $\forall x. \forall y. \exists w. \neg p(x, y) \lor \neg p(x, z) \lor p(x, w)$

Alternately,

$$F'_3$$
: $\forall x. \exists w. \forall y. \neg p(x, y) \lor \neg p(x, z) \lor p(x, w)$

Note: In F_2 , $\forall y$ is in the scope of $\forall x$, therefore the order of quantifiers must be $\cdots \forall x \cdots \forall y \cdots$

$$F_4 \Leftrightarrow F \text{ and } F'_4 \Leftrightarrow F$$

Note: However $G \Leftrightarrow F$

$$G$$
 : $\forall y. \exists w. \forall x. \neg p(x, y) \lor \neg p(x, z) \lor p(x, w)$

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There does not exist an algorithm for deciding if a FOL formula F is valid, i.e. always halt and says "yes" if F is valid or say "no" if F is invalid.



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On the other hand,

• PL is decidable

There exists an algorithm for deciding if a PL formula F is valid, e.g., the truth-table procedure.

Similarly for satisfiability

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Decision Procedures